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METHODS FOR CONSTRUCTION OF PERIODIC MOTIONS IN PIECEWISE LINEAR SYSTEMS

(A survey of Soviet Works)

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The development of self oscillations and forced oscillations in non-linear systems comes to the statement of proper conditions of existence for stable periodic solutions of non-linear differential equations which govern the motion in question.

Here we shall not discuss the problems connected with conditions of periodic solutions stability. Our interests are to be restricted to the evaluation of periodic regimes or to the indication of conditions for their existence */.

To construct the periodic regimes a variety of approximate methods is widely used up to the date , these methods based on the assumption that the system under discussion slightly differs from some other system which can be described by already known

*/ In this survey, only general problems of this kind are considered, the order of equations not in view. Various specific problems for the second and the third order of equations were left out of the scope.

" generating " periodic solution.

This assumption leads to the following mathematical deduction ; the periodic regime to be found does formally slightly differ from generating, usually garmonic solution. This statement though of use in different problems of mechanics, electrical oscillations and theory of automatic control turns out to appear not sufficiently accurate and corresponding to the nature of phenomena in debate.

Some attention was drawn in this sense to certain methods of exact solutions evaluation , for non-linear differential equations. Exact methods are in need for verification of applicability and accuracy of the approximate ones as well as for the presentation of periodic regimes when the assumption noted above is not the case and applicability of approximate methods cannot be proved.

Exact methods were being developed only for special classes of differential equations because of absense of regular ways to check up the periodic solutions for any generalization of the form of non-linear differential equation.

Methods of construction of exact solution for piecewise linear differential equations have been worked out in great detail.

Three directions can be outlined here. These are I)Method of " sewing " together the solutions of separate linear systems which form the whole piece-wise problem. These solutions are presented either in the form of the sum of exponents or, if it is undesirable to seek the roots of some secular equations, in the form of power or trigonometric series. 2) Integral

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equations method. 3) The periodic regimes are presented in the form of trigonometric series and no garmonic is neglected. The coefficients of these series are found out from the considerations connected to the piecewise linearity of characteristics.

It is natural, indeed, that all the methods described lead to the same results.

§ 1. Statement of the Problem. The Type of Periodic Motion.

This report concernes the dynamic systems whose motions are guided by the system of differential equations

$$\dot{\eta} = B\eta + h f(\sigma), \quad \sigma = j'\eta \qquad (I.1)$$

Here η - n - vector to be found, B = $n \times n$ matrix of given constants, h and j - $n \times 1$ coloumns of constants known as well. $f(\sigma)$ denotes a piecewise linear function given as follows : the equations

$$f(\sigma) = \alpha_k \sigma + \beta_k \qquad (I.2)$$

ascertain r straight lines in the plane σ, f ; on each of these lines there are known the values σ_{sk} of these values being such that when during the motion of representing point along the s-th line σ becomes equal to σ_{sk} just at this moment the representing point passes at once up to the k-th line and moves further along the

latter **/. Fig.I shows the examples of piecewise characteristics, these examples met in non-linear oscillations problems.

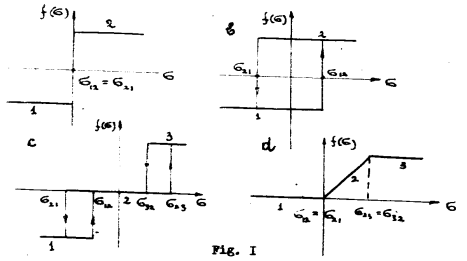

Fig. I

Fig. 2 presents an example of a characteristic constructed of three straight lines ; the inscriptions show the numbers and corresponding values of σ , the arrows denote the directions of transitions.

**/ The possibility of this "jump" is surmized, i.e. that the instantaneous transition of the representing point from the s-th to the k-th line is not in contradiction with the equations considered. If, however, $\sigma_{sk} = \sigma_{ks}$, it may happen that just after the jump the velocity of the representing point changes its sign, and the point does not actually move along the k-th line and glides down to the s-th. The "slippering" regimes like this are out of consideration hereafter; we put the motion of the representing point after the jump from the s-th line to the k-th to continue along the latter in future.

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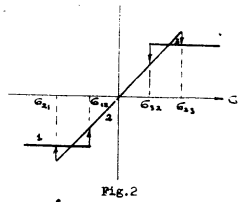

Fig.2

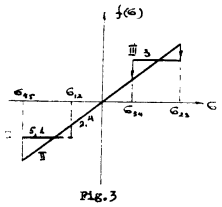

Fig.3

When $f(\sigma)$, i.e.(I.2) and transition conditions, are known, various periodic solutions occur possible, these solutions differing by the order of passing through separate branches of the characteristic in the course of one period of the periodic motion.

The sequence of straight lines which combine the characteristic for a period may be ascertained by set of numbers of the lines passed over in turn. For example, the sequence

I 2 3 2 I

denotes that in the course of a period the motion started on the line I prolonges on 2 , passes up to 3, turnes over to 2 and then to I where it comes up to initial value of the coordinate and so on. But in the case of Fig. 2 except the noted regime the following regimes are possible , for instance

I 2 I , I 2 3 2 I 2 I ; I 2 3 2 3 2 I ; I 2 3 2 I 2 3 2 I
2 3 2

etc. Let us call the inscription like this the formula or

the periodic regime, this formula denoting the type of the regime. The regimes of the same formula are of equal type though they may differ by the length of the period, the spectrum and so on.

When constructing the periodic solutions, their type have to be prescribed and corresponding regimes outlined. To examine all conceivable regimes, one must separately develop every regime of each type. Here we certainly come to the problem of the limiting number of various types of periodic solutions for the system (I.I) with given r . Some results of this kind may be found in [I2-I4] .

We shall further mark the straight lines according to their sequence in the periodic regime formula, so that the same line may be mentioned several times ; below from this point in the periodic regime formula the numbers follow the natural order I 2 3 ...τ , the abscisses of the transition points having the indexation

$$\sigma_{12}, \sigma_{23}, \ldots, \sigma_{\tau-1,\tau}$$

the numbers I ,τ to be related to the same straight line. For instance, the periodic regime of Fig. 3, previously shown on Fig.2, will be presented now by the designation I2345 and the line I receives the numbers I and 5 , the line II - numbers 2 and 4 , III - 3 .

Equations (I.I) - (I.2) do not contain t explicitly so that the origin for t may be chosen arbitrarily. Let the origin be placed in the moment when the representing point starts its motion along the first (the r-th) line of

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the periodic regime formula, the value of σ being initially $\sigma_{\tau-1,\tau}$, and the abscisses σ_{12}, ... , $\sigma_{\tau-2,\tau-1}$ corresponding to the moments t_2, ... , $t_{\tau-1}$. Up to this notation, $t_\tau = T$ means the period of the regime in consideration.

The representing point moves along the 1-st (the τ - th) line during the time interval ($t_1=0, t_2$), the 2-nd - during the interval (t_2, t_3) etc., the (τ-1)th line is being passed for ($t_{\tau-1}$, $t_\tau=T$) time.

To summarize, the motion in the time interval (t_ℓ , $t_{\ell+1}$) is governed by the system of the differential equations

$$(\eta^\ell)^{\cdot} = B\eta^\ell + h(\alpha_\ell \sigma + \beta_\ell), \quad \sigma^\ell = j'\eta^\ell; \ \ell = 1,2,\dots,\tau-1 \quad (I.3)$$

The problem is to find the continuous periodic regime for (I.1) - (I.2). Hence the solutions of (I.3) are to be chosen so that the conditions of conjunction

$$\eta^{\tau-1}(T) = \eta^1(0), \ \eta^1(t_2) = \eta^2(t_2), \dots, \ \eta^{\tau-2}(t_{\tau-1}) = \eta^{\tau-1}(t_{\tau-1}) \quad (I.4)$$

are satisfied.

The solutions of all the systems (I.3) provide certain relations among the vectors $\eta^\ell(t_\ell)$ and $\eta^\ell(t_{\ell+1})$, so that $2\tau-2$ vector equations for just that number of unknowns will take place. Having determined the noted unknowns and expressed with their aid (τ-1) quantities known from the start

$$\sigma_{\ell,\ell+1} = j'\eta^\ell(t_{\ell+1}) = j'\eta^{\ell+1}(t_{\ell+1}); \ \ell = 1,2,\dots,\tau-2, \ \sigma_{\tau 1} = j'\eta^{\tau-1}(T) = j'\eta^1(0) \quad (I.5)$$

will come up to some equation containing $\tau-1$ ... $t_2 \dots , T$. These turn out to be the ... the periods. The construction of these equations ... the main problem to be delivered. Having been ... there exist the positive solutions ($t_2, \dots , T$ ... should evaluate the quantity $\eta(t)$ and consequently $\sigma(t)$ for every group of these solutions. This is in fact ... end of the story for one should verify that $\sigma(t)$ ... does actually meet our supposition not to have the ... $\sigma_{\ell,\ell+1}$ within the interval t_ℓ , $t_{\ell+1}$. Lastly, there ... some problems of stable regimes selection, the exceptions of "slippering" periodic solutions and so on. ... however, leaves out of its scope the items just ... and we restrict ourselves only with the formal ... the question - the presentation of some recent ... equations construction.

The process of evaluation just described is based ... conjunction conditions (I.4) for the solutions ... ons parts of a piecewise characteristic. ... that of " sewing " and it was already mentioned ... this method may be developed through ... according to certain form of the solution for (I.3) ; these modifications are a) the "sewing " of solutions ... sed through the roots of the secular equations ; b) the "sewing " of the trigonometric series , c) that for the power series.

To elude the sewing process, one has to ... from

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system (1.I) to some integral equation for $\theta(t)$. The ... ment of differential equations (I.3) within separate ... is now xcluded, the conjunction conditions (I.4) ... automatically held. When the piecewise function $f(\theta)$ ... mbined of parallel lines, particularly as it is for ... systems, then it occurs of no trouble indeed to eva... $\theta(t)$ out of the integral equation and to put down the ... equations. In general however, it seems impossible ... proceed with the "closed" solution of the resultant Fredholm equation.

The Fourier series, which express the solutions $\eta(t)$ within the interval $(\tau_\ell, \tau_{\ell+1})$, are of the period T and ... it will be shown below, present the functions of time, vanishing when $0 < t < \tau_\ell$ and $\tau_{\ell+1} < t < T$.

Hence the sum

$$\eta(t) = \overset{1}{\eta}(t) + \dots + \overset{r-1}{\eta}(t)$$

becomes the general trigonometric representation of the exact solution for the piecewise linear problem.

The periodic regime can be in fact projected in a form of a single trigonometric series without "sewing", and a computable set of Fourier coefficients is expressed ... with the finite number of parameters $\tau_2, \dots, T$. This ... is founded upon peculiar properties of certain piecewise characteristics of special type (those of such ...) ... one or two given straight lines).

§ 2. The Representation of Solutions for the Homogeneous Linear System of Differential Equations with Constant Coefficients.

We shall now consider the homogeneous linear system of differential equations. Following the notation of § I, we put

$$\dot{x} = Bx + \alpha h j' x = Qx \qquad (2.1)$$

The determinants of the matrices $pE - B$ and $pE - Q$ will be assigned as

$$D(p) = |pE - B|, \quad \Delta(p) = |pE - Q| = |pE - B - \alpha h j'| \qquad (2.2)$$

and the eigenvalues of the matrices B and Q, that is the roots of polynomials of the n-th power $D(p), \Delta(p)$, will correspondingly be λ_q and μ_q.

To settle the formula connecting these polynomials let us introduce the coloumns $H(p)$, $J(p)$ of the polynomials $H_s(p)$, $J_s(p)$; the former being the determinants which come out of $D(p)$ when the s-th coloumn is substituted by the numbers $h_1, \dots, h_n$; the same way holds for the latter with the s-th line instead of the coloumn and $j_1, \dots, j_n$ instead of $h_1, \dots, h_n$. The result is

$$H_s(p) = \sum_{m=1}^{n} h_m D_{ms}(p), \quad J_s(p) = \sum_{m=1}^{n} j_m D_{sm}(p) \qquad (2.3)$$

Now, introducing the polynomial

$$M(p) = -j' H(p) = -h' J(p) = -\sum_{s=1}^{n} \sum_{\kappa=1}^{n} h_s j_\kappa D_{s\kappa}(p) \qquad (2.4)$$

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[illegible]

$$\Delta(p) = [illegible] + \alpha M(p) \qquad (2.5)$$

[illegible] $H_s J_r$ [illegible] $\alpha h J$ [illegible] out down in the [illegible]

$$H_s(p) J_r(p) = \sum_{m=1}^{n} \sum_{\ell=1}^{n} h_m j_\ell \, \mathcal{D}_{ms}(p) \mathcal{D}_{r\ell}(p) \qquad (2.6)$$

[illegible] the last expression, let us use the relation between algebraic complements of $\mathcal{D}(p)$

$$\mathcal{D}_{ms}(p) \mathcal{D}_{r\ell}(p) = \mathcal{D}_{m\ell}(p) \mathcal{D}_{rs}(p) + \mathcal{D}(p) \mathcal{D}_{mr,s\ell}(p) \qquad (2.7)$$

[illegible] $\mathcal{D}_{mr,s\ell}(p)$ [illegible] product of $(-1)^{m+s+r+\ell} \operatorname{sgn}(r-m) \operatorname{sgn}(\ell-s)$ [illegible] determinant resulting from $\mathcal{D}(p)$ when the m-th and r-th lines, s-th and ℓ-th columns are thrown away : the case $m = r$ and $s = \ell$ corresponds to $\mathcal{D}_{mr,s\ell} = 0$.

The last relation in view, as well as (2.5) and (2.4), [illegible] proceed to

$$Z_{sr}(\mu_q) = \frac{H_s(\mu_q) J_r(\mu_q)}{M(\mu_q)} = -\left[\mathcal{D}_{rs}(\mu_q) + \alpha \sum_{m=1}^{n} \sum_{\ell=1}^{n} h_m j_\ell \mathcal{D}_{mr,s\ell}(\mu_q)\right] \qquad (2.8)$$

[illegible] expression on the right side presents when $r = s$ [illegible] $n-1$-th power polynomial in μ_q with the unity coefficient [illegible]. If $r \neq s$ this expression is the [illegible] of the power no higher than $n-2$. Having recalled [illegible] known equalities (for simple roots μ_q)

$$\sum_{q=1}^{n} \frac{\mu_q^m}{\Delta'(\mu_q)} = \begin{cases} 0 & m = 0, 1, \ldots, n-2 \\ 1 & m = n-1 \end{cases}$$

we get

$$-\sum_{q=1}^{n} \frac{Z(\mu_q)}{\Delta'(\mu_q)} = -\sum_{q=1}^{n} \frac{H(\mu_q) J'(\mu_q)}{\Delta'(\mu_q) M(\mu_q)} = E \qquad (2.9)$$

It is easy to verify the identities

$$\mu_q H(\mu_q) = \beta H(\mu_q) - \alpha h M(\mu_q), \quad \mu_q J(\mu_q) = \beta' J(\mu_q) - \alpha j M(\mu_q) \qquad (2.10)$$

These relations in mind together with (2.8), we evaluate scalar products $J'(\mu_\ell) H(\mu_q)$

$$J'(\mu_\ell) H(\mu_q) = \begin{cases} 0 & \mu_\ell \neq \mu_q \\ -M(\mu_q) \Delta'(\mu_q) & \mu_\ell = \mu_q \end{cases} \qquad (2.11)$$

Now, let us proceed to the fundamental matrix of solutions $x(t)$ for the system (2.1), this matrix becoming unitary when $t = 0$

$$x(0) = E \qquad (2.12)$$

We have

$$x(t) = e^{Qt} = -\sum_{q=1}^{n} e^{\mu_q t} \frac{Z(\mu_q)}{\Delta'(\mu_q)} \qquad (2.13)$$

For the differential equations system conjugate to (2.1)

$$\dot{y}(t) + (\beta' + \alpha j h') y(t) = 0 \quad \text{or} \quad \dot{y}(t) + Q' y(t) = 0 \qquad (2.14)$$

the solution with unitary matrix of initial conditions will be

$$y(t) = e^{-Q't} = -\sum_{q=1}^{n} e^{-\mu_q t} \frac{Z'(\mu_q)}{\Delta'(\mu_q)} \qquad (2.15)$$

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The transponed matrix $y'(t)$ is

$$y'(t)=e^{-Qt}=-\sum_{q=1}^{n}\frac{Z(\mu_q)}{\Delta'(\mu_q)}e^{-\mu_q t} \tag{2.16}$$

so that

$$x(t)y'(t)=E \tag{2.17}$$

This easy to verify with the aid of (2.11).

The matrix

$$X(t-t^*)=\begin{cases}-e^{Q(T+t-t^*)}(e^{QT}-E)^{-1} & 0\le t<t^*\le T\\ -e^{Q(t-t^*)}(e^{QT}-E)^{-1} & 0\le t^*<t\le T\end{cases} \tag{2.18}$$

presents for $0\le t\le T$ the periodic regime (with the period of T) for the system (2.I). This regime has a jump when $t=t^*$, this jump prescribed by a unitary matrix

$$X(+0)-X(-0)=E \tag{2.19}$$

The Sylvester theorem provides the presentation of the matrix (2.18) in terms of eigenvalues of the Q - matrix. Using (2.15), we put

$$X(t-t^*)=\begin{cases}\sum_{q=1}^{n}\frac{e^{\mu_q(T+t-t^*)}}{e^{\mu_q T}-1}\frac{Z(\mu_q)}{\Delta'(\mu_q)} & 0\le t<t^*\le T\\ \sum_{q=1}^{n}\frac{e^{\mu_q(t-t^*)}}{e^{\mu_q T}-1}\frac{Z(\mu_q)}{\Delta'(\mu_q)} & 0<t^*<t<T\end{cases} \tag{2.20}$$

The half-sum of left-side and right-side values of the matrix $X(t-t^*)$ taken at the jump point, will be denoted as $X(0)$; then

$$X(0)=-\tfrac{1}{2}\operatorname{ctg}\frac{QT}{2}=\frac{1}{2}\sum_{q=1}^{n}\operatorname{ctg}\frac{\mu_q T}{2}\frac{Z(\mu_q)}{\Delta'(\mu_q)} \tag{2.21}$$

and

$$X(+0)=X(0)+\tfrac{1}{2}E,\quad X(-0)=X(0)-\tfrac{1}{2}E \tag{2.22}$$

The matrices (2.18) or (2.20) can easily be put into the trigonometric series form. Meaning in mind the simple fractions expansion

$$\frac{e^{a/2}}{e^{z}-1}=\sum_{\nu=-\infty}^{\infty}\frac{e^{2\nu\pi i a}}{z-2\nu\pi i}\qquad 0<a<1 \tag{2.23}$$

we receive

$$\frac{1}{T}\sum_{\nu=-\infty}^{\infty}\frac{e^{i\omega\nu(t-t^*)}}{\mu_q-i\omega\nu}=\begin{cases}\dfrac{e^{\mu_q(t-t^*)}}{e^{\mu_q T}-1} & t>t^*\\ \dfrac{e^{\mu_q(T+t-t^*)}}{e^{\mu_q T}-1} & t^*<t<T\end{cases} \tag{2.24}$$

where $\omega T=2\pi$. Hence, taking use of the Heaviside expansion:

$$\sum_{q=1}^{n}\frac{Z(\mu_q)}{\Delta'(\mu_q)}\frac{1}{\mu_q-i\omega\nu}=-\frac{Z(i\omega\nu)}{\Delta(i\omega\nu)} \tag{2.25}$$

we get

$$X(t-t^*)=-\frac{1}{T}\sum_{\nu=-\infty}^{\infty}\frac{Z(i\omega\nu)}{\Delta(i\omega\nu)}e^{i\omega\nu(t-t^*)} \tag{2.26}$$

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The well-known property of trigonometric series brings

$$X(0) = -\frac{1}{T}\sum_{\nu=-\infty}^{\infty}\frac{Z(i\omega\nu)}{\Delta'(i\omega\nu)} \qquad (2.27)$$

the last relation can also be evaluated from (2.2I) with the help of the hyperbolic cotangent expansion

$$ctg\frac{\mu_q T}{2} = \frac{2}{T}\sum_{\nu=-\infty}^{\infty}\frac{1}{\mu_q - i\nu\omega}$$

Let us treat the matrix

$$Y'(t-t^*) = \begin{cases} -(e^{QT}-E)e^{-Q(T+t-t^*)} & 0 \le t < t^* \le T \\ -(e^{QT}-E)e^{-Q(t-t^*)} & 0 \le t^* < t \le T \end{cases} \qquad (2.28)$$

This matrix is periodic with the period of T and has a jump when $t = t^*$. (2.I8) necessitates immediately

$$X(t-t^*)Y'(t-t^*) = E,\ X^{-1}(t-t^*) = Y'(t-t^*) \qquad (2.29)$$

and, in particular,

$$X^{-1}(+0) = Y'(+0) = E - e^{QT},\ X^{-1}(-0) = Y'(-0) = e^{-QT} - E \qquad (2.30)$$

The expressions for these matrices after (2.I3) and (2.I6) through the eigenvalues of the Q-matrix will be

$$X^{-1}(+0) = \sum_{q=1}^{n}\left(e^{\mu_q T}-1\right)\frac{Z(\mu_q)}{\Delta'(\mu_q)},\quad X^{-1}(-0) = \sum_{q=1}^{n}\left(1-e^{-\mu_q T}\right)\frac{Z(\mu_q)}{\Delta'(\mu_q)} \qquad (2.3I)$$

Thus, the matrices $X^{-1}(+0)$ and $X^{-1}(-0)$ are expressed in terms of the coefficients of (2.I) system, these expressions being the power series in T as well as certain forms of the roots of the secular equation $\Delta(\rho) = 0$. These roots do actually enter the (2.27) representations of these matrices.

Given the matrix (2.I8) of solutions, will we construct now the periodic function of time with the period T

$$\xi_i(t) = X(t-t_\ell)\xi_i(t_\ell+0) - X(t-t_{\ell+1})\xi_i(t_{\ell+1}-0) \qquad (2.32)$$

and let this function be considered within the time interval

$$t_\ell + 0 < t < t_{\ell+1} - 0 \qquad (2.33)$$

contained inside of $(0, T)$. Up to this interval, the function (2.32) coincides with the solution of (2.I); it will be seen from below, this function vanishes when t stays within $(0, T)$, but outside of (2.33).

We have

$$\frac{1}{2}\xi_i(t_\ell+0) = X(0)\xi_i(t_\ell+0) - X(t_\ell - t_{\ell+1})\xi_i(t_{\ell+1}-0)$$

and also

$$\xi_i(t_\ell - 0) = -\frac{1}{2}\xi_i(t_\ell+0) + X(0)\xi_i(t_\ell+0) - X(t_\ell - t_{\ell+1})\xi_i(t_{\ell+1}-0)$$

so that the first of these relations gives $\xi_i(t_\ell - 0) = 0$

Exactly the same token provides

$$\frac{1}{2}\xi_i(t_{\ell+1}-0) = X(t_{\ell+1}-t_\ell)\xi_i(t_\ell+0) - X(0)\xi_i(t_{\ell+1}-0),\ \xi_i(t_{\ell+1}+0) = 0$$

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[illegible]

$$[illegible] E)\xi(t_\ell+0) = [illegible] t_\ell - t_{\ell+1})\xi(t_{\ell+1}-0) \qquad [illegible]$$

$$[illegible] X(t_{\ell+1}-t_\ell)\xi(t_\ell+0)$$

$$[illegible]$$

[illegible] homogenious linear differential equations system with [illegible] initial conditions has no solutions except trivial [illegible] together with (2.36)

$$[illegible] \quad 0<t<t_\ell,\ t_{\ell+1}<t<T \qquad (2.37)$$

[illegible] that the values $\xi(t)$ in the initial [illegible] of the interval (2.33) are connected through (2.34) [illegible].

[illegible] methods of presentation of the solution $\xi(t)$ [illegible] now from above stated.

[illegible] The presentation presupposing the eigenvalues μ_{ij} the matrix Q to be known. Then we have after (2.35), [illegible] and (2.31)

$$\xi(t_{\ell+1}-0) = Y'(+0)\,X(t_{\ell+1}-t_\ell)\xi(t_\ell+0)$$

or after the matrix multuplication

$$\xi(t_{\ell+1}-0) = x(t_{\ell+1}-t_\ell)\xi(t_\ell+0) \qquad (2.38)$$

This might have been awaited as (2.12) show [illegible] $x(t)$ exactly realizes the transformation of the initial values into the instantenuous ones; altogether, by (2.3[illegible]

$$\xi(t) = [X(t-t_\ell) - X(t-t_{\ell+1})x(t_{\ell+1}-t_\ell)]\xi(t_\ell+0)$$

and it is easy to ascertain the matrix in brackets [illegible] equal to $x(t-t_\ell)$, so that

$$\xi(t) = x(t-t_\ell)\xi(t_\ell+0) \qquad (2.39)$$

To sum up, this method of evaluation requires only the matrix (2.13) of solutions $x(t)$ to be known. We can set the solution (2.39) be vanishing out of the interval (2.33) and having a period of T.

2^0. The evaluation with the aid of the power series. The formulas (2.38) and (2.39) can be rewritten in the form.

$$\xi(t_{\ell+1}-0) = e^{Q(t_{\ell+1}-t_\ell)}\xi(t_\ell+0)$$
$$\xi(t) = e^{Q(t-t_\ell)}\xi(t_\ell+0) \qquad (2.40)$$

and the calculation proceeds by the power series through the initial coefficient matrix Q of the (2.1) system. Let us recollect now that this process requires the determination of subsequent powers of Q up to the (n-1)th included; $Q^2, Q^{n+1}, \ldots$ etc. are expressed through $Q, Q^2, \ldots\ldots Q^{n-1}$ after the well-known Cayley theorem.

3^0. The evaluation through trigonometric series. To evaluate $\xi(t)$ after (2.32) one should take care of the presentations of the matrices $X(t-t_\ell)$ and $X(t-t_{\ell+1})$ in the form of trigonometric series (2.26).

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$\xi(t_{\ell+1}-0)$ may be expressed through $\xi(t_\ell+0)$ provided one has calculated the matrix

$$\left[X(0)+\frac{1}{2}E\right]^{-1} \qquad (2.41)$$

using the value (2.27) of the matrix $X(0)$. This phase of the calculation can however be avoided if the use is made of the first formula in (2.40). So far as it is, we keep in mind some "mixed way", which exploitates the trigonometric series for the presentation of the solution $\xi(t)$ and the power series for the installation of dependance between its values in the initial and ending points of the interval (2.33).

§3. The "sewing" method. The trigonometric series presentation.

As it has already been stated above, we discuss here the problem of construction for the continuous periodic regime of the system described in the time interval $(t_\ell, t_{\ell+1})$ by the system of differential equations (I.3).

So long as the moments $t_2, \ldots, t_{r-1}, T$ are given, one is able to build up in every of the intervals the $\eta^\ell(t)$ solution periodic with the period T. The methods of § 2 should thereby be taken in mind. We take the solution $\eta^\ell_0(t)$ corresponding to the free term β_ℓ in the right sides of (I.3)

$$\eta^\ell_0(t) = \beta_\ell \frac{H(0)}{\Delta^\ell(0)} \qquad (3.1)$$

with necessary alterations when $\Delta^\ell(0)=0$. We put

$$\eta^\ell(t) = \eta^\ell_0(t) + \xi^\ell(t) \qquad (3.2)$$

where the solution $\xi^\ell(t)$ has one of the forms of § 2. The index ℓ denotes that $\alpha = \alpha_\ell$ in the expression of the matrix Q . Now, one has to meet the conditions (I.4) and to ascertain the form of the period equations (I.5).

I. The method based on the known eigenvalues μ_q of the matrices Q^ℓ . The solution $\xi(t)$ is taken after (2.39). The equations (I.4) convert to the form

$$\begin{aligned}
&\xi^1(T) - \xi^1(0) = -\beta_{r-1}\frac{H(0)}{\Delta^1(0)} + \beta_1\frac{H(0)}{\Delta^1(0)}\\
&\xi^1(t_1) - \xi^2(t_1) = -\beta_1\frac{H(0)}{\Delta^1(0)} + \beta_2\frac{H(0)}{\Delta^2(0)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.3)\\
&\xi^{r-1}(t_{r-1}) - \xi^r(t_{r-1}) = -\beta_{r-1}\frac{H(0)}{\Delta^{r-1}(0)} + \beta_r\frac{H(0)}{\Delta^r(0)}
\end{aligned}$$

The following connections among the vectors $\xi^\ell(t_\ell)$ and $\xi^\ell(t_{\ell+1})$ should be added

$$\xi^\ell(t_{\ell+1}) = X^\ell(t_{\ell+1}-t_\ell)\,\xi^\ell(t_\ell), \quad \ell = 1,2,\ldots,r-1 \qquad (3.4)$$

All the unknown coloumns $\xi^1(t_1), \ldots, \xi^{r-1}(t_{r-1}), \xi^r(T)$ are subsequently excluded from (3.3) and (3.4) . The result will stay as a linear system for the coloumn $\xi^1(0)$ */. For instance , for the piecewise linear characteristic

*/ The corresponding determinant will be different from zero for the breaking characteristic. This determinant vanishes for the continuous characteristic like that of Fig. I d. For the case one has to alter the order of calculations.

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of Fig.4, one has

$$\xi^1(t_2) = x^1(t_2)\xi^1(0),\ \xi^2(T) = x^2(T-t_2)\xi^1(t_2)$$

the equations (4.3) being formulated as follows

$$\xi^2(T) = \xi^1(0) + \beta_1\frac{H(0)}{\Delta^1(0)} - \beta_2\frac{H(0)}{\Delta^2(0)}$$
$$\xi^2(t_2) = \xi^1(t_2) + \beta_1\frac{H(0)}{\Delta^1(0)} - \beta_2\frac{H(0)}{\Delta^2(0)} \qquad (3.5)$$

The unknown coloumn $\xi^1(0)$ is described by the equality

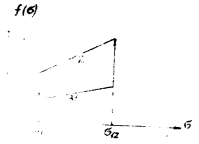
$$\xi^1(0) = \left[x^2(T-t_2)x^1(t_2) - E\right]^{-1}\left[x^2(T-t_2) - E\right]\left(\beta_2\frac{H(0)}{\Delta^2(0)} - \beta_1\frac{H(0)}{\Delta^1(0)}\right)$$

The period equations will be

$$\sigma_{23} = j^1\xi^1(0) - \beta_1\frac{M(0)}{\Delta^1(0)},\quad \sigma_{12} = j^1x(t_2)\xi^1(0) - \beta_1\frac{M(0)}{\Delta^1(0)}$$

The evaluation of matrices $x^\ell(t_{\ell+1} - t_\ell)$ follows the formula (2.13)

$$x^\ell(t_{\ell+1} - t_\ell) = -\sum_{q=1}^{n} e^{\ell\mu_q(t_{\ell+1}-t_\ell)}\frac{Z(\mu_q^\ell)}{\Delta'(\mu_q^\ell)} \qquad (3.6)$$



of the application of the power series. All mentioned above does in fact hold except one has to use the expressions

$$x^\ell(t_{\ell+1} - t_\ell) = e^{\hat{Q}^\ell(t_{\ell+1}-t_\ell)}$$

instead of (3.6). E.g. (3.5) will take form

$$\xi^1(0) = \left(e^{\hat{Q}^2(T-t_2)}e^{\hat{Q}^1(t_2)} - E\right)^{-1}\left(e^{\hat{Q}^2(T-t_2)} - E\right)\left(\beta_2\frac{H(0)}{\Delta^2(0)} - \beta_1\frac{H(0)}{\Delta^1(0)}\right) \qquad (3.8)$$

3. The trigonometric series utilized.

The form of the connections (3.4) becomes more complex. Now, these will be

$$\xi^\ell(t_{\ell+1}) = \left[\overset{\ell}{X}(0) + \tfrac{1}{2}E\right]^{-1}X(t_{\ell+1} - t_\ell)\xi^\ell(t_\ell),\quad \ell = 1,2,\ldots,2-1 \qquad (3.9)$$

where $\overset{\ell}{X}(t_{\ell+1} - t_\ell)$ and $\overset{\ell}{X}(0)$ given by the trigonometric series (2.26), (2.27)

$$\overset{\ell}{X}(t_{\ell+1} - t_\ell) = -\frac{1}{T}\sum_{\omega=-\infty}^{\infty}\frac{\overset{\ell}{\Delta}(i\omega\gamma)}{\overset{\ell}{\Delta}(i\omega\gamma)}e^{i\omega\gamma(t_{\ell+1}-t_\ell)},\quad \overset{\ell}{X}(0) = -\frac{1}{T}\sum_{\omega=-\infty}^{\infty}\frac{\overset{\ell}{\Delta}(i\omega\gamma)}{\overset{\ell}{\Delta}(i\omega\gamma)} \qquad (3.10)$$

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One has so far to calculate the inverse matrices not once as before but for $(\tau-1)$ times.

The equations (3.2) and (3.3) provide the coloumns $\dot{\xi}(t_2)$, $\dot{\xi}(t_3), \ldots, \dot{\xi}(T)$ to be one after another expressed through $\dot{\xi}(0)$, the latter to be presented by help of $t_2, \ldots, t_\tau = T$. Thus every coloumn $\dot{\xi}(t)$ will after (2.32) be put down as a trigonometric series of the period T, this series being a solution for $t_\ell < t < t_{\ell+1}$ of the system (2.I) and vanishing outside of the interval just mentioned. Let us take now the trigonometric series $\eta_0(t)$ of the period T which denotes the function equal to the constant (4.I) in every of intervals noted above. Then the matrix-coloumn

$$\eta(t) = \eta_0(t) + \dot{\xi}(t) + \cdots + \dot{\xi}^{\tau-1}(t) \qquad (3.II)$$

does actually give the trigonometric presentation (I.6) of the periodic solution for the piecewise linear system in question. It is easy to verify this solution to be unique.

For example, the system of relation to be considered when $\tau = 3$ (Fig.4) will contain:

a)two equations of connection for the initial and ending values of $\dot{\xi}$ and $\ddot{\xi}$ coloumns

$$\dot{\xi}(t_2) = [\dot{X}(0) + \tfrac{1}{2}E]^{-1}\dot{X}(t_2)\dot{\xi}(0), \quad \ddot{\xi}(t_2) = (\ddot{X}(0) + \tfrac{1}{2}E)^{-1}\ddot{X}(T-t_2)\ddot{\xi}(t_2) \qquad (3.I2)$$

b)two requirements (4.5) for the periodic regime to be continuous. The coloumns $\ddot{\xi}(0)$ and $\ddot{\xi}(t_2)$ are evaluated from these four conditions. After this one comes up to the equations for the periods

$$\ddot{\xi}(0) - \beta_1 \frac{M(0)}{\ddot{\Delta}(0)} = \sigma_{23}, \quad \ddot{\xi}(t_2) - \beta_1 \frac{M(0)}{\ddot{\Delta}(0)} = \sigma_{12} \qquad (3.I3)$$

It occurs of no difficulties indeed to evaluate the trigonometric presentation of the function

$$\eta_0(t) = \begin{cases} \beta_1 \dfrac{H(0)}{\ddot{\Delta}(0)} & 0 < t < t_2 \\ \beta_2 \dfrac{H(0)}{\ddot{\Delta}(0)} & t_2 < t < T \end{cases}$$

This one has the form

$$\eta_0(t) = \frac{T}{2\pi i}\left(\beta_1 \frac{H(0)}{\ddot{\Delta}(0)} - \beta_2 \frac{H(0)}{\ddot{\Delta}(0)}\right) \sum_{\nu=-\infty}^{\infty} \frac{e^{i\omega\nu t}}{\nu}\left(1 - e^{-i\omega\nu t_2}\right)$$

so that every term $\eta_0(t)$, $\dot{\xi}(t)$ in (4.II) is determined.

& 4. The periodic regimes projected in the Fourier series form.

We shall suppose throughout this section that σ coincides in (I.I) with one of the variables, say, . The generalization to the case $x = \sigma -$ linear combination of the variables is trivial. Firstly, let us exclude from (I.I) all the variables except x, that is will we pass from this system to so called leading equation

$$D(p)x = M(p)y \qquad y = f(x) \qquad (4.I)$$

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where

$$\mathcal{D}(p^*) = a_0 p^{*n} + \dots + a_n \qquad M(p^*) = b_0 p^{*n} + \dots + b_n \tag{4.2}$$

are the polynomials with constant coefficients of the power not less than by one unity than that of $\mathcal{D}$; and p^* - the operator of the generalized time derivative.

The sense of it will be as follows. Were the functions x and y smooth enough, p^*x and p^*y would be treated simply as $\dot{x}$ and $\dot{y}$. But in the case considered y is discontinuous and consequently the same is true for $\dot{x}$ and the higher derivatives of x. That is why we are to use here the p^* as the generalized derivative operator given by the formula

$$p^*F(t) = \sum_q \varrho_q \delta(t-t_q) = pF(t) + \sum_q \varrho_q \delta(t-t_q) \tag{4.3}$$

where the moments of "jumps" for $F(t)$, ϱ_q - the amplitudes of the jumps, that is $\varrho_q = F(t_q+0) - F(t_q-0)$, δ - the Dirac function.

Having put (4.2) into (4.1) and compared the terms that contain δ - functions, those that contain δ - functions, we receive the equations

$$\mathcal{D}(p)x = M(p)y, \qquad y = f(x) \tag{4.4}$$

that describe the motion everywhere except the t_q -moments. We have simultaneously up to n conditions for the jump

$$\begin{aligned} a_0\xi_0 &= b_0\eta_0 \\ a_0\xi_1 + a_1\xi_0 &= b_0\eta_1 + b_1\eta_0 \\ &\dots\dots\dots\dots\dots\dots\dots \\ a_0\xi_{n-1} + \dots + a_{n-1}\xi_0 &= b_0\eta_{n-1} + \dots + b_{n-1}\eta_0 \end{aligned} \tag{4.5}$$

where

$$\xi_\kappa = \left(\frac{d^\kappa x}{dt^\kappa}\right)_{t=t_q+0} - \left(\frac{d^\kappa x}{dt^\kappa}\right)_{t=t_q-0} \qquad \eta_\kappa = \left(\frac{d^\kappa y}{dt^\kappa}\right)_{t=t_q+0} - \left(\frac{d^\kappa y}{dt^\kappa}\right)_{t=t_q-0}$$

The initial (I.1) is equivalent either to (4.4) with the generalized derivative or to the usual differential equation (4.4) with the conditions (4.5) for the jumps added at all the jump points.

To proceed, we shall project the solution as the Fourier series

$$x = \sum_{\tau=-\infty}^{\infty} \alpha_\tau e^{i\tau\omega t} \tag{4.6}$$

where according to the generalized derivative determination

$$\mathcal{D}(p^*)\sum_{\tau=-\infty}^{\infty} \alpha_\tau e^{i\tau\omega t} = \sum_{\tau=-\infty}^{\infty} D(i\tau\omega)\alpha_\tau e^{i\tau\omega t}$$

We shall call the complex function

$$W(i\omega) = \frac{M(i\omega)}{D(i\omega)}$$

the frequency characteristic according to the control theory terminology. If x has the form (4.6) when

$$y = \sum_{\tau=-\infty}^{\infty} \beta_\tau e^{i\tau\omega t}$$

then

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$$d_r = \beta_r W(ir\omega) \qquad (4.7)$$

Consider firstly the simplest case of the relay type characteristic (Fig.5), that is constisting of two straight lines $y = +1$, $y = -1$

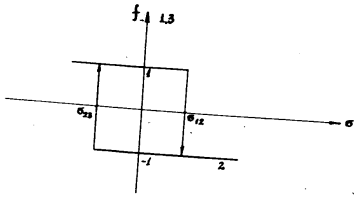

Fig.5

with $\sigma_{12} = \varkappa$ and $\sigma_{22} = -\varkappa$ and let us try to find the periodic regime I2I provided the comutation occurs only at the edges of half a period, that is when $t = \frac{T}{2}$ and $t = T$, the latter value being the period to be found. Putting the origin for t to the moment when $x = \sigma_{12} = \varkappa$, we come up to the alteration of y illustrated by Fig.6, and $y(t)$ may be expanded into the series

$$y(t) = \frac{4}{\pi} \sum_{r=1}^{\infty}{}' \frac{1}{r} \sin r\omega t \qquad (4.8)$$

the trait denoting the summation over odd values of r.

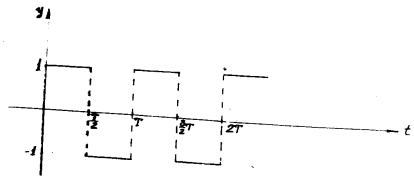

Fig. 6

Then, taking (4.7) in mind, we shall receive the oscillation of x

$$x(t) = \frac{4}{\pi} \sum_{r=1}^{\infty}{}' \frac{1}{r} \left|W(ir\omega)\right| \sin\left[r\omega t + \arg W(ir\omega)\right]$$

If the oscillation is so chosen as to correspond to the Fig.6, then two conditions are to be met:

a) $x(0) = \varkappa$

b) $x(t) > -\varkappa$ for $0 < t < \frac{T}{2}$

The condition (a) immediately leads to the finite equation depending only on the single unknown T:

$$\sum_{r=1}^{\infty}{}' \frac{1}{r} \left|W(ir\omega)\right| \sin\left[\arg W(ir\omega)\right] = \frac{\pi\varkappa}{4}$$

or

$$\sum_{r=1}^{\infty}{}' \frac{1}{r} \operatorname{Im} W(ir\omega) = \frac{\pi\varkappa}{4} \qquad (4.9)$$

All the real numbers ω satisfying (4.9) determine $T = \frac{2\pi}{\omega}$ which quantities may serve the periods sought for.

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These will actually be the case provided the function $x(t)$ with T just mentioned does meet the condition (b). Thus the selection after the condition (b) is performed simultaneously with the evaluation of the periodic regimes themselves.

Had we been interested in more complex regimes, i.e. that of I2I2I or some other relay characteristic, the analogous story would be true because for any periodic regime the function $y(t)$ is periodic and known beforehand except of some unknown moments of relay " switching ". Hence, given ℓ such switches according to the known regime formula, one will have the coefficients α_r $(r=-\infty,\dots,\infty)$ of the Fourier expansion for $y(t)$ depending only on ℓ numbers, e.g. on the times of the motion between the relay switches $\tau_1, \tau_2, \dots, \tau_{r-1}$.

Now let us pass to more complex case when the characteristic is combined of two arbitrary straight lines non-parallel neither to each other nor to the ordinate axis.

The (4.I) equation will be written as follows

$$D(p^*)\bar{x} = M(p^*)\bar{y}, \quad \bar{y} = f(\bar{x}) \tag{4.I0}$$

We suppose the function $\bar{y} = f(\bar{x})$ to be constructed of two non-parallel lines

$$\bar{y} = \tau_1\bar{x} + s_1, \quad \bar{y} = \tau_2\bar{x} + s_2$$

and the transitions between the branches to take place with $x = const$ in the moments when it firstly occurs that $\bar{x} = \sigma_{12}$

and $\bar{x} = \sigma_{23}$ (Fig.4).

Consider the affine coordinate transformation

$$\bar{x} = \alpha x + \beta y + \varkappa, \quad \bar{y} = \gamma x + \delta y + \lambda \tag{4.II}$$

This transforms (4.I0) into

$$L(p^*)x = N(p^*)y + \varepsilon, \quad \gamma x + \delta y + \lambda = f(\alpha x + \beta y + \varkappa) \tag{4.I2}$$

where

$$L(p^*) = \alpha D(p^*) - \gamma M(p^*) = a_0 p^{*n} + a_1 p^{*n-1} + \dots + a_n$$

$$N(p) = \delta M(p^*) - \beta D(p^*) = b_0 p^{*n} + b_1 p^{*n-1} + \dots + b_n$$

$$\varepsilon = -M(0)\lambda - D(0)\varkappa = const$$

Note that if $\alpha \neq 0$ and $\beta \neq 0$ then the powers L and N are equal to each other even if the M power is less than that of D.

If one chooses the coefficients of (4.II) transformation according to the formulas

$$\gamma = k_1\alpha \qquad \delta = k_2\beta$$

$$\varkappa = \frac{s_1 - s_2}{\tau_1 - \tau_2} \qquad \lambda = \frac{\tau_2 s_1 - \tau_1 s_2}{\tau_2 - \tau_1}$$

(here α and β are arbitrary, but $\alpha \neq 0$ and $\beta \neq 0$), then the straight lines $\bar{y} = \tau_1\bar{x} + s_1$ and $\bar{y} = \tau_2\bar{x} + s_2$ do correspondingly

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coincide with coordinate axes and and the parallel "transition lines" convert into the parallel sloping straight lines passing correspondingly through the points

$$Q_2 = \frac{1}{\beta}\left(\sigma_{23} - \frac{S_1 - S_2}{z_1 - z_1}\right),\ Q_2' = \frac{1}{\alpha}\left(\sigma_{23} - \frac{S_1 - S_2}{z_2 - z_1}\right)$$

and

$$Q_1 = \frac{1}{\alpha}\left(\sigma_{12} - \frac{S_1 - S_2}{z_2 - z_1}\right),\ Q_1' = \frac{1}{\beta}\left(\sigma_{12} - \frac{S_1 - S_2}{z_2 - z_1}\right)$$

For example, instead of Fig.4 ,we get after the transformation in x, y coordinates the characteristic shown at the Fig. 7 and the periodic solutions of the equations (4.10) with the characteristic of Fig. 4 pass into the periodic

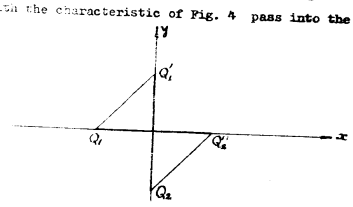

Fig. 7

solutions of (4.12) with the characteristic of Fig.7 and conversely.

Let us look for the periodic regimes of the type I2I. (Any other type of solution may be found by the same token). We project the solutions in the form of complete Fourier series

$$x = \sum_{\tau=-\infty}^{\infty} \alpha_\tau e^{i\omega\tau t},\quad y = \sum_{\tau=-\infty}^{\infty} \beta_\tau e^{i\tau\omega t}$$

Consider the function

$$N(p^*)y = \sum_{\tau=-\infty}^{\infty} N(i\tau\omega)\beta_\tau e^{i\tau\omega t} = \sum_{\tau=-\infty}^{\infty} \nu_\tau^* e^{i\tau\omega t}$$

where $\nu_\tau^* = N(i\tau\omega)\beta_\tau$ - the Fourier coefficients of the periodic function $N(p^*)y$.

Were all the values ν_τ^* been known the coefficients β_τ under seeking would at once be evaluated from

$$\beta_\tau = \frac{\nu_\tau^*}{N(i\tau\omega)} \qquad (4.13)$$

and then one would know all the α_τ

$$\alpha_\tau = \frac{\nu_\tau^*}{L(i\tau\omega)}\ (\tau \neq 0),\ \alpha_0 = \frac{\nu_0^* + \mathcal{E}}{L(0)} \qquad (4.14)$$

The direct evaluation of the coefficients ν_τ^* turns out to fail. But it is easy to calculate the Fourier coefficients γ_τ of the series

$$N(p)y = N(p)\sum_{\tau=-\infty}^{\infty} \beta_\tau e^{i\tau\omega t} = \sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{i\tau\omega t}$$

Here $p = \frac{d}{dt}$ means the usual derivative operator (the Dirac function in the jump points not taken into account) this operator non absurd only out of the jump points; hence the

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operation over the series $N(p)\sum_{-\infty}^{\infty}\beta_\tau e^{i\tau\omega t}$ cannot be realized by the differentiation of every term. By all means, however,

$$\nu_\tau = \frac{1}{T}\int_0^T N(p)y e^{-i\tau\omega t}dt = \frac{1}{T}\left[\int_0^{t_1} N(p)y e^{-i\tau\omega t}dt + \int_{t_1}^{t_2} N(p)y e^{-i\tau\omega t}dt\right] \quad (4.15)$$

Now, to set to the calculation of ν_τ, let us note that during the time of one transition (at the regime I2I) on the characteristic of Fig.7, the x and y variables change according to Fig.8.

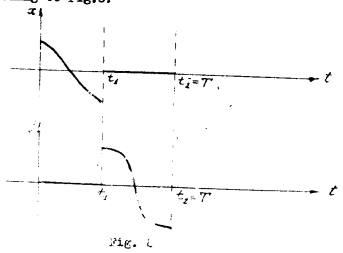

Fig. 8

In the interval $0<t<t_1$ we have $y=0$. This is actually guaranteed provided

$$N(p)y = 0$$

with the boundary conditions

$$y(t_1-0) = \dot{y}(t_1-0) = \ldots = y^{(n-1)}(t_1-0) = 0 \quad (4.16)$$

Hence, if the conditions (4.16) are satisfied, the first integral in (4.15) vanishes.

In the interval $t_1<t<t_2$ we put $x=0$ that is $L(p)x=0$ if the requirements

$$x(t_2-0) = \dot{x}(t_2-0) = \ldots = x^{(n-1)}(t_2-0) = 0 \quad (4.17)$$

are secured. But the equations $L(p)=0$ and (4.12) where p^* is changed by p, lead to the following

$$N(p)\,y = \mathcal{E}$$

Therefore

$$\nu_\tau = -\frac{1}{T}\int_{t_1}^{t_2}\mathcal{E}e^{-i\tau\omega t}dt = \frac{\mathcal{E}}{2\pi i\tau}\left(e^{-i\tau\omega t_2} - e^{-i\tau\omega t_1}\right) \quad (4.18)$$

Now, given ν_τ, one has to calculate ν_τ^*. The formula which connects these quantities, has the form */

$$\nu_\tau^* = \nu_\tau + \frac{1}{T}\left[e^{-i\tau\omega t_1}\sum_{\kappa=0}^{n-1}\eta_\kappa m_\kappa(i\tau\omega) + e^{-i\tau\omega t_2}\sum_{\kappa=0}^{n-1}\eta_\kappa m_\kappa(i\tau\omega)\right] \quad (4.19)$$

where

*/ This formula is obtained as follws : we set some connection between the operations $p^*F(t)$ and $pF(t)$, where $F(t)$ is periodic. To check this, let us put the series $\sum_{k=-\infty}^{\infty}\delta(t-\tau T)$ instead of $\delta(t-t_1)$ in (4.3), the summation in (4.3) being performed only within a period ; analogous relations are put down for $p^{*k}F(t)$ ($k=1,2,\ldots,n-1$). The formulae so obtained are multiplied by $\beta_0, \beta_1, \ldots, \beta_n$ correspondingly and summed up. Comparing the coefficients by equal harmonics, one should take in mind that in the case considered the jumps may occur only by t_1 and t_2 that is q has two values only : $q=1$ and $q=2$.

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$m_k(i\tau\omega) = b_{n-k-1} + b_{n-k-2}(i\tau\omega) + \ldots + b_0(i\tau\omega)^{n-k-1}$

and η_k^1 and η_k^2 – the jumps of $d^k y$ in the moments t_1 and t_2 correspondingly.

Having replaced (4.18) into (4.19) and the resulting expression into (4.13) and (4.14) we gather the terms with η_k^1, η_k^2 and find x and y in the Fourier expansions form

$$x = \sum_{k=0}^{n-1} \left[R_k(t-t_1)\eta_k^1 + R_k(t-t_2)\eta_k^2 \right] + R(t) \qquad (4.20)$$

$$y = \sum_{k=0}^{n-1} \left[S_k(t-t_1)\eta_k^1 + S_k(t-t_2)\eta_k^2 \right] + S(t)$$

where

$$R_k(t) = \frac{1}{T}\sum_{\tau=-\infty}^{\infty} \frac{m_k(i\tau\omega)}{L(i\tau\omega)} e^{i\tau\omega t}, \qquad S_k(t) = \frac{1}{T}\sum_{\tau=-\infty}^{\infty} \frac{m_k(i\tau\omega)}{N(i\tau\omega)} e^{i\tau\omega t}$$

$$R(t) = \sum \frac{\nu_\tau + \varepsilon'}{L(i\tau\omega)} e^{i\tau\omega t}, \qquad S(t) = \sum \frac{\gamma_\tau}{N(i\tau\omega)} e^{i\tau\omega t}$$

and ε' denotes ε if $\tau = 0$ and 0 has other values.

The expansions (4.20) have only $2n+2$ unknowns instead of infinite number of coefficients α_k and β_k. These unknowns are to be the jumps η_k^1 and η_k^2 ($k = 0,1,\ldots,$) and times t_1 and $t_2 = T = \frac{2\pi}{\omega}$. Besides, $2n$ unknowns (all the η_k^1 and η_k^2) enter linearly into (4.20) and only two unknowns t_1 and t_2 – nonlinearly. To determine these $2n+2$ unknowns, we have $2n$ relations (4.16) and (4.17), and the characteristic of Fig. 7 provides additional connections

$$\eta_0^1 = Q_1 \qquad \text{and} \qquad \eta_0^2 = Q_1' \qquad (4.21)$$

Hence, having required the series (4.20) to meet (4.16) and (4.17), / and solved the resulting equations relative to η_0^1 and η_0^2 we make them equal to Q_1 and Q_1' in accordance with (4.21) and obtain two transcendental equations for unknown times t_1 and t_2, that is the period equations.

The roots of these equations $t_1 = t_1^*$ and $t_2 = t_1^*$ determine only the "pretenders" to possible periodic regimes. Having evaluated all the η_k^1 and η_k^2 and put them into (4.20), one actually has to find the series $x(t)$ and $y(t)$ and draw them graphically for $0 < t < t_2 = T$. Then it is necessary to persuade that the regime has in

/ For this sake, one has to differentiate (4.20) $n-1$ times; in all the Fourier coefficients of the expansions R_k, S_k, R and S the power of denominator is higher than that of a numerator this feature accelerating the convergence of the series. If the differentiation is of $(n-1)$ times, some part of these coefficients may occur to have the power of numerator higher than that of denominator, that is the derivatives of x and y of one order high enough may be expressed by divergent series. The convergence has then be secured by any method.

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fact the formula I2I that is there occur no additional switches within the period.

Note that the period equations as well as in the relay case, are expressed not through the integrals of the corresponding linear equations but through the coefficients of those equations. Natural generalization of this method to more complex regimes and characteristics with arbitrary number of straight lines parallel to two lines given is left out of the scope of the present survey.

§5. The reduction to the integral equation.

The system of the differential equations (I.I) will be put down in the following form

$$\dot{\eta} = B\eta + \alpha h j'\eta + h f_*(\sigma), \quad \sigma = j'\eta \tag{5.I}$$

The piecewise function $f_*(\sigma)$ will now be determined in another way than before, namely, by the relations

$$f_*(\sigma) = (\alpha_k - \alpha)\sigma + \beta_k, \quad k = 1,2,\dots \ell \tag{5.2}$$

where the numbers I and ℓ denotes the same line as before.

In the periodic regime sought for, $f_*(\sigma)$ is periodic of the period T as well as the $\eta(t)$ — solution of (5.I) itself. The latter may be presented by means of already introduced matrix $X(t-\tau)$ in the form of the definite integral

$$\eta(t) = \int_0^T X(t-\tau) f_*(\sigma(\tau))\, d\tau \tag{5.3}$$

Now, the quantity $\sigma(t)$ will be presented by two ways corresponding to the presentations (2.20) and (2.26) of the matrix $X(t-\tau)$. The formulae (2.8) and (2.4) should be taken into account.

The first presentation conjectures the knowledge of the eigenvalues of the matrix Q, this presentation being

$$\sigma(t) = -\sum_{q=1}^{n} \frac{M(\mu_q)}{\Delta'(\mu_q)} \frac{e^{\mu_q t}}{e^{\mu_q T} - 1} \left[\int_0^t f_*(\sigma(\tau)) e^{-\mu_q \tau} d\tau + e^{\mu_q T} \int_t^T f_*(\sigma(\tau)) e^{-\mu_q \tau} d\tau \right] \tag{5.4}$$

The second one gives $\sigma(t)$ in the form of the trigonometric expansion

$$\sigma(t) = \frac{1}{T} \sum_{\nu=-\infty}^{\infty} \frac{M(i\omega\nu)}{\Delta(i\omega\nu)} e^{i\omega\nu t} \int_0^T f_*(\sigma(\tau)) e^{-i\omega\nu\tau} d\tau \tag{5.5}$$

Thus, to determine the function $\sigma(t)$ we have an integral equation of the form

$$\sigma(t) = \int_0^T K(t-\tau) f_*(\sigma(\tau))\, d\tau \tag{5.6}$$

with the kernel $K(t-\tau)$, the function $K(x)$ being as follows

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$$K(\tau)=\frac{1}{T}\sum_{\nu=-\infty}^{\infty}\frac{M(i\omega\nu)}{\Delta(i\omega\nu)}e^{i\omega\nu x}=\begin{cases}-\sum_{q=1}^{n}\dfrac{M(\mu_q)}{\Delta'(\mu_q)}\dfrac{e^{\mu_q x}}{e^{\mu_q T}-1} & x>0\\ -\sum_{q=1}^{n}\dfrac{M(\mu_q)}{\Delta'(\mu_q)}\dfrac{e^{\mu_q(T+x)}}{e^{\mu_q T}-1} & x<0\end{cases} \qquad (5.7)$$

In the simplest case, when all the straight lines forming the characteristic are parallel to each other the formula (5.6) presents direct solution of the problem. In fact, if $\alpha_k=\alpha$ (for all the values of k) then it follows after 5.5

$$\sigma(t)=\frac{1}{2\pi i}\sum_{\nu=-\infty}^{\infty}\frac{M(i\omega)}{\nu\Delta(i\omega\nu)}e^{i\omega\nu t}\sum_{k=1}^{\tau-1}\beta_k\left(e^{-i\omega\nu t_k}-e^{-i\omega\nu t_{k+1}}\right) \qquad (5.8)$$

where $t_1=0$, $t_\tau=T$. The power of the numerator (relative to ν) is at least by two not less than that of the denominator, so that the expansion (5.8) converges uniformly and presents a continuous function of t.

The period equations will be

$$\sigma_{t_{\ell+1}}=\frac{1}{2\pi i}\sum_{\nu=-\infty}^{\infty}\frac{M(i\omega\nu)}{\nu\Delta(i\omega\nu)}\sum_{k=1}^{\tau-1}\beta_k\left[e^{i\omega\nu(t_{\ell+1}-t_k)}-e^{i\omega\nu(t_{\ell+1}-t_{k+1})}\right]$$

$$\sigma_{t_{\tau+1}}=\frac{1}{2\pi i}\sum_{\nu=-\infty}^{\infty}\frac{M(i\omega\nu)}{\nu\Delta(i\omega\nu)}\sum_{k=1}^{\tau-1}\beta_k\left(e^{-i\omega\nu t_k}-e^{-i\omega\nu t_{k+1}}\right) \qquad (5.9)$$

$$\ell=1,2,\ldots,\tau-2$$

These determine $(\tau-1)$ unknowns $t_2, t_3, \ldots t_\tau=T$. In particular, for the relay systems, all the links of the piecewise characteristic are parallel to the σ-axis

then $\alpha=0$ and it is necessary to change Δ by Σ in (5.8) and (5.9) in accordance with (2.5).

In the general case of a piecewise characteristic, the analysis of the integral equation (5.7) is reduced only to the infinite system of linear equations. Having put

$$\gamma_\nu=\frac{1}{T}\int_0^T f_\sigma(\sigma(\tau))\,e^{-i\omega\nu\tau}\,d\tau, \quad \nu=0,\pm1,\pm2,\ldots \qquad (5.10)$$

one receives

$$\sigma(t)=\sum_{\nu=-\infty}^{\infty}\frac{M(i\omega\nu)}{\Delta(i\omega\nu)}\gamma_\nu\,e^{i\nu\omega t} \qquad (5.11)$$

and having set this expression into (5.10), comes up to the equations just mentioned. Setting $\alpha=\alpha_1$, one gets

$$\gamma_\nu=\frac{1}{2\pi i\nu}\sum_{\ell=1}^{\tau-1}\beta_\ell\left(e^{-i\omega\nu t_\ell}-e^{-i\omega\nu t_{\ell+1}}\right)+$$

$$+\frac{1}{2\pi i}\sum_{s=-\infty}^{\infty}\frac{M(i\omega s)\gamma_s}{(s-\nu)\Delta(i\omega s)}\sum_{\ell=2}^{\tau-1}(\alpha_\ell-\alpha_1)\left(e^{i\omega(s-\nu)t_{\ell+1}}-e^{i\omega(s-\nu)t_\ell}\right) \qquad (5.12)$$

For example, for the characteristic of Fig.4 ($\tau=3$) the period equations will be

$$\sigma_{23}=\sum_{\nu=-\infty}^{\infty}\frac{M(i\omega\nu)}{\Delta(i\omega\nu)}\gamma_\nu, \quad \sigma_{12}=\sum_{\nu=-\infty}^{\infty}\frac{M(i\omega\nu)}{\Delta(i\omega\nu)}\gamma_\nu e^{i\omega\nu t_2} \qquad (5.13)$$

the system (5.12) rewritten as follows

$$\gamma_\nu=\frac{\beta_1-\beta_2}{2\pi i\nu}\left(1-e^{-i\omega\nu t_2}\right)+\frac{\alpha_2-\alpha_1}{2\pi i}\sum_{s=-\infty}^{\infty}\gamma_s\frac{M(i\omega s)}{\Delta(i\omega s)(s-\nu)}\left(1-e^{i\omega(s-\nu)t_2}\right) \qquad (5.14)$$

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In practical calculations, there is left in (5.II) only finite number of terms. We come up then to approximate solutions realized as exactly as given beforehand. The first approximation, the zeroth and the first harmonics conserved, is a well known method of harmonic balance.

In the work [15], the integral equation (5.6) is reduced to that of a Fredholm type by virtue of the function of time equal to α_k when $t_{k+0} < t < t_{k+1} - 0$ introduced into consideration. To solve this equation, the general method of resolvent may be applied.

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ТЕОРИЯ БИФУРКАЦИЙ ДИНАМИЧЕСКИХ СИСТЕМ ВТОРОГО ПОРЯДКА И ЕЕ ПРИМЕНЕНИЕ К ИССЛЕДОВАНИЮ НЕЛИНЕЙНЫХ ЗАДАЧ ТЕОРИИ КОЛЕБАНИЙ

I

This paper deals with the effective methods of the qualitative investigation of the autonomous dynamical systems of the second order. Those methods are based on the theory of bifurcations of the dynamical systems developed by A.A. Andronov and his collaborators. The application of these methods to certain concrete problems is considered.

The dynamical systems suppose to have the form

$$\frac{dx}{dt} = P(x, y, \lambda_1, \ldots, \lambda_n), \quad \frac{dy}{dt} = Q(x, y, \lambda_1, \ldots, \lambda_n) \quad (A)_\lambda$$

where $P(x, y, \lambda_1, \ldots, \lambda_n)$ and $Q(x, y, \lambda_1, \ldots, \lambda_n)$ are analytical functions of the variables x and y in certain regions of the plane and analytical functions of the parameters λ_i in a certain region of the values of these parameters •). These dynamical systems, inspite of being special, are known to have

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•) For the development of the theory of bi-

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played a great role in the problem of an adequate mathematical description of a number of nonlinear phenomena in different fields of physics and engineering. A number of problems in mechanics, radio physics, radio engineering, problems in the theory of electrical machines etc., lead to this kind of systems.

Here, as well as elsewhere in this paper, we are interested in those phenomena to which correspond the qualitative properties of the trajectories of the dynamical system i.e., trajectories in the corresponding phase space.

If the idealization adopted in the description of the corresponding dynamical system explicitly reveals the main features of the phenomena under consideration then the mathematical investigation of the dynamical system "mathematical model" of the phenomena under consideration - may be rather fruitful. A.A. Andronov paid a great attention to this fact.

Such mathematical investigation (we mean that it is the qualitative investigation of the dynamical system, as much as we are interested only in those properties which are expressed by the qualitative properties of the trajectories) very often allows to draw the non-trivial conclusions concerning the physical problem under consideration or to give original recommendations for the construction of a technical device.

The dynamical systems which are obtained in considering physical and engineering problems always contain a certain number of parameters. Then there naturally arises the problem of investigating any possible changes of the qualitative structure of the phase-space that occurs with the changes of the parameters.

This investigation represent the essence of the theory of bifurcations of the dynamical systems developed by A.A. Andronov and his school. The role of the theory of bifurcations, however, is not reduced to this only.

The qualitative investigation of the system $(A)_\lambda$ is comparatively well developed for the cases I) when such system is close to the linear conservative system (the method of small μ) or to the nonlinear conservative system, 2) when it is a piece-linear system, 3) when it has a particular form and graphical methods (method by Lienard and another) would be quite sufficient to investigate it. It is not quite usual, however, that we can reduce the problem under consideration to one of the above cases.

[illegible] it is not necessary to assume that functions P and Q are analytical. [illegible] that a [illegible], however, in view of the fact that [illegible] [illegible]

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Thus the problem that faces us is that of the development of regular methods or even sufficiently effective ways of the qualitative investigation of the general non-linear dynamical systems $(A)_\lambda$. These methods must give an opportunity of getting the partition of the parameter space into regions with different qualitative behavior of the trajectories, corresponding to the different behavior of either physical or engineering problem under consideration.

These methods may be developed by the theory of bifurcations. Before describing these methods we shall remind of the main facts of the theory of bifurcations (see [7, II]). In the system $(A)_\lambda$ among all the trajectories - states of equilibrium, limit- cycles and separatrices (saparatrices of the saddle points and the separatrices of the multiple states of equilibrium. The relative position of the singular trajectories determines the qualitative structure of the partition of phase surface into trajectorics. When considering the changes (bifurcations) of the qualitative structure it is sufficient to follow the changes of the singular trajectorics. (See [2,4-6]).

The main concept, on which the theory of bifurcations or "the theory of the change of the dynamical systems with the change of the right-hand sides" is based, is the concept of the structurally stable dynamical system introduced by A.A. Andronov and L. Pontrjagin [1-4] and also the concept of the non-structurally stable systems with different degrees of non-structural stability [7] and, at last, the concept of bifurcations of the dynamical system [2,4]. All these concepts are quite natural in their applications as well as from purely mathematical point of view.•)

We do not give any exact definitions here; we only explain what these concepts mean in general, not aiming at exactness. First of all we shall remind that the dynamical system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \qquad (A)$$

is called structurally stable if its qualitative structure does not change with all sufficiently small changes of its right-hand sides with which there also occur rather small changes of the derivatives of these right - -hand sides. If the system (A) is structurally stable then all the systems in which the right-hand sides and their derivatives are sufficiently close to the right-hand sides of the system and their derivatives are also structurally stable. As it is known, all states of equilibrium of structurally stable

•) The exact definition of a structurally stable system and systems with different degrees of non-structural stability is given by introducing Banach space of the dynamical systems.

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systems are simple (i.e., they are nodes, foci and saddle points, all limit cycles characteristic exponents do not vanish and the separatrices do not go from a saddle point, to the same saddle point or to another one. Together with structurally stable systems the investigation of non-structurally stable systems and their classification according to the degree of non-structural stability is of great importance for applied knowledge [7,8].

The systems "of the first degree of non-structural stability", in particular, being relatively structurally stable among the set of non-structurally stable systems" are of great interest. It is quite natural to consider the conservative dynamical systems as systems of an infinite degree of non-structural stability.

A dynamical system of the first degree of non-structural stability must have the only singular trajectory of the first degree of non-structural stability, i.e. the trajectory of one of the following types: I) a double state of equilibrium $O(x_0,y_0)$ in which $P'_x(x_0,y_0)+Q'_y(x_0,y_0)\neq 0$ (the saddle-node point (see fig.I.); 2) a fine (multiple) focus for which the first Liaponov's value is not vanishing (fig.2a); 3) a double limit-cycle (fig.3a),; 4) a separatrix going from one saddle point to another (fig. [illegible], a separatrix [illegible] saddle point $O(x_0,y_0)$ [illegible]

this saddle point $P'_x(x_0,y_0)+Q'_y(x_0,y_0)\neq 0$ (fig.5a)

Let us emphasize that while the conditions distinguishing among the dynamical systems the structurally stable systems are conditions of the type of unequalities (a structurally stable systems may have only such of equilibrium for which

$$D = \begin{vmatrix} P'_x(x_0,y_0) & P'_y(x_0,y_0) \\ Q'_x(x_0,y_0) & Q'_y(x_0,y_0) \end{vmatrix} \neq 0$$

there may be only such limit-cycles for which the characteristic exponent $h\neq 0$ etc., the conditions distinguishing the non-structurally stable systems are conditions of the type of equalities *). A non-structurally stable system must have either a state of equilibrium for which $D=0$, or a fine (multiple) focus for which $Q'_x(x_0,y_0)+Q'_y(x_0,y_0)=0$ or limit-cycles having the characteristic exponent $h=0$; or a separatrix going from one saddle point to another one or to the same saddle-point.

Let us return to the system of the type $(A)_\lambda$, i.e., the system containing parameters and explain what is called "a bifurcation".

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*) It should be noted that not for all these conditions there are analytical expressions; for instance, there is no such expression available for the condition that the separatrix goes from one saddle point to another.

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If the system (A)$_\lambda$ corresponding to a certain point $(\lambda_1^\circ, \lambda_2^\circ, \ldots, \lambda_n^\circ)$ of the parameter space is non-structurally stable and with certain change of the parameters there changes the qualitative structure, then it is said that a bifurcation takes place and the values $(\lambda_1^\circ, \lambda_2^\circ, \ldots, \lambda_n^\circ)$ are called "the bifurcational" values of the parameters.

Thus, for example, there may be bifurcations at which the multiple states of equilibrium generate some states of equilibrium or disappear (see [2,4,8-II], bifurcations at which there appear or disappear limit-cycles etc. The bifurcations at which the system corresponding to the values of λ_i° is a system of the first degree of nonctructural stability may be naturally called the main simplest bifurcations.

The consideration of these bifurcations allows, in particular, to distinguish the simplest cases of the "generation" of limit -cycles [8,IO,II]. As it is known, there are the following cases:

a) The generation from a fine (multiple) focus (fig. 2a,2b)

b) The generation from a double limit-cycle(fig.3a, 3b)

c) The generation from a separatrix of the saddle point forming a loop(fig.5a, 5b) if in the saddle point the value

$$\sigma = P_x'(x_0,y_0,\lambda_1^\circ,\ldots,\lambda_n^\circ) + Q_y'(x_0,y_0,\lambda_1^\circ,\ldots,\lambda_n^\circ) \neq 0 \qquad (1)$$

In the last case the loop is either "stable" or "unstable" depending on the fact which of the unequalities

$$\sigma < 0, \quad \sigma > 0$$

holds. This when $\sigma < 0$ from the loop there may generate the only stable limit-cycle and when $\sigma > 0$ there may generate the only unstable limit cycle (see fig.5a, 5b) •)

d) The generation from the separatrix of the saddle-node point $O(x_0,y_0)$ issuing from it and tending to it (fig.6a,6b).

In this case there always appear the only limit-cycle the stable one if

$$\sigma = P_x'(x_0,y_0,\lambda_1^\circ,\ldots,\lambda_n^\circ) + Q_y'(x_0,y_0,\lambda_1^\circ,\ldots,\lambda_n^\circ) < 0$$

and the unstable one if $\sigma > 0$.

Let us assume for the sake of simplicity,

•) In case the dynamical system under consideration has the closed path consisting of separatrices of the saddle points and the saddle points and the path contains more than one saddle point. In this case the corresponding dynamical system is evidently the system of the degree of non-structural stability higher than the first. Then the following statement holds:

Let $O_1(x_1,y_1), O_2(x_2,y_2), \ldots, O_n(x_n,y_n)$ be the coordinates of the saddle points contained by the path Δ and $\sigma_i = P_x'(x_i,y_i) + Q_y'(x_i,y_i)$, $i = 1,2,\ldots,n$ In the case $\sigma_i < 0$, $i = 1,2,\ldots,n$ "the path is stable" in the case $\sigma_i > 0$ $i = 1,2,\ldots,n$ "path is unstable", then in the first case from the path there may appear the only stable limit-cycle and in the second case the only unstable limit cycle.

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that the dynamical system under consideration contains only two parameters λ_1 and λ_2 so that "the space" of the parameters is a plane and for this simple assumption let us illustrate some facts of the theory of bifurcations.

If the problem under consideration is of such kind that in the corresponding dynamical system there are values of the parameters at which the system is structurally stable, then there is apparently a whole region of the values of the parameters, the region of the plane (λ_1, λ_2) for which this system is also structurally stable (if follow from the very definition of"the structural stability of a system•)

The values of the parameters at which the system under consideration is non-structurally stable, i.e., the bifurcational values of the parameters satisfy, as we have mentioned, the relations of the type of equalities. Each of these conditions determines a certain line in the parameter plane being, as a rule, a boundary of the regions to which correspond structurally stable systems with different qualitative structures.

When we intersect such a line in the parameter space, then in dynamical systems there apparently takes place a bifurcation. To the points of such a line there apparently correspond dynamical systems of the first degree of non-structural stability or of a higher degree of non-structural stability (conservative systems, in particular) owing to the depence of the right-hand sides of the dynamical system on the parameters .•) But then to the points of intersections of two such lines there certainly correspond the dynamical system of the degree of non-structural stability higher than the first.

If there is known ,the set of all the bifurcational values of the parameters,or if

•) There are problems which lead to the consideration of the dynamical systems which are non-structurally stable at all values of the parameters of this system.For example, this takes place when the problem is described by Hamilton's systems. In this paper we consider only those dynamical systems which ar certain values of the parameters and consequently for certain regions of the values of the parameters and consequently for certain regions of the values of the parameters are structurall stable.

•) At the dependence of the right-hand sides of the dynamical system on the parameters,it is quite natural to consider it most "general" or "normal" when to the systems of the first degree on non-structural stability,and only to these systems, in the n- dimensional parameter space there correspond the points in the manifold of n-I dimensions, to the systems of the second degree of non-structural stability there correspond points in the manifold of n-2 dimensions etc.

In the concrete cases, however,which are considered in this paper the value of parameters filling the manifold of the n-I dimensional number often correspond to the systems of the degree of non-structural stability higher than the first and to the conservative systems, in particular.

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it is proved that they are absent) and the character of all the bifurcations in passing through all the bifurcational values and besides if there is known the qualitative structure of the dynamical system with some parti - cular values of the parameters, THEN, TAKING INTO consideration the continuity, it is possible to show that having these data, we can determine the qualitative structure for any point in the whole parameter space.•)

Thus the set of the bifurcational values evidently partitions the parameter space into regions of the values of the parameters corresponding to the same structurally stable qualitative structures. Thus if this situation takes place in a concrete non-linear system then the qualitative investigation is considered to be complete.

From what has been said it is clear that the development of the general methods of the establishment of the set of the bifurcational values of the parameters is a very important and actual problem in developing the methods of the qualitative investigation. As we have seen some bifurcational values can be defined from certain relations between the parameters

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•) One might draw the analogies between the considerations given here and Morse's ideas. The consideration of these analogies however would exceed the limit of this paper.

of the type of equalities.

Analytical expressions, however, cannot be given for the whole set of the bifurcational values. These expressions exist only for the bifurcational values corresponding to the non-simple states of equilibrium. Nevertheless it is possible sometimes to prove in the indirect way the existence or absence of the bifurcations of one kind or another.

As an example of one of such ways there may be regarded methods of establishing the existence of the bifurcations which take place with the change of the parameters from those corresponding to one well-known qualitative structure to those corresponding to another well-known qualitative structure and the methods of reducing to contradiction the assumption of the existence of this or that bi - furcation as well.

In the cases considered below such methods are used for the qualitative investigation of concrete systems in which the vector field rotates to one side with the increase of one of the parameters, for example λ_i. For these systems the difference

$$R = \left(\frac{dy}{dx}\right)_{\lambda_i^{(2)}} - \left(\frac{dy}{dx}\right)_{\lambda_i^{(1)}} \qquad (\lambda_i^{(2)} > \lambda_i^{(1)})$$

remains constant either on the whole phase surface, or on a certain part of it.

At this investigation of great importance is the ability to establish uniquely the qualitative structure of the system

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even at certain concrete values of the parameters. For this purpose to determine the existence and relative position of the singular trajectories (and in a number of cases to prove the absence of bifurcations) the well known classical methods of qualitative analysis, are of use. To these methods belongs the application of different criterion on the existence of limit-cycles (Bendixon Dulac's criterion and their generalizations); the consideration of systems close to conservative, the investigation of the disposition of the isoclines and the behavior of the trajectories in the regions between them and also different versions of the investigation of topographic systems and differential equations of comparison etc.

Let us make some remarks concerning the cases when the sets of the bifurcational values of the parameters are given not analytically, by the expression of the type of the equality, but only by the demonstration of the very fact of the existence of these sets in some intervals of the change of the parameters λ_i, $\lambda_i^{(1)} < \lambda_i < \lambda_i^{(2)}$. In these cases more exact quantitative evaluations for the points of the set of the bifurcational values of the parameters may be received by the numerical methods.

Indeed as it is shown in [12] for each structurally stable system (A) there is not vanishing value δ_0 the so-called "geometrical measure of the non-structural stability" of the system. All singular trajectories of the

structurally stable system and consequently, its qualitative structure may be determined by an approximatively construction of the singular trajectories and by the final number of operations.

Sufficient exactness of the approximation depends upon the value $\delta_0 > 0$. For any non-structurally stable system the value $\delta_0 = 0$ and it is impossible to determine the establishment of the qualitative structure of the non-structurally stable system by the approximation of its singular trajectories.

The qualitative structures of the non-structurally stable systems may be in a sense, "approximated" by the qualitative structures of the close structurally stable systems with arbitrary small geometrical measure of structural stability.

The bifurcational value of the parameter λ_κ of the system $(A)_{\lambda_\kappa}$ the existence of which is established at some interval of the change of the parameter λ_κ, $\lambda_\kappa^{(1)} < \lambda_\kappa < \lambda_\kappa^{(2)}$ (then the values λ_i ; when $i \neq \kappa$ are fixed), may be approximated with the required exactness to the values of the parameters

$$\lambda_\kappa^{(1)} < \lambda_\kappa^{(1)} < \lambda_\kappa^{(11)} < \lambda_\kappa < \lambda_\kappa^{(21)} < \lambda_\kappa^{(2)})$$

of the structurally stable systems $(A)_{\lambda_\kappa^{(1)}}$ and $(A)_{\lambda_\kappa^{(2)}}$ close to the non-structurally stable system $(A)_{\lambda_\kappa}$ (then the exactness of the "approximation" is characterized by the value of the interval $(\lambda_\kappa^{(1)}, \lambda_\kappa^{(2)})$ and "measure of the structural stability" of structurally stable sy-

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be cycles, surrounding the cylinder (cycles of the 2nd kind). To the cycles of the second kind there correspond the solutions $y(x)$ of the equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

which are periodic in x. Some ways facilitating the quantitative investigation of the non-linear system on the phase, are described in the work [27].

Here we shall dwell on the systems (3) - - (6), the system (3) being investigated in detail. In this investigation requiring the study of all the main bifurcations there may be clearly shown the role of the theory of bifurcations in working out the effective methods of the qualitative investigation of concrete nonlinear systems. For the system (4) and (5) we shall only enumerate any possible bifurcations, and for the system (6) we shall describe all possible qualitative structures and their change with the transition from one region of the parameter space into another.

S Y S T E M (3)

This system is equivalent to the equation:

$$\frac{d\rho}{d\varphi} = \frac{2\rho(1-\mu\rho\sin\varphi)}{\rho\cdot\cos\varphi} \qquad (7)$$

Here φ is the angle which the direction of velocity of the centre of gravity makes with the horizon, ρ is the value proportional to the velocity square of the centre of gravity of the aeroplane, parameters λ and μ are proportional to the values of the traction of the propeller and the resistance of the medium respectively.

According to the physical sense of the variables the rectilinear uniform motion of the aeroplane corresponds to the stable states of equilibrium of the system (3), the periodic motions of the aeroplane on the wavy curves correspond to the limit-cycles of the first kind and the loops correspond to the cycles of the second kind.

The equation (7) and the system (3) were investigated in a number of works [16], [17], [28-30].

Here we give a brief account of the full investigation [17] of the system (3).

Let us begin with the investigation of the main qualitative characteristics: states of equilibrium, separatrices of the saddles points and limit-cycles, having (preliminary) singled out a conservative case of the system.

If $\lambda=\mu=0$ there occurs a conservative case of the system, Zhukowsky's case [22]. The general integral of the system has the form: $\rho^{1/2}\cos\varphi - \frac{1}{3}\rho^{3/2} = const$.

The state of equilibrium ($\varphi=0$, $\rho=1$) is the centre. The separatrix goes from the saddle point ($-\frac{2}{3}$, 0) to the saddle point

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$(+\frac{\pi}{2}, 0)$. The other integral curves represent closed curves surrounding either the state of equilibrium, or the cylinder.

This case is considered in details in [4].

The non-conservative cases of the system (3) ($\mu \neq 0$, $\lambda \neq 0$).

THE STATES OF EQUILIBRIUM. In the finite part of the phase cylinder $\rho \geq 0$ there are not more than four states of equilibrium of system

$$O_1(-\tfrac{\pi}{2}, 0);\ O_2(\tfrac{\pi}{2}, 0),\ O_3(\varphi_3, \rho_3);\ O_4(\varphi_4, \rho_4), \tag{8}$$

$$-\frac{\pi}{2} < \varphi_{34} = \arcsin\frac{\lambda \mp \mu\sqrt{1+\mu^2-\lambda^2}}{1+\mu^2} < \frac{\pi}{2},\quad 0 \leq \rho_{34} = \frac{\lambda\mu \pm \sqrt{1+\mu^2-\lambda^2}}{1+\mu^2}$$

O_1 and O_4 -saddle points, O_2-saddle point or node, O_3 -focus or node.

Bifurcations of the system associated with the change of the number of states of equilibrium take palce for the points of the curves

$$1+\mu^2-\lambda^2 = 0\ ,\ \lambda = 0 \tag{9}$$

of the parameter planes (μ, λ). The curves (9) divide the parameter plane (μ, λ) into three regions I-III with different number of equilibrium states (see fig.7).

For the points of the regions I, II, III there are three, four and two states of equilibrium of the system respectively.

When $\lambda = 1$ the note O_2 and the saddle point O_4 merge; when $1+\mu^2-\lambda^2 = 0$ the node O_3 and the saddle point O_4 merge. With these values of parameters the system is

non-structurally stable.

The system has another type of bifurcations, accosiated with the states of equilibrium for the points of the curve of the parameter plane

$$R_1 \equiv \lambda - \frac{3\mu}{\sqrt{1+4\mu^2}} = 0 \qquad (\mu = \sqrt{\tfrac{1}{2}}) \tag{10}$$

(The heavy line on fig.7).

For the points of this curve the focus O_3 is fine (multiple), so that the system is non-structurally stable. At the intersection of the curve(IO) in the parameter plane the focus O_3 of the corresponding system changes its stability from the unstability at $R_1 > 0$ to the stability at $R_1 < 0$ (in this case as it is shown below, from the fine (multilpe) focus there emerges the limit cycle).

The curve $\lambda = \lambda_*(\mu)$ (a dotted line in fig. 7) separates the region $\lambda < \lambda_*(\mu)$ of the values of the parameters for which the state of equilibrium O_3 is the focus of the region $\lambda > \lambda_*(\mu)$ for which O_3 is the node. This curve is not bifurcational.

THE SEPARATRICES OF THE SADDLE POINTS. The possible positions of the separatrices of the saddles points are determined on the base of the equality for the tangents of the separatrices in the saddle points and the signs of the derivatives $d\rho/dt$ and $d\varphi/dt$ in the regions of the phase plane between the isoclines of the horizontal slopes [4]

$$\rho = 0,\quad \lambda - \mu\rho - \sin\varphi = 0 \tag{II}$$

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and the isocline of the vertical slopes.

$$\rho \cdot \cos\varphi = 0 \qquad (12)$$

Fig.8-9 represent one of the possible positions of the separatrices of the saddle point S_i for the points of the regions I-III (fig.6) of the parameter plane respectively. The dotted lines represent the isoclines and the uninterrupted lines represent the separatrices of the saddles point (the separatrices are designated in fig.8_a,8_b-9).

When the separatrices merge the system(3) passes through the non-structurally stable state(we mean that the non-structural stability of the system is not bound up with the separatrices $\rho=0$ going from saddle point to saddle point at all values of the parameter μ and $\lambda<1$. Then the following closed paths Δ_i $(i=1-4)$ may be formed consisting of the separatrices of the saddle points.

The separatrix going from the saddle point O_1, to the saddle point O_2 and part $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$ $\left(-\pi \le \varphi \le -\frac{\pi}{2}, \frac{\pi}{2} \le \varphi < \pi\right)$ of the integral curve $\rho=0$ form the closed path $\Delta_1(\Delta_2)$.

When the separatrices S_2 and S_3 (S_2 and S_4) merge the path $\Delta_3(\Delta_4)$ are formed. The closed path Δ_1 and Δ_2 are always unstable since the saddle values of the saddle points included in the path satisfy the unequality

$$\sigma_1 > 0, \ \sigma_2 > 0, \ldots \qquad (13).$$

The path $\Delta_3(\Delta_4)$ changes its stability on the curve

$$\lambda = \lambda^* = \frac{3\mu}{\sqrt{1+4\mu^2}}, \quad \mu > \sqrt{\tfrac{1}{2}} \qquad (13)$$

since the saddle value σ_4 of the saddle point vanishes on this curve. The path $\Delta_3(\Delta_4)$ if it exists at $\lambda<\lambda^*$ is stable, at $\lambda>\lambda^*$ it is unstable.

The system having the path $\Delta_3(\Delta_4)$ under conditions (13) is non-structurally stable, the degree being higher than the first.

The consideration of the rotation (R) of the vector field of the equation (7) with the increase of one of the parameters

$$R_\lambda = \left(\frac{d\rho}{d\bar{\rho}}/\lambda_2 - \left(\frac{d\rho}{d\bar{\rho}}\right)/\lambda_1 = 2\rho/(\lambda_2-\lambda_1)/(\rho-\cos\varphi) \qquad (14)$$

$$R_\mu = \left(\frac{d\rho}{d\bar{\rho}}\right)_{\mu_2} - \left(\frac{d\rho}{d\bar{\rho}}\right)_{\mu_1} = -2\rho/(\mu_2-\mu_1)/(\rho-\cos\varphi)$$

determines the behavior of the separatrices with the change of the parameters.

The following statement is being proved

a) If there are bifurcational values (μ_0, λ_0), for which in the part $\rho-\cos\varphi>0$ of the phase surface the separatrix goes from saddle point to saddle point, then in the parameter plane is an arc passing through the point (μ_0, λ_0) for which there also takes place the bifurcation of this kind.

LIMIT-CYCLES. The investigation of the moment of the generation of the limit-cycles from the state of equilibrium and the separatrices, and also at the disappearance of the saddle-like singular point of the type of saddle + knot allows to find out the behavior of the

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cycles with the change of the parameters.

It is stated by the definition of the sign of the first Liapounov's value that the boundary (IO) of the stability region in the parameter plane is [24] "unsafe". With the intersection of the boundary accompanied by the change of λ from the values greater than λ_0 $(R, >0)$ up to the values less than λ_0 $(R, <0)$ (the curve $R,=0$ is shown in fig. 7) from the state of equilibrium there appears the only unstable limit-cycle (see fig. I2). Then the limit-cycles of the first and the second kind may appear from the closed paths Δ_i $(i=1-4)$ consisting of the separatrices of the saddle points. For establishing any possible phase structures in this problem there is used the statement of the theory of bifurcations that the stability of the path Δ_i, determines the stability of the cycle appearing form it (see (I)).

The only stable limit-cycle of the second kind appears from the separatrix of the saddle-node point this multiple state of equilibrium disappears. It occurs at the transition for $\mu > \frac{\sqrt{5}}{2}$ from the values of the parameters satisfying the equility $1+\mu^2-\lambda^2=0$ (the boundary of the region II in fig.II) to the value satisfying the unequality $1+\mu^2-\lambda^2<0$

Besides the above mentioned cases of the cycles generation in the given system there appears the double (half-stable) limit-cycle of the second kind from the "thickening" of the trajectories.

The following statements are being proves: b) if there are bifurcational values of the parameters (μ_0, λ_0) for which the system (3) has a double cycle of the second kind not merging with the path consisting of the separatrices of the saddle points, then in the parameter plane (μ, λ) there is an arc $\lambda=\lambda(\mu)$ passing through the point (μ_0, λ_0) ($\lambda(\mu)$ - is one-valued and increasing function of μ), for the points of which the system also has a double limit-cycle. c) Two cycles of the second kind of the system (3)(stable and unstable) if they exist at $\mu=\mu_0$, $\lambda=\lambda_0$ approach each other with the increase of the parameter μ.

POSSIBLE QUALITATIVE STRUCTURES AND THE PARTITION OF THE PARAMETER SPACE

Any possible qualitative structures with λ and μ not vanishing but sufficiently small are determined by N.N.Bautin in his work [16] . The system is investigated as close to nonlinear system ($\lambda=\mu=0$). It is found out that the qualitative structures change with the change of the relation $\kappa=\frac{\lambda}{\mu}$.

For the points of the straight plane (this case of the gliding line $\lambda=0$ of the parameter flight has been considered before in the works [29], [30]. the system has no periodic motions. All the trajectories except the separatrix of the saddle-point $(-\frac{\mu}{2}, 0)$ tend to the stable state of equilibrium O_3.

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The possible qualitative structures of the system (3) at any values of λ and μ are established in [17] by the investigation of the bifurcational states of the system and consideration of the continious dependence of the structure on the parameters. (Below we take into consideration the changes of the qualitative structure of the system on the phase surface, as we did before, and simultaneously the motion of the point representing this system in the parameter space). This investigation is possible due to the properties (I4) of the rotated vector field of equation (7) with the increase of the parameter. The application of the differential equation comparisons and Bendixon-Dulac's criterion on the cylinder allows to distinguish the regions of the parameter plane (regions G_i in fig.I0) with certain qualitative features. For the points of the regions G_2 and G_6 qualitative structures of the system (3) are determined uniquely. For the points of the regions G_6 the system has the only stable limit-cycle of the second kind. For the points of the region G_2 the system has no cycles. When passing with the increase of μ from the region G_6 to the region G_2 there are established all the bifurcational states of the system passed by it.•)

The cases $\lambda<1$ and $\lambda>1$ are considered separately.

Let us fix a certain $\lambda<1$ and follow the bifurcations of the system with the increase of the parameter μ from the points of the region G_6 (fig.I0 and I2a) to the points of the region G_2 (fig.I0 and I2e). The system passes the first bifurcational state at $\mu=\mu_1$ at the intersection in the parameter plane of the boundary $\lambda=3\mu/\sqrt{1+4\mu^2}$ of the region I (fig.II) (see (I0)), then from the multiple state of equilibrium O_3 there appears the unstable limit-cycle (fig.I2). The system has the second bifurcational state when $\mu=\mu_2$ at which the separatrices S_1 and S_2 merge (fig.I2). Then there are formed the closed unstable paths Δ_1 and Δ_2 from which according to the theory (see (I) and (I3)•), only the unstable limit-cycle can appear. Taking into consideration this fact and the properties (I4) of the rotated vector field we have the following:

According to the statement a) there is the curve $\ell_2\ (\mu=\mu_2(\lambda))$ of the parameter plane (it bounds the region 2 on the right, -fig.II), and when reaching this curve with the increase of μ the unstable limit-cycle of the first kind [illegible] with the closed path Δ_1 on the phase surface.

•) Then the qualitative structures of the system (3) for certain region of the parameter plane [illegible] without taking into consideration [illegible] cycles which may to appear [illegible] limit-cycles generated with [illegible] of the trajectories. This region is bound on the parameter plane (fig.II) on the left side of the straight line $\lambda=\mu\sqrt{8}$ on the right side by a dotted curve and on the top by the straight line $\lambda=3/2$.

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With the subsequent increase of $\mu > \mu_2$ from the closed path Δ_2 (consisting of the separatrices S_1, S_2 and $\rho=0$ $(-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2},\ \varphi \le -\frac{\pi}{2})$ there appears the only unstable cycle of the second kind (fig.I2e).

The curve ℓ_2 and also the bifurcational curves, shown below *), are drawn through the points obtained approximately from the qualitative integration of the close structurally stable systems.

Thus when $\mu > \mu_2$ the system has two cycles of the second kind (stable and unstable). With the increase of the parameter μ the cycles approach each other (according to the statement c)) and owing to the existence of the region G_2, there is the only bifurcational value μ_3 so that when $\mu = \mu_3$ the two cycles merge into a half-stable cycle (fig.I2g), and when $\mu < \mu_3$ they disappear. The curve $\mu = \mu_3(\lambda)$ it exist according to the statement b) bounds the region 3 (fig.II) on the right. It is drawn by means of the major curve which is represented in fig.II by the dotted line.

In this case $\lambda > 1$ there is used the same method of consideration of the bifurcations. It is stated that at the transition from region 5 to the regions I0 and II of the values of the parameters (see fig.II) the qualitative structures of the system change as it is shown in fig.I3a -I3k.

Here besides the bifurcations existing in the case $\lambda < 1$ there are four bifurcations, as follows: I) For the points of the curve, separating the regions 6 and 7 in the parameter plane, (fig II) the separatrices S_2 and S_3 merge, forming the closed path Δ_3 surrounding the state of equilibrium (fig.I3b); 2) For the points of the curve separating the regions 7 and 8 (fig. ii) the separatrices S_1 and S_2 merge (fig.I3g); 3) For the points of the curve separating the regions 8 and 9 the separatrices S_2 and S_3 merge forming the closed path Δ_4 surrounding the cylinder (fig.I3k); 4) On the curve $1+\mu^2-\lambda^2=0$ separating the region II fig.II the states of equilibrium the saddle point and the node merge forming the multiple state of equilibrium the saddle-node (fig.I3i). When this multiple state of equilibrium disappears, for $\lambda > \sqrt{3}/2$ there appears from the separatrix of the saddle-node the limit-cycle of the second kind (fig. I3j).

The change of the qualitative structures of the system (3) with the increase of μ with constant $\lambda \le 1$ is represented in fig.I2 with $\lambda > 1$ - in fig.I3. To the boundary curves of the regions I-II of the parameter planes (fig.II) there correspond the non-structurally stable cases of the system, which are also represented in fig.I2 and I3.

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*) These curves of the bifurcational values of the parameters are given not analytically, by the expression of the type of the equality, but only by the demonstration of the very fact of their existence in some intervals of the parameter ν.

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S Y S T E M (4)

The system (*) is equivalent to the equation

$$\frac{dy}{dx} = \frac{-\varepsilon(x+y^2)}{-\frac{1-\beta}{1+\beta x} + y}$$

(where $0 \le \varepsilon \le \infty$, $0 \le \beta \le \frac{1}{3}$ are parameters). Systems (4) in the real part of the phase plane has [18] two states of equilibrium the focus or the node ($x_1 = -1$, $y_1 = 1$) and the saddle point ($x_2 = \frac{1}{2} - \frac{1}{\beta} + \sqrt{\frac{1}{\beta} - \frac{3}{4}}$, $y_2 = -\frac{1}{2} + \sqrt{\frac{1}{\beta} - \frac{3}{4}}$). The system (4) has two bifurcations: 1) from the multiple focus (x_1, y_1) corresponding to $\varepsilon = \varepsilon_1(\beta)$ in converting the focus from unstable into stable there appears the only unstable limit-cycle. The boundary of the region of stability in the parameter space (ε, β) is "a non-safe boundary" [24], 2) the unstable limit-cycle with the increase of ε merges at $\varepsilon = \varepsilon_2(\beta)$ with the loop, consisting of the separatrix, going from the saddle point (x_2, y_2) to the same saddle point. With the further increase of the parameter ε the loop disintegrates and the limit-cycle disappears, the system becomes stable " in global ".

The establishment of the character of bifurcations in this problem allowed to determine the erroneous statement adopted in engineering that the stability of the system in "local" always ensures its stability in "global".

S Y S T E M (5)

The system (5) is equivalent to the equation

$$\frac{dS}{d\Theta} = \left[T - \sin\Theta - z\sin 2\Theta - \kappa(1 - \delta\cos 2\Theta)S\right] / S,$$

where κ, z, T, δ are parameters, whose real values satisfy the unequalities:

$$\kappa \ge 0, \quad z \ge 0, \quad T \ge 0, \quad 0 \le \delta \le 1.$$

The system (5) with $T_1(z) < T$ has [18,19] two states of equilibrium (one saddle point and one focus or node), with $T_2(z) < T < T_1(z)$ it has four states of equilibrium (two saddles points and two focuses or nodes) / $T_1(z)$ and $T_2(z)$ are certain functions of z /.

The bifurcational states of the system are connected 1) with the bifurcations of the state of equilibrium. With $0 \le z \le \frac{1}{2}$ and $T = T_1(z)$ ($z > \frac{1}{2}$ and $T = T_2(z)$, $T = T_1(z)$) the states of equilibrium, the node and the saddle point merge into a multiple state of equilibrium of the type the saddle-node and with the increase of the parameter they disappear; 2) with the bifurcations of the separatrices and the limit-cycle of the second kind.

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With $T = T_o(\kappa, \delta, \varepsilon)$ the separatrices of the saddle-point merge forming in the half-plane $S \geqslant 0$ a loop surrounding the cylinder. With the increase of the parameter T the only stable limit-cycle of the second kind appears from this loop.

In this case the synchronous regime of work corresponds to the stable state of equilibrium. When the stable states of equilibrium disappear the synchronous motion of the machine is impossible. The asychronous motion of the machine correspond to the stable cycle of the second kind.

The investigation of the phase portait of the system (5) explains the presence and the condition of the existence of the asynchronous motion of the machine and also the possibility of the restoration of synchronism after the asynchronous motion.

SYSTEM (6)

This system is equivalent to the equation

$$\frac{dz}{d\varphi} = \frac{\gamma - \sin\varphi - \lambda(1 - d\cos\varphi)z}{z}$$

where φ is the difference of the phases of oscillations of the generators. γ, λ, d are positive parameters proportional to initial difference of frequencies, damping and delay respectively. The system (6) in its form is similar to the system (5) with $\varepsilon = 0$. The values of the parameter d, however, which are physically significant allow the new qualitative structures [21].

In the given case to the stable state of equilibrium there corresponds the regime of synchronism, to the stable limit-cycle of the first kind corresponds the stationary regime of pulsations in the system when the frequence approach to the stable limit-cycle of the second kind corresponds the stationary regime of pulsations in the system, at which certain constant difference of frequencies is periodically repeated and the difference of phases is increased unrestrictedly. The system (6) with $\gamma < 1$ has two states of equilibrium: the focus or the node ($\varphi_1 = \arcsin\gamma$, $z_1 = 0$) and the saddle point ($\varphi_2 = \pi - \arcsin\gamma$, $z_2 = 0$).

In [21] by way of the investigation of the bifurcational states of the system with the continuous change of the parameter γ there are determined all possible phase structures of the system (fig.14) and the regions of the parameter space (γ, λ, d) corresponding to them (fig.15). Five bifurcational surfaces, the plane $\gamma = 1$ and the surfaces u, v, w, D corresponding to the non-structurally stable cases of the system partition the parameter space into regions corresponding to different structurally stable qualitative structures *).

*) The qualitative structures of the system

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(In fig.I4 the numbers of the corresponding regions of the parameter space are shown in small circles).

For the points of the plane $\gamma = 1$ (separating the region I from the regions 2 and 4 (fig.I5) two states of equilibrium, the saddle point and the node merge into a multiple state of equilibrium the saddle-node (fig. I4 δ and I4 e).

With $\gamma > 1$ the state of equilibrium disappears. (Fig.I4a). From the loop of the separatrix of the saddle-node (fig.I4δ) at the intersection with the increase of γ of the boundary separating the regions I and 2 (fig.I5) there appears the only stable limit-cycle of the sedond kind (fig.I4a). For the points of the surface W separating the regions 2 and 4 (see fig.I5 and I4Δ), 5 and 3 (see fig.I5u and I4$\varkappa$) the separatrices of the saddle point merge forming in the half-plane $z \geq 0$ the closed path surrounding the cylinder (the path of the second kind). At the intersection of the surface W with the increase of γ (the transition into the region 4 or 5 corresponds to it) from the closed path there appears the only stable limit-cycle of the second kind (fig.I4$\varkappa$ and I4u).

The surfaces D, u, v are situated in the part $d \geq 1$ of the parameter space. The surface D is determined by the equality

$$d = \frac{1}{\sqrt{1-\gamma^2}}$$

. For the points of this surface of the state of equilibrium the focus is fine (multiple). At the intersection of the surface D with the increase of d and the constants λ and γ or with the decrease of γ and the constants λ and d (the transition into the region 3 or 5 (fig.I5) corresponds to it) the focus from the stable becomes unstable and the only stable limit-cycle of the first kind appears from it (fig. I4z or I4u). Here the boundary of the region of stability is "a safety boundary" [24]. With this bifurcation which inevitably takes place with the increase of the parameter d (delay) in the system AFC the automatic oscillations arise when the frequencies approach.

For the points of the surface v, separating the regions 5 and 6 of the parameter space the separatrices of the saddles points merge forming the closed path surrounding the state of equilibrium (fig.I4λ). With the decrease of γ with the transition into the region 6 the stable cycle of the first kind merges with this path and disappears (fig.I4M). And at last, for the points of the surface u separating the regions

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(6) for the values of the parameters satisfying the unequality $d > 1$ are determined not taking into consideration the even number of cycles of the first kind which may to appear from the doubles limit-cycles generating with the thickening of the trajectories.

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6 and 7 of the parameter space the separatrices of the saddle merge forming the closed path of the second kind in the half-plane $z \leq 0$ (fig.I4*н*). (On the fig.I5 the surfaces u and v are represented approximately). At the intersection of the surface u with the decrease of γ (i.e. at the transition into the region 7) from the loop of the second kind in the half-plane $z<0$ ther appears the only stable limit-cycle of the second kind (fig.I4*о*).

Thus we can see in fig.I4*а*-I4*о* the change of the qualitative structures of the system (6) with the transition of the point representing the system in the parameter space from one region of the values of the parameter to the others (fig.I5).

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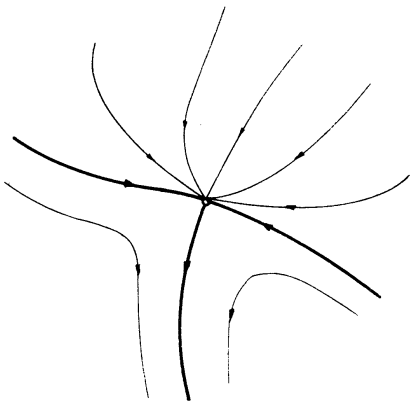

Fig. 1

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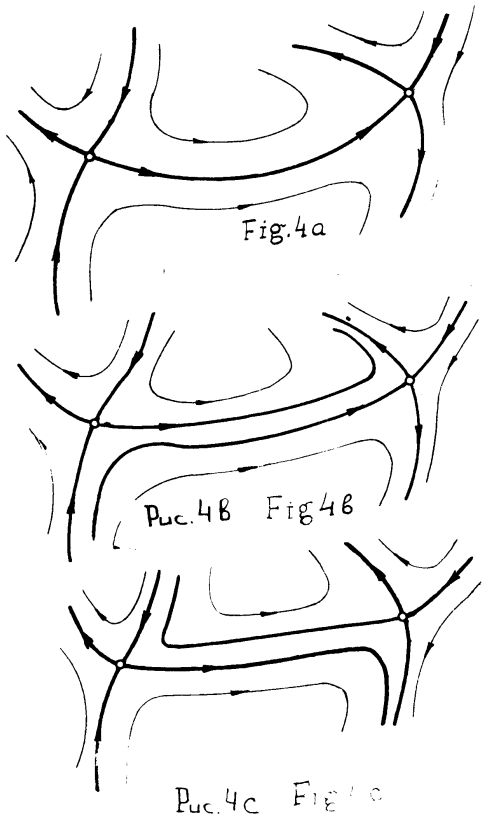
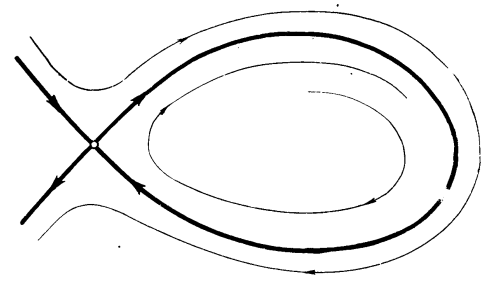


Fig. 4a

Рис. 4в Fig 4в

Рис. 4с Fig 4c



Фиг. 5а.

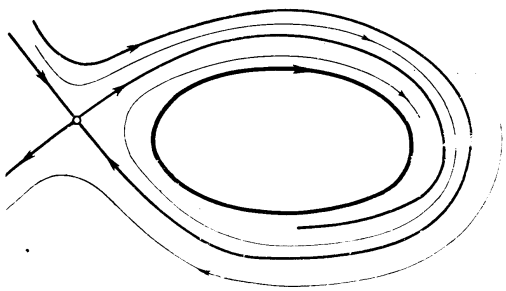


Fig. 5в

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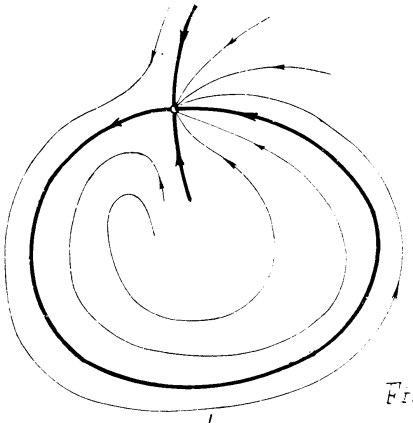

Fig. 6a

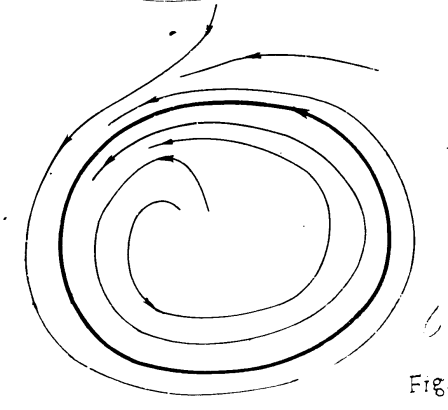

Fig 6

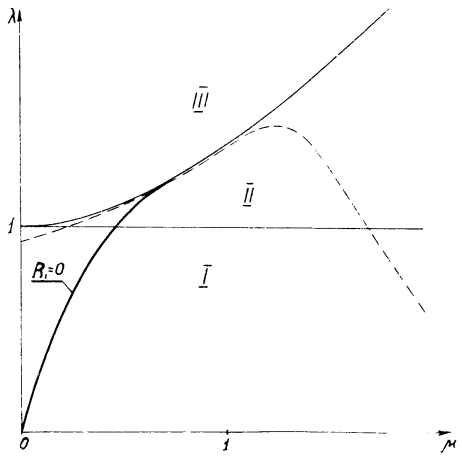

Fig. 7

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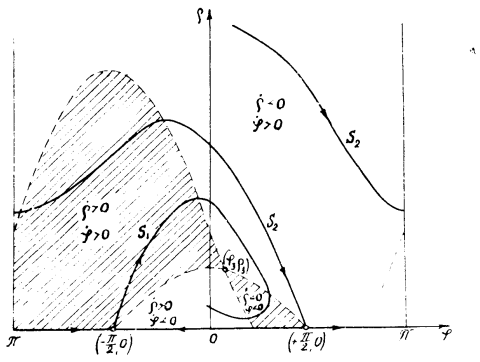

Fig. 8a

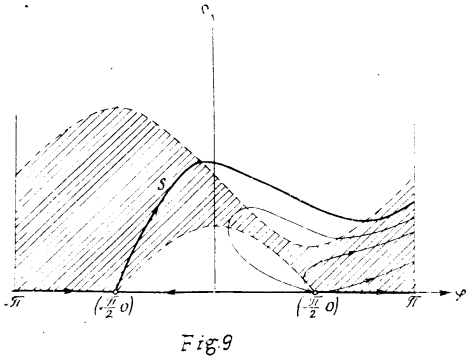

Fig 9

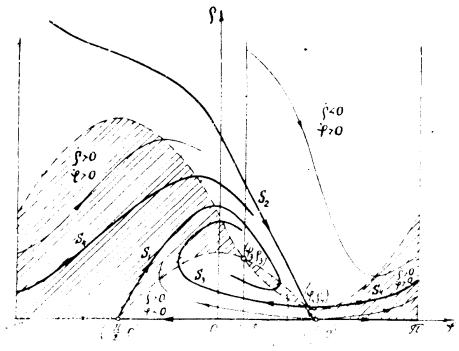

Fig 8b

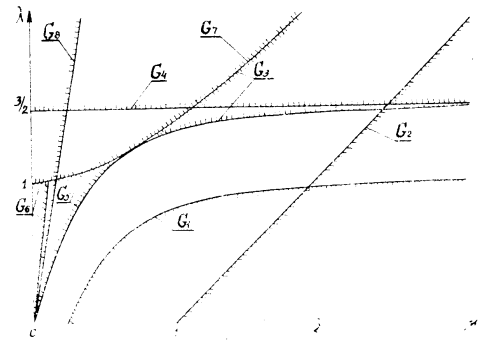

Fig 10

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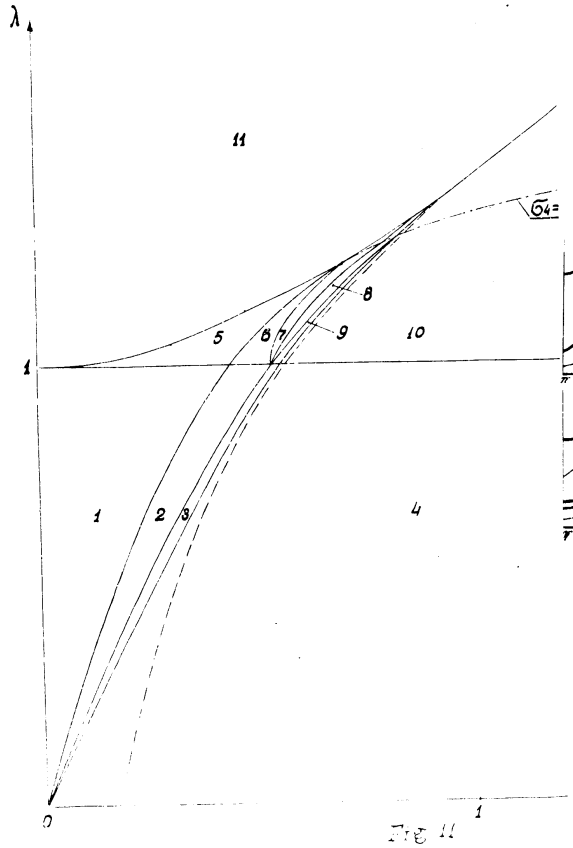

Fig 11

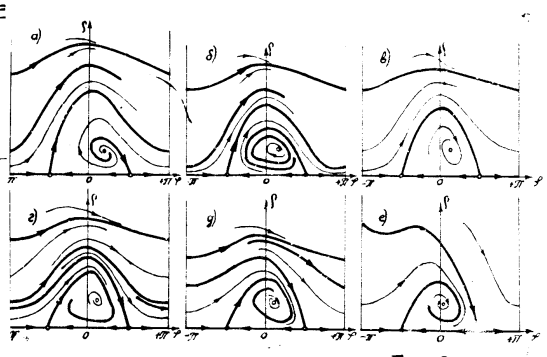

Fig 12

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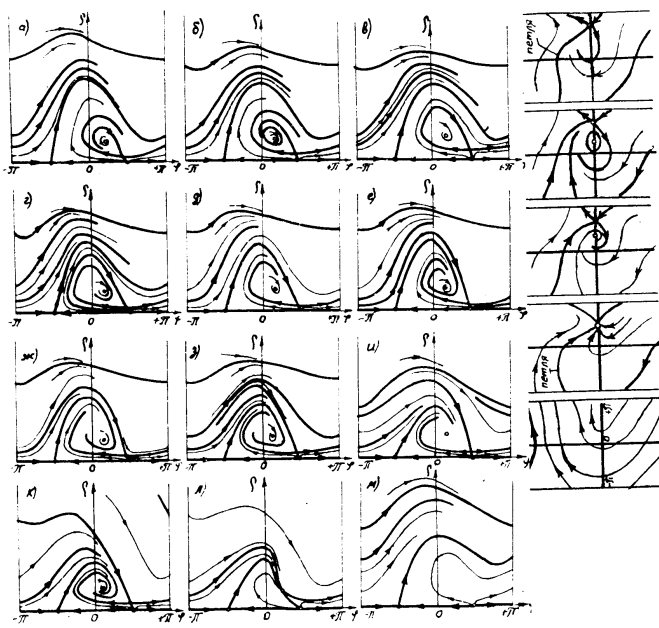

Fig 13

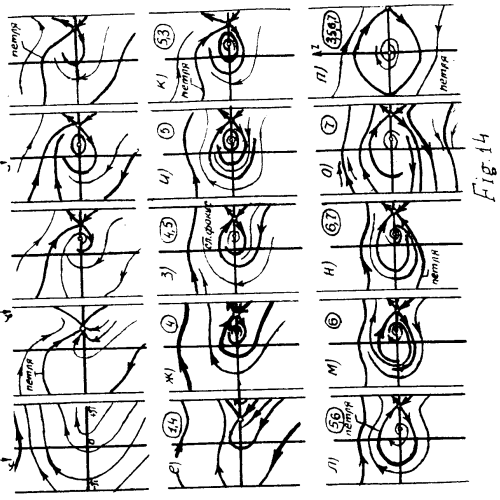

Fig 14

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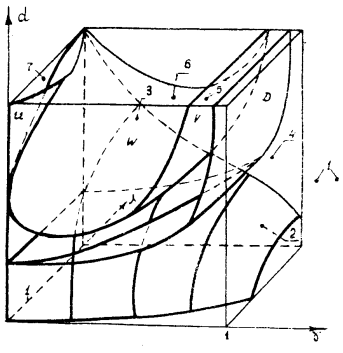

Fig 15

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Н. А. КАРТВЕЛИШВИЛИ, Я. К. ЛЮБИМЦЕВ

ПРИМЕНЕНИЕ МЕТОДОВ ТЕОРИИ КОЛЕБАНИЙ К ЗАДАЧАМ УСТОЙЧИВОСТИ СТАЦИОНАРНЫХ РЕЖИМОВ ГИДРОЭЛЕКТРИЧЕСКИХ СТАНЦИЙ И ЭНЕРГЕТИЧЕСКИХ СИСТЕМ

I. Statement of the Problem

Any a.c. machine, strictly speaking, is a system with periodically varying parameters /Ref. I/. However, under symmetrical conditions the equations for its transient behavior can be reduced to ones having constant coefficients. We have in mind just this case, or cases which boil down to the latter with sufficient accuracy. They comprise the majority of cases that are of practical interest /Ref. 2/. The general case of transients in unsymmetrical systems has not been studied much as yet.

A power system consisting of stations interconnected through a network, each of which comprises several units having rather complex-regulation equipment, is essentially a non-linear oscillatory system with constant parameters /in view of the above restrictions/. There are hundreds and even thousands of degrees of freedom in such systems; only the simplest cases contain several or tens of degrees of freedom. Moreover, there is no way of taking a limit, which would transform this oscillatory system to one with distributed parameters, i.e., that would enable this large number of simultaneous ordinary differential equations to be expressed by partial differential equations.

Oscillatory processes in power systems, and in particular the stability of their steady-state operation can be investigated by decomposition the initial system into several simpler oscillatory systems having concentrated parameters so that the behaviour of the initial system can be judged from their behaviour. Mathematically this amounts to dividing a large number of ordinary differential equations into smaller sets, with the solutions of the former being judged from solutions of the latter.

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Division of the power system into several independent oscillatory systems is justifiable on the basis of the following physical premises.

I. When studying transients in headwater systems at hydroelectric stations and in the boilers of steam power stations, it is assumed that the turbine governors do not not possess any time delay, that is to say, they always maintain exact correspondence between the mechanical and electrical torques /the hypothesis of ideal regulators/. With hydraulic turbines the inertia of the water in the penstock pipe, that is, hydraulic impacts, are neglected as well..The hypothesis of ideal regulators enables us to study these processes, which we will call hydraulic processes, completely disregarding transients in the remaining elements of the power system.

2. When studying regulation of turbines /mechanical processes/, it is assumed on the hand that the water level in the equalizing basins /and in the headwater basins or in the headrace, if there are none of the former/ of hydroelectric stations and the pressure in the boilers of steam power stations remain constant throughout the process being investigated; on the other hand the electrical ties between the generators in the system are taken to be absolutely rigid /that is, all the generators are supposed to have the same electrical speed all the time/. As a result, we abstract ourselves from the power swings between the machine rotors and from transients in all devices standing prior to the penstock pipe.

3. When investigating mutual power swings of machine rotors, that is, electromechanical transients in the system, the mechanical torques created by the turbines are taken to be constant. This excludes the necessity of accounting for turbine regulation processes and also transients in all the devices standing before the turbines.

It can be shown that in making all these assumptions we account for only some of the time constants in the problem, different ones being included in the different problems. The remaining time constants are either taken to be infinitely large or infinitely small. As a result, we discard some terms in the equations describing transient processes or some of the equations themselves.

These assumptions are ordinarily taken for granted when investigating transients in power systems. Only recently works have appeared, which consider within what limits such an approach is permissible. Prior to these studies, which will be discussed below, hydraulic mechanical and electromechanical processes were strictly divided and each of these three types of processes were considered independent of the others.

However the simplification achived as a result of these restrictions turns out to be insufficient in practise: therefore, in the case of electromechanical processes the power swing angles between the generators at one station or even at several nearby stations are neglected, as well, that is, each such group of generators is represented by a single "equivalent" machine with recourse to insufficiently grounded assumptions. It can be shown by a rather simple analysis that this assumption is also tantamount to neglecting several time constants in the equations.

Neglecting small time constants as a rule does not lead to errors. However, many cases are known where this completely distorts the results. Moreover, when a certain factor is neglected, it might be necessary to neglect ano-

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Division of the power system into several independent oscillatory systems is justifiable on the basis of the following physical premises.

I. When studying transients in headwater systems at hydroelectric stations and in the boilers of steam power stations, it is assumed that the turbine governors do not not possess any time delay, that is to say, they always maintain exact correspondence between the mechanical and electrical torques /the hypothesis of ideal regulators/. With hydraulic turbines the inertia of the water in the penstock pipe, that is, hydraulic impacts, are neglected as well..The hypothesis of ideal regulators enables us to study these processes, which we will call hydraulic processes, completely disregarding transients in the remaining elements of the power system.

2. When studying regulation of turbines /mechanical processes/, it is assumed on the hand that the water level in the equalizing basins /and in the headwater basins or in the headrace, if there are none of the former/ of hydroelectric stations and the pressure in the boilers of steam power stations remain constant throughout the process being investigated; on the other hand the electrical ties between the generators in the system are taken to be absolutely rigid /that is, all the generators are supposed to have the same electrical speed all the time/. As a result, we abstract ourselves from the power swings between the machine rotors and from transients in all devices standing prior to the penstock pipe.

3. When investigating mutual power swings of machine rotors, that is, electromechanical transients in the system, the mechanical torques created by the turbines are taken to be constant. This excludes the necessity of accounting for turbine regulation processes and also transients in all the devices standing before the turbines.

It can be shown that in making all these assumptions we account for only some of the time constants in the problem, different ones being included in the different problems. The remaining time constants are either taken to be infinitely large or infinitely small. As a result, we discard some terms in the equations describing transient processes or some of the equations themselves.

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[illegible], too, so as to obtain the correct result. For [illegible], the hypothesis of ideal regulators turns out to be correct when investigating the stability of steady-state operation of network hydraulic systems only when the time constant of the penstock pipe /that is, hydraulic impact - Ref.7/ is neglected or when hydraulic impact as well as the inertia of the water mass in the equalizing reservoir are taken in consideration /ref. 4/. Otherwise, entirely false result is obtained that the system is basically unstable.

Finally, we should bear in mind that qualitative changes may occur in transient processes as the number of degrees of freedom increase alongside with the development and consolidation of power systems spreading over vast areas. Here, the processes may be somewhat similar to wave phenomena when the disturbance is propogated from one end of the consolidated system to the other. By attempting at any cost to reduce a complex system to a system with a small number of equations it is impossible from the outstart to analyse phenomena of this type.

The premises for separating complex non-linear dynamical systems into simplere systems are founded in Ref. 6 and 7 where the idea is fostered of breaking down the resulting motion of these systems into "fast" and "slow" motion. This idea is based on the analysis of relaxational oscillations. There are purely mathematical works /see Ref. 7, 8, 9, 10/ directly connected to this subject.

The equations for transients in a power system may be expressed in the form

$$T_i \frac{dx_i}{dt} = \varphi_i(x_1,\dots,x_n,m), \quad i=1,2,\dots,n, \qquad /I.1$$

where x_i are generalized dimensionless coordinates; m is a small parameter.

Usually with $m=0$ certain φ_i have particular characteristics and the set of equations /I.1/ can be brought to the form $(\lambda+\mu+\nu=n)$:

$$T_i \frac{dx_i}{dt} = f_i(x_1,\dots,x_\lambda,y_1,\dots,y_\mu,z_1,\dots,z_\nu) \quad i=1,2,\dots,\lambda, \qquad /I.2/$$

$$mT_j'' \frac{dy_j}{dt} = h_j(x_1,\dots,x_\lambda,y_1,\dots,y_\mu,z_1,\dots,z_\nu) \quad j=1,2,\dots,\mu, \qquad /I.3/$$

$$m^{\alpha_\kappa} T_\kappa'' \frac{dz_\kappa}{dt} = g_\kappa(x_1,\dots,x_\lambda,y_1,\dots,y_\mu,z_1,\dots,z_\nu) \quad \kappa=1,2,\dots,\nu. \qquad /I.4/$$

Here $\alpha_\kappa > 0$ and can take on various for different k.

The generalized coordinates y_j and z_κ undergo quik changes during the transient process dy_j/dt and especially dz_κ/dt are large/, while x_i varies slowly. By neglecting small time constants, that is, assuming $m=0$, we obtain the degenerative set of equations:

$$T_i \frac{dx_i}{dt} = f_i(x_1,\dots,x_\lambda,y_1,\dots,y_\mu,z_1,\dots,z_\nu), \qquad /I.5/$$

$$0 = h_j(x_1,\dots,x_\lambda,y_1,\dots,y_\mu,z_1,\dots,z_\nu), \qquad /I.6/$$

$$0 = g_\kappa(x_1,\dots,x_\lambda,y_1,\dots,y_\mu,z_1,\dots,z_\nu). \qquad /I.7/$$

This is tantamount physically to the assumption that "fast" motion is damped out instantaneously on the hypersurfaces of equations /I.6/ and /I.7/, which represent stable equilibrium states for fast motion provided all the roots ρ of the equation /see Ref.8/:

$$\begin{vmatrix} \frac{\partial h_1}{\partial y_1}-T_1'\rho & \dots & \frac{\partial h_1}{\partial y_\mu} & \dots & \frac{\partial h_1}{\partial z_1} & \dots & \frac{\partial h_1}{\partial z_\nu} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial h_\mu}{\partial y_1} & \dots & \frac{\partial h_\mu}{\partial y_\mu}-T_\mu'\rho & \dots & \frac{\partial h_\mu}{\partial z_1} & \dots & \frac{\partial h_\mu}{\partial z_\nu} \\ \frac{\partial g_1}{\partial y_1} & \dots & \frac{\partial g_1}{\partial y_\mu} & \dots & \frac{\partial g_1}{\partial z_1} & \dots & \frac{\partial g_1}{\partial z_\nu} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial g_\nu}{\partial y_1} & \dots & \frac{\partial g_\nu}{\partial y_\mu} & \dots & \frac{\partial g_\nu}{\partial z_1} & \dots & \frac{\partial g_\nu}{\partial z_\nu} \end{vmatrix} = 0$$

have a negative real part for all y_j and z_κ satisfying equations /I.6/ and /I.7/. If this condition is satisfied, "fast" motion in processes starting out from states described by the approximate "slow" motion equations /I.5/ - /I.7/ is negligible. However, the question of processes starting out from other states, that is, the stability prob-

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lem in the large sense of "fast" motion, remains unanswered and requires further study.

The above premises, as well as several other results found in Ref. 7, 8, 9, I0, when appropriately developed.are sufficient, apparently, for simlifying complicated systems of equations in the problem being considered, provided the degrees of freedom of the load motors are neglected, that is if the load is represented by a so called equivalent shunt, or if its static characteristics are used. The problem of constructing resultant dynamic load characteristics for sub-stations can be solved in principle: however, in reality it requires an experimental approach. This is not only because the number of motors /and other loads/ is never known with sufficient accuracy even as concerns their rating distribution, not even mentioning their characteristics.

2. Division of the characteristic equation for small oscillations into low degree equations

The characteristic equation for the system obtained by linearizing /I.I/ has the form /Ref. II/:

$$m^{M_0}F_{N_0}(p)+m^{M_1}F_{N_1}(p)+\dots+m^{M_{n-1}}F_{N_{n-1}}(p)+F_{N_n}(p)=0, \qquad /2.I/$$

where $M_0>M_1>\dots>M_{n-1}>0$ are irreducible rational numbers; $F_{N_k}(p)$ is a polynomial p of degree $N_k\,(N_k>N_{k+1})$ some of whose coefficients may be zero except for the coefficient to the p^{N_k} term. "Slow" motion is characterized by the roots of equation /2.I/ whose approximate values are determined as the roots of the equation $F_{N_n}(p)=0$. The approximate values of the roots corresponding to "fast" motion are found as the roots of the equationsderived from /2.I/ when taking $p=q/m^{\sigma}$, removing negative powers of m and then setting $m=0$. Here σ are rational numbers dependent on M and N; they are determined according to the rules listed in Ref. II. Thus, the characteristic equation is broken down into several independent small-degree equations. This operation may be carried out without expanding the characteristic determinant along its row. The characteristic determinant can also be broken down when /I.I/ contains difference as well as differential equations /for example, when accounting for elastic hydraulic impact/.

This method enabled us to determined where the hypothesis of ideal regulators and the hypothesis of constant mechanical turbine torques are applicable in the study of small oscillations. The first hypothesis is valid for sufficiently large time constants /amounting to tens of seconds/ of derivational headwork structures in hydraulic systems when the turbines have governors with follow-up /Ref. I2,I3/ and rather fast so-called secondary regulation. With slow secondary regulation /Ref. I3/ or transient feedback /Ref. I4,I5/ the hypothesis of ideal regulators results in stronger conditions for stability in the small sense for the steady-state operation of hydraulic headwork systems than they are in actuality. The assumption of constant mechanical turbine torque is valid provided the power system is not operating near the so-called static limit of transfer capacity. Otherwise, turbine regulation has to be accounted for when studying electromechanical processes, and if the time constants of the headwork derivational structures are small /amounting to several seconds/, fructuations of the liquid mass in these structures and in the equalizing reservoirs also have to be included.

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This may occur, for example, in channel installations when a rather short penstock pipe in the body of the dam terminated in an equalizing reservoir before the power house affords headwater derivation. The equations for electromechanical, mechanical and hydraulic processes in this case cannot be separated into independent sets.

We have not as yet dealt with the possibility of breaking down the equations for electromechanical processes themselves into simple sets. This question was studied in Ref. I6 and I7. In complicated cases this process of division is essentially simplified when the method applied is used in conjunction with matrix transformations of the initial linearized equations /Ref. I7/.

Several studies were conducted /Ref. I8, I9, 20, 2I, 22/ dealing with a particular aspect of the problem of simplifying linearized sets, namely, the reduction of similar type generators to a single "equivalent" machine. Their quint essence in the final analysis also amounts to separating the motion of the system; more specifically, "fast" motion is taken to be the power swings of the machines within one station or group of stations, while "slow" motion corresponds to power swings between the machines of different groups of stations. However, the various ingenuous attacks made in these studies are not generalized to a sufficient extent, which would permit them to be applied in solving the problem as a whole.

3. Dividing sets of non-linear equations into simple sets

The loss of stability in the large sense, that is, the transient stability of a power system is associated with short circuits, which lead to instantaneous changes in the scheme and parameters of the system. After some time the faulted line is cleared; this results in a new change in the scheme and parameters. The system must assume a new steady-state under the latter conditions for normal operation to take place. In the above the perturbed values of the generalized coordinates are the values which they take on at the moment when the short circuit is cleared. Therefore, the transient stability problem can be broken down into the following four stages:

I. The set of equations for the short circuit transients are simplified or separated.

2. The same is done to the equations for the transients after clearing the short circuit.

3. Regions of stability in the large sense are constructed for the steady-state operating conditions after clearing the fault.

4. The perturbed values of the generalized coordinates at the instant the fault is cleared are compared with the region of stability in the large sense for the steady-state operating conditions after clearing the fault.

Separation of the sets of equations into those for electromechanical and mechanical processes is dealt with in Refs. 23, 24 and 25. For the sake of illustration a simple case will be considered of two interconnected generators. The purpose of this idealization is not to account for all the factors influencing the process, but show in the simplest form the significant features of the method of solution.

We will assume that the generator emf's are constant throughout the transient; changes in the mechanical torques of the generators will be accounted for, however, assuming that the turbines are equipped with speed governors having

follow up.

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The differential equations for the problem have the following form /in the perturbed state/:

$$T_1\frac{dS_1}{dt}=-\frac{\alpha_1(2\varphi_{10}S_1+S_1^2)}{1-\alpha_1}(1-\gamma_{10}-\mu_1+c_1)-\frac{F_{10}-\alpha_1\varphi_{10}^2}{1-\alpha_1}\mu_1-$$
$$-c_1(2\varphi_{10}S_1+S_1^2)-B_1[\sin(\delta_0+\delta-\alpha)-\sin(\delta_0-\alpha)],$$
$$T_2\frac{dS_2}{dt}=-\frac{\alpha_2(2\varphi_{20}S_2+S_2^2)}{1-\alpha_2}(1-\gamma_{20}-\mu_2+c_2)-\frac{F_{20}-\alpha_2\varphi_{20}^2}{1-\alpha_2}\mu_2- \qquad /3.1/$$
$$-c_2(2\varphi_{20}S_2+S_2^2)+B[\sin(\delta_0+\delta-\alpha)-\sin(\delta_0+\alpha)],$$
$$T_3\frac{d\delta}{dt}=S_1-S_2,\quad T_4\frac{d\mu_1}{dt}=f_1(S_1-\rho_1\mu_1),\quad T_5\frac{d\mu_2}{dt}=f_2(S_2-\rho_2\mu_2).$$

Usually the time constants T_i satisfy the following inequalities:

$$T_3 \ll T_4, T_5 \ll T_1, T_2 \qquad /3.2/$$

/in real conditions $T_3 = 0.003I8$; T_4, T_5 amount to tenths or whole seconds and T_1, T_2 to tenths of a second/. In view of inequalities /3.2/ some of the differential equations can be considered as containing a small parameter to its derivatives. Therefore, the entire phase space for the set can be broken down into regions for faster and slower motion. If we do not as yet take note of the degree of the right hand side in the equation, the fastest variable will be δ, after which come μ_1 and μ_2 with the slowest variables being S_1 and S_2. This prompts us to use the theory of discontinuous oscillations /Ref. 5, 6/ for approximately analysing the dynamical system being considered. The approximate equation for "fast" motion take the form:

$$T_3\frac{d\delta}{dt}=S_1-S_2,\quad S_1=const,\quad S_2=const,$$

whence $\delta=(S_1-S_2)\frac{t}{T_3}+const$. Substituting this result into the first two equations /3.I/, we obtain equations which are typical for a dynamical system having a quickly rotating phase /Ref. 27/, provided the difference S_1-S_2 is not very small. The peculiarity of this case is that the "fast" motion δ can damp out only due to a slow decrease in the difference between the slowest variables S_1-S_2 themselves. Moreover, this difference may not only tend to zero, but also may change its sign. This circumstance requires that the analysis be started not by studying the behaviour of δ, but by examining μ_1 and μ_2 whose motion damps out much faster and independent of δ.

Let the functions of f_1 and f_2 given by the characteristics of the servomotors satisfy the conditions

$$f_i(\delta_i)\delta_i>0,\quad f_i(0)=0 \qquad (i=1,2)$$

and be lenearized. In the region of fast motion μ_1 and μ_2 the differential equations for μ_i may be approximately written as follows:

$$T_4\frac{d\mu_1}{dt}=f_1(S_1-\rho_1\mu_1),\quad S_1=const,$$
$$T_5\frac{d\mu_2}{dt}=f_2(S_2-\rho_2\mu_2),\quad S_2=const.$$

The points on the phase spaces $f_i=0$ are equilibrium states for the approximate equations of fast motion /Ref.5/. These equilibrium states are stable since at all points of the substances $f_i=0$, $\frac{\partial f_i}{\partial\mu_i}<0$ /we take $\rho_i>0$/. Therefore, if at the initial moment the moving point is not near the subspaces $f_i=0$, it must reach these subspaces when travelling along the trajectories for "fast" motion. After this, it will travel slowly inside of them. In view of the restrictions imposed on the function f_i the motion in the subspaces $f_i=0$ will be described by the first three equations in /3.I/ with $\mu_i=\frac{S_i}{\rho_i}$. As a result, if the initial set of differential equations was of the fifth order, after eliminating the "fast" motion of μ_1 and μ_2, it is reduced to the third order.

The equilibrium state $S_1=S_2=\delta=0$ is stable in the small sense if the corresponding Routh-Hurowitz conditions are satisfied. The sufficient conditions for stability in the large sense can be determined by means of one of the alternatives in Lapunov's direct method. If the initial deflections S_1, S_2 and δ lie in the region constructed in this way they will damp out with time. Therefore, the perturbations

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α_1 and α_2 will also damp out.

In Ref. 26 it is also assumed that a voltage regulator is installed on one the equivalent generators. The initial set of differential equations is of the eighth order in this case and of fourth order when simplified.

This approach may give effective results in much more complicated cases, also, whenever the time constants can be divided into a large number of groups as to their order of magnitude. It is only necessary that the corresponding equilibrium states for fast motion satisfy the stability criteria.

Whether or not the perturbed values of the generalized coordinates acquired at the instant the fault is cleared lie inside or outside the regions /which are constructed as described above/ can be ascertained by numerical integration of the transient equations during the fault. This is proposed, for example, in Ref. 27. However, this problem can apparently be solved in a better way with recourse to theory of stability over a definite time interval.

The method of separating motion into fast and slow also enables us to ground the hypothesis of the ideal regulator in calculating the stability of a large system with headworks and equalizing reservoirs.

In this connection Ref. 28 considers the operation of an isolated hydroelectric installation with headwater derivation and a simple cylindrical equalizing reservoir, which contains one water wheel unit having a constant electrical resistive torque.

The set of differential equations for the problem with an inertialess regulator having follow-up is of the fifth order and contains five time constants: T_1 for the reservoir, T_2 for the derivation, T_3 for the water-wheel unit, T_y for the damping and T_5 for the servomotor.

If they satisfy the following inequality

$$T_4, T_5 \ll T_3 \ll T_1, T_2$$

the hypothesis of the ideal regulator is completely valid.

If the time constants have the following relationship /which is more frequently encountered in practise/,

$$T_y, T_5 \ll T_1, T_3 \ll T_2$$

the hypothesis of the ideal regulator will be justified practically, provided the regulation system, which in itself is stable, possesses a small /theoretically an infinitely small/ slope in its characteristic /astatic regulation/.

What was said in § 2, 3 explains the conditions for dividing the initial set of differential equations into simpler sets. This division of the equations corresponds to separating the sum total power system transient into components of a different physical nature. Thus we obtain and base the conditions permitting us to consider separately of hydraulic processes in the headwater section of a hydroelectric station, the stability of electromechanical swings of the rotors in the units, and the stability of turbine regulation.

Below are given the results of an over-all examination of electromechanical and hydraulic processes in certain simple dynamical systems, which constitute the components of a divided power system. The initiative for investigating these dynamical systems by studying their phase space belongs to Academician A.A.Andronov.

4. Electromechanical processes

While the stability of steady-state operation of hydraulic systems with headworks and of turbine regulation was investigated from the outstart by small sense stability methods, for a long time the criterian of appraising the sign of only the

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brium state after several revolutions about the phase cylinder / .

Let us consider in more detail a salient-pole machine. When the second harmonic in the torque and the change in the damping as a function of the angle are accounted for, the motion of the rotor in a salient-pole synchronous motor can be determined by investigating the following equations /Ref. 34/,

$$\frac{dS}{dt} = D - \sin\delta - z\sin 2\delta - \kappa(1 - b\cos 2\delta)S,$$
$$\frac{d\delta}{dt} = S \qquad /4.2/$$

which are equivalent to the equation

$$\frac{dS}{dt} = \frac{D - \sin\delta - z\sin 2\delta}{S} - \kappa(1 - b\cos 2\delta) \qquad /4.3/$$

These equations were examined in Ref. 32 and 33. We will briefly relate the results of these studies, taking into account that parameter b for a salient-pole synchronous motor does exceed unity. In the more general case when $f(\delta)$ can change in sign $(b>1)$ the results in Ref. 35 are applicable.

If the function $f(\delta)$ can change in sign, self oscillations of the first kind as well as other qualitative characteristics of the system appear.

With $\kappa=0$ the conservative case of equation /4.2/ holds true and /4.3/ has the following general solution:

$$S^2 = 2D\delta + 2\cos\delta + z\cos 2\delta + C.$$

The equilibrium states of the system are either centers or saddle-points.

With $\kappa \neq 0$ the coordinates of the equilibrium states of equations /4.2/ are independent of κ. They are determined as in the case $\kappa = 0$ by the equalities

$$D - \sin\delta - z\sin 2\delta - \kappa(1 - b\cos 2\delta)S = 0, \quad S = 0 \qquad /4.4/$$

The space of parameters κ, z, D, b is divided into regions I, II, III /fig. I/ at whose points equations /4.2/ have 0,2 and 4 equilitrium states, respectively, with $\delta = \delta_i$ and $S = 0$.

These regions are defined by the inequalities:

I $\quad D > D_1(z) = \frac{\sqrt{32z^2 - 2 + 2\sqrt{1+32z^2}}\,[3 + \sqrt{1+32z^2}]}{32z}$,

II $\quad D_1 \geq D > D_2(z) = \frac{\sqrt{32z^2 - 2 - 2\sqrt{1+32z^2}}\,[3 + \sqrt{1+32z^2}]}{32z}$,

III $\quad D \leq D_2(z)$.

The character of the equilibrium states is determined by the roots of the corresponding characteristic equation

$$\lambda_{1,2} = -\frac{\kappa}{2}(1 - b\cos 2\delta_i) \pm \sqrt{\frac{\kappa^2}{4}(1 - b\cos 2\delta_i)^2 - 2z\cos 2\delta_i - \cos\delta_i}.$$

The equilibrium states $(\delta_1, 0)$ and $(\delta_3, 0)$ for which

$$\alpha = -(\cos\delta_i + 2z\cos 2\delta_i) > 0$$

are saddle points; the equilibrium states $(\delta_2, 0)$ and $(\delta_4, 0)$ for which $\alpha < 0$ with $\kappa = 0$ are centers and with $\kappa > 0$ singular points or focal points.

With $D < D_2(z)$ two stable equilibrium states [illegible] $(\delta_2, 0)$ and $(\delta_4, 0)$.

Possible bifurcations of the equilibrium states [illegible] focal points $(\delta_2, 0)$ [illegible] complex [illegible] of the inequality [illegible] $0 < b < 1$ they do not change [illegible] bifurcations of the equilibrium states [illegible] to the [illegible] of equilibrium states of saddle point $(\delta_1, 0)$ [illegible] singular point $(\delta_2, 0)$ for $D > D_1(z)$ and [illegible] saddle [illegible] and singular point $(\delta_4, 0)$ [illegible] for $D > D_1(z)$ and for $D > D_2(z)$ complex equilibrium states [illegible] $(\delta_{34}, 0)$ disappear.

Separatrixes of saddle points. The [illegible] separatrixes at the saddle points are determined [illegible] /4.5/ with $i = 1, 3$. As the [illegible] gents to the separatrixes [illegible] clockwise direction [illegible]

$$\partial\lambda_{1,2}/\partial\kappa < 0$$

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The difference in the direction of the vector fields in equation /4.3/ with $\kappa=\kappa_2$ and $\kappa=\kappa_1(\kappa_2>\kappa_1)$ is negative.

$$R=\left(\frac{dS}{d\delta}\right)_{\kappa=\kappa_2}-\left(\frac{dS}{d\delta}\right)_{\kappa=\kappa_1}=-(\kappa_2-\kappa_1)(1-\beta\cos 2\delta)<0 . \qquad /4.7/$$

By comparing the vector fields of equation /4.3/ for $\kappa\neq 0$ and $\kappa=0$, and inequality /4.6/ we obtain the location of the separatrixes of saddle points for sufficiently small values of κ other than zero. In fig. 2 the separatrixes of the conservative case are indicated by dashed lines for points of region III $(\mathcal{D}<\mathcal{D}_2(z))$ /fig. I/ and by solid lines for sufficiently small values of κ other than zero. For non-zero values of κ that are not small the location of the saddle point separatrixes may be established by evaluating parts of the separatrixes.

Construction of the curves without contact[1st] the parts of the separatrixes results in the following

I. Under the conditions:

$$\mathcal{D}_2(z)<\mathcal{D}\le\mathcal{D}_1(z), \qquad /4.8/$$

$$\kappa\le\frac{1}{2\pi(1+\beta)}\left[\sqrt{4\pi\mathcal{D}+\rho(\delta_2)-\rho(\delta_1)}-\sqrt{\rho(\delta_2)-\rho(\delta_1)}\right]$$

or

$$0<\mathcal{D}<\mathcal{D}_2(z), \quad \kappa\le m,$$

$$\rho(\delta)=2\mathcal{D}\delta+2\cos\delta+2\cos 2\delta,$$

where m is the smaller of the quantities

$$m_1=\frac{\sqrt{\rho(\delta_2)-\rho(\delta_3)}-\sqrt{\rho(\delta_2)-\rho(\delta_1)}}{(1+\beta)(\delta_1-\delta_3)}, \quad m_2=\frac{\sqrt{\rho(\delta_4)-\rho(\delta_1+2\pi)}-\sqrt{\rho(\delta_4)-\rho(\delta_3)}}{(1+\beta)(\delta_3-\delta_1+2\pi)}$$

the saddle point separatrixes are located as in the case for small κ. Under the conditions of /4.8/ the saddle point separatrixes having a positive slope in the half plane $S>0$ satisfy the inequality

$$S(\delta+2\pi)>S(\delta). \qquad /4.9/$$

2. Under the conditions

$$0\le\mathcal{D}\le\mathcal{D}_1(z), \quad \kappa\ge\frac{2\sqrt{\pi\mathcal{D}}}{(1-\beta)(\delta_1-\delta_2)} \qquad /4.10/$$

inequality /4.9/ is not satisfied for any of the saddle point separatrixes.

In accordance with /4.6/ and /4.7/ for $0\le\mathcal{D}\le\mathcal{D}_1(z)$, when transcending from the values of κ satisfying the conditions of /4.8/ to the values of κ satisfying the conditions of /4.10/, the equations /4.2/ must pass through the bifurcational state /and do so only once when $\kappa=\kappa_0(\mathcal{D},\beta,z)$/ at which the separatrixes coincide, thereby forming a closed contour of the II kind /including the cylinder/.

Figure 3 shows how the location of the saddle point separatrixes varies with the parameter κ under the conditions $\mathcal{D}_2\le\mathcal{D}\le\mathcal{D}_1$.

The "saddle points" quantities /Ref.5/ σ_i for saddle points δ_1 and δ_3 are negative: $\sigma_i=\mathcal{D}_\delta'(\delta_i,2)+\mathcal{Q}_\delta'(\delta_i,0)=-\kappa(1-\beta\cos 2\delta_i)<0$. Therefore, the closed contour of the II kind composed of the saddle point separatrixes is stable and only a single stable ultimate cycle of the II kind may appear from it.

Ultimate cycles: In order to solve the problem concerning the existence of ultimate cycles the Bendikson-Duliac criterion on the cylinder is applied /Ref. 5/. Since the Bendikson function for the equation /4.2/ $B=-\kappa(1-\beta\cos 2\delta)$ with $\kappa>0$ and $\beta\le 1$ does not change its sign over the entire phase plane, the equations /4.2/ with $\kappa>0$ do not have ultimate cycles of the I^{st} kind, they cannot have more than one cycle of the IInd kind, they cannot have a semi-stable ultimate cycle of the IInd kind, they do not have a cycle of the IInd kind in the half-plane $S<0$, they do or do not have a cycle of the IInd kind in the half plane $S>0$ depending on the located of the saddle point separatrixes.

With $\kappa>0$ and $\mathcal{D}>\mathcal{D}_1$ for the trajectory passing through the point $S=0$ and $\delta=\frac{\pi}{2}$ inequality /4.9/ is satisfied; moreo-

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ver, the inequality $S(\delta+2\pi)<S(\delta)$ is satisfied for the trajectories passing in the plane $S>\mu$ /when μ is the maximum ordinate of the isoclinic line /4.4/. Therefore, with $\kappa>0$ and $\mathcal{D}>\mathcal{D}_1$ the equations /4.2/ have a IInd order cycle.

Thus, equations /4.2/ have only one stable ultimate cycle of the second kind with $\mathcal{D}>\mathcal{D}_1(\varepsilon)$ and with $\kappa\leq\kappa_0$, $0\leq\mathcal{D}\leq\mathcal{D}_1$. They do not have cycles for $\kappa>\kappa_0$, $0\leq\mathcal{D}\leq\mathcal{D}_1$. The values for the quantity κ_0 can be made more precise by using numerical methods of calculation /Ref. 36/.

Division of the space of the parameters: The space of the parameters $\kappa, \varepsilon, \mathcal{D}, \beta$ is divided into the following 5 regions corresponding to different rough qualitative structures.

I. $\mathcal{D}>\mathcal{D}_1(\varepsilon)$ - no equilibrium states but a stable ultimate cycle of the IInd kind exists.

2. $\kappa\leq\kappa_0$, $\mathcal{D}_2\leq\mathcal{D}\leq\mathcal{D}_1$ - one stable equilibrium state, one saddle point and one cycle of the IInd kind exist.

3. $\kappa\leq\kappa_0$, $0\leq\mathcal{D}\leq\mathcal{D}_2$ - two stable equilibrium states, two saddle points, one cycle of the IInd kind exist.

4. $\kappa>\kappa_0$, $\mathcal{D}_2\leq\mathcal{D}\leq\mathcal{D}_1$ - one stable equilibrium state and one saddle point exist, but there are no cycles.

5. $\kappa>\kappa_0$, $0\leq\mathcal{D}\leq\mathcal{D}_2$ - two stable equilibrium states and two saddle points exist, but there are no cycles.

NOTE that in these studies the damping torque was taken to be linearly dependent on the slip. Later Mamikonyantz /Ref. 37, 38/ and Sirotnikov /Ref. 39/ investigated the relationships determining synchronous torques. It turned out that due to the magnetic inertia in the machine circuits the induction torque for a varying slip may markedly differ in certain cases from the value it would have had at the given slip if the latter was constant. In these cases the torque also depends on the derivatives of the slip, that is to say, the system will have more than two generalized coordinates, and we are confronted with the necessity of using a phase space having three and more dimensions. In 1957-1958 the direct method of Liapunov was employed for studying electromechanical processes in synchronous machines by Janko-Trinitski /Ref.27/ and Andreyiuk /Ref.40,41/ whose work was quite similar, but apparently done independently.

The first integral of the transient equations for the conservative case was taken to be the Liapunov function V. For the case of two interconnected machines it is

$$V=A_1 S_1^2+A_2 S_2^2+f(\delta_1,\delta_2).$$

In unconservative cases sign-forming quadratic forms of certain variables are added to it /change in the e.m.f.'s the currents, etc./. Although the region bounded by the surface $V=$ const inside of which $V\geq 0$ and $\frac{dV}{dt}\leq 0$ /which we will conditionally call the region of stability/ turns out to be rather wide when Liapunov's function is constructed in this way, it is nevertheless not so wide to be able to judge whether or not the machine can fall back step after running at a slip.

Unfortunately, other approaches in constructing Liapunov's function have the same drawback, viz the method proposed by Aizerman used in Ref. 24, and the method proposed by one of the authors /Ref. 42/. Therefore, it is very important for practical purposes to find a way of constructing Liapunov's functions giving less degenerative conditions for stability in the large sense.

All that was said above reemphasizes the necessity of working out methods for the overall qualitative investigation of systems having different numbers of degrees of freedom similar to the way we can do it now for systems with a

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phase surface.

5. Hydraulic processes

a. A hydroelectric station with cylinrical equalizing reservoir /fig. 4/

In this and all subsequent cases it is assumed that the turbines have ideal regulators. Moreover, the inertia of the water mass in the equalizing reservoir and in the penstock pipe is not accounted for.

The discrete idealization of hydraulic processes is grounded in Ref. 43, where the stability of a hydraulic system in the small sense is investigated when accounting for wave phenomena in the headwater derivation with the turbine efficiency η constant and also dependent on the head H. The characteristic equation for the problem is as follows:

$$-\alpha\rho+\beta+(\varepsilon\rho-\delta)e^{\rho\tau}=0,$$

where $\alpha, \beta, \varepsilon, \delta, \tau$ - are system parameters

By means of the method of D-division we obtain a generalization of Tom's known stability criteria pertinent to the case of discrete idealization. Figure 5 presents a diagram for the stabile region of the system as a function of parameters F and L. It was constructed for $\eta = const$ and given values of the other system parameters. From this diagram we see that wave processes in the derivation can be neglected. This is done in all cases below.

The problem of stability in the large sense is solved for the case of η being a linear function of H, and also $\eta = const$. In the latter case the equations for the problem in dimensionless form are as follows /Ref. 43/:

$$\frac{dy}{dx}=-\varepsilon(x+y^2), \quad \frac{dx}{dt}=y-\frac{1-\beta}{1+\beta x}, \quad y>0, \qquad /5.1/$$

$$\frac{dy}{dx}=-\varepsilon(x-y^2), \quad \frac{dx}{dt}=y-\frac{1-\beta}{1+\beta x}, \quad y<0, \qquad /5.2/$$

where in addition to the designations in fig. 4 we

have $x=\frac{z}{h_o}$, $y=\frac{Q}{Q_o}$, $t=\frac{Q_o\tau}{Fh_o}$, $\beta=\frac{h_o}{H_o}$, $\varepsilon=\frac{gF^2h_o^2}{LQ^2}$.

Q_o being the hydro station discharge during steady-state operation, h_o the loss in head in the derivation for a discharge of Q_o and τ stating for time.

By linearizing these equations we obtain Tom's stability conditions in the form

$$\beta<\frac{1}{3}, \quad \varepsilon>\frac{\beta}{2(1-\beta)}. \qquad /5.3/$$

In Ref. 43 and also in the works cited below, stable states and hydraulic transients are investigated by studying the phase surface of the dynamical system.

The differential equations

$$\frac{dy}{dx}=\frac{\varepsilon(x+y^2)}{\frac{1-\beta}{1+\beta x}-y}, \qquad /5.4/$$

which are equivalent to equations /5.1/ and /5.2/ are analysed using the qualitative theory of differential equations on a surface. Equations /5.1/ for $0<\beta<\frac{1}{3}, \varepsilon>0, 1+\beta x>0$ have two equilibrium states O_1 /a focal or singular point, which is stable or unstable/ and O_2 /a saddle point/. The behaviour of the saddle point separatrix is shown on fig. 6. Two lemmas are proved with the aid of Duliac's criterion and curves without contact. They determine the values of the parameters, for which the system beforhand is known not to have ultimate cycles /the system is stable in the large sense for region III on fig. II, and unstable for region I/.

The boundary of the stability region is dangerous. On the boundary of this region the only unstable ultimate cycle /whose interior will be a large sense stability region for the given equilibrium state/ appears from the focal point, which changes stability relationships with increasing ε. For a certain value of ε this unstable ultimate cycle sticks to the separatrix loop running from saddle

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to saddle point
point. The division of the phase plane into trajectories and the division of the space of the parameters is shown in fig. 7 to II. In fig. II the curve $\mathcal{R} = \beta_2/(1-\beta_1)-\varepsilon = 0$ corresponds to the boundary for stability in the small sense, while the dashed curve $\mathcal{R}_1 = 0$ /obtained by means of graphical integration/ corresponds to the case of separatrixes running from saddle point to saddle point.

For the case of reverse flow $(y < 0)$ it is proved starting out from equations /5.2/ that ultimate cycles in the region $1+\beta x > 0$ of the phase plane cannot exist.

Finally a "pieced together" system is considered when in the process of motion $y \gtrless 0$. We see that such a system either has an ultimate cycle /greater or equal to the ultimate cycle of the system for $y > 0$/ or is stable in the large sense without ultimate cycles, or is unstable. In the plane of the coefficients for the "pieced together" system the region of values of the parameters for which the cycle exists decreases, as does the region of parameters corresponding to large sense stability without ultimate cycles.

When η is a linear function of the head, the qualitative pictures are similar to the those with $\eta = const$.

A quantitative evaluation of large sense stability /see Ref. 44/ for cylindrical and so-called pneumatic reservoirs, which was carried out by means of Lyapunov's direct method, shows that if the left hand side given by [illegible] exceeds the right hand side given by [illegible], the large sense stability region covers the entire region of practically possible initial deviations from steady-state operation.

[illegible] reservoir [illegible] the equations for the problem are written as follows:

$$\frac{dy}{dt} = -\varepsilon\left(x + y|y| + \alpha\frac{dx}{dt}\left|\frac{dx}{dt}\right|\right),$$
$$\frac{dx}{dt} = y - \frac{1-\beta}{1+\beta x+\beta\alpha\frac{dx}{dt}\left|\frac{dx}{dt}\right|}, \qquad /5.5/$$

where $\alpha = \frac{h_0}{h_0}$, h_0 being the loss in head in the resistance of the reservoir when a discharge Q_0 passes through it. Only the cases $1+\beta x+\beta\alpha\frac{dx}{dt}\left|\frac{dx}{dt}\right| > 0$ can occur in reality. The conditions for small sense stability are the same as Tom's conditions for a simple cylindrical reservoir. Let us introduce a new variable $u = dx/dt$. The problem amounts to investigating the non-linear differential equation:

$$\frac{du}{dx} = \frac{(1-\beta)\beta u - \varepsilon\{(x+\alpha u^2)(1+\beta x+\beta\alpha u^2)+[u(1+\beta x+\beta\alpha u^2)+(1-\beta)]^2\}}{u[(1+\beta x+\beta\alpha u^2)+2\alpha\beta(1-\beta)u]} \qquad /5.6/$$

The signs to be taken are evident from equation /5.5/.

Equation /5.6/ has two singularities in the region of the phase plane of interest. They lie on the piecing line: $O_1(x_1,u_1)$ which is a stable or unstable singular or focal point, and $O_2(x_2,u_2)$, which is a saddle point. In solving the problem it is necessary to determine the qualitative structure for division of the phase plane into trajectories near the singular point O_1 as a function of the system parameters. Considering that the point O_1 lies on the piecing line U=0, the usual algorithm for calculating the first Lyapunov quantity /see Ref. 46/ determing the character of the stability region is not applicable here. However, by generalizing Bowtin's method for determining the nature of stability region boundaries for dynamical systems described by "pieced together" differential equations /Ref.47/, it is shown that in the given case the stability region boundary will be dangerous.

The following new variables are introduced for analy-

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sing the qualitative structure with the division of the phase plane into trajectories in the large sense, and not only in the vicinity of the equilibrium state:

$$x_1 = \frac{1+\beta x}{(1-\beta)^{2/3}(\alpha\beta)^{1/3}}, \quad u_1 = \frac{(\alpha\beta)^{1/3}}{(1-\beta)^{1/3}} u$$

As result we obtain the differential equation

$$\frac{du_1}{dx_1} = \frac{u_1 - A\{(x_1 - B \pm u_1^2)(x_1 \pm u_1^2)^2 \pm C[u_1(x_1 \pm u_1^2)+1]^2\}}{u_1[(x_1 \pm u_1^2)^2 \mp 2u_1]}, \qquad /5.7/$$

containing three parameters

$$A = \frac{\varepsilon\alpha}{\beta}(\alpha\beta)^{1/3}(1-\beta)^{2/3}, \quad B = \frac{1}{(\alpha\beta)^{1/3}(1-\beta)^{2/3}}, \quad C = \frac{1}{\alpha}.$$

Since the right hand side of /5.7/ contains the ratio of the "pieced" polynomial of the sixth degree to the "pieced" polynomial of the fifth degree, it is difficult to directly determine the number of singularities that exist. Nomograms are used for this purpose /for $u_1 > 0$, $u_1 < 0$ and $u_1 = 0$ /. It turns out that there are a maximum of seven singularities with only two of them corresponding to the equilibrium states / O_1 and O_2 /.

In order to determine the structure of rough cases determining system stability, the simplest bifurcational pictures are constructed logically, which appear with an increase of the parameter $A(\alpha)$ for set values of B and C. More precise cases were not considered. The selection of possible logical cases is facilitated by the fact that the "saddle point" quantity $P_x' + Q_y'$ /Ref.5/ determining in the bifurcational case the stability of the loop of the separatrix running from saddle point to saddle point is always negative for the singularity at O_2. Such a logical examination showed that in the vicinity of equilibrium state O_1 large sense instability can occur, as well as one ultimate cycle /which is stable or unstable/, two ultimate cycles /with the unstable one inside the stable one/ and finally large sense stability.

c. A differential reservoir /Ref. 48, 49/: Let us restrict ourselves to the case when the water level in the central riser lies all the time between the openings connecting the riser to the tank / ω is the area of the openings: α is the discharge factor/ and the crest of the riser weir /fig. 12/. In this case the equations are as follows:

$$\frac{dy}{dt} = -\varepsilon(x_2 + y/\gamma), \quad \frac{dx_1}{dt} = \kappa\sqrt{|x_2 - x_1|}\,\mathrm{sign}(x_2 - x_1),$$

$$\frac{dx_2}{dt} = y - \frac{1-\beta}{1+\beta x_2} - \alpha\kappa\sqrt{|x_2 - x_1|}\,\mathrm{sign}(x_2 - x_1),$$

where

$$\varepsilon = \frac{gf F_2^2 h_0^2}{L Q_0^2}, \quad \alpha = \frac{F_1'}{F_2'}, \quad \kappa = \frac{\alpha\omega\sqrt{2gh_0}}{Q_0}$$

The case $\alpha\kappa\omega$ corresponds to a reservoir with resistance, while the case $\alpha = 0$, $\omega = 0$ ($\kappa = 0$) corresponds to an ordinary cylindrical reservoir.

In the region of direct flow ($y > 0$) equations /5.8/ have two singularities $O_1(y = -x_1 = -x_2 = 1)$, $O_2\left(y = \frac{\beta + \sqrt{3\beta(4-3\beta)}}{2\beta}, x_1 = x_2 = \frac{2-\beta-\sqrt{3\beta(4-3\beta)}}{2\beta}\right)$ which correspond to steady-state operating conditions in the headwater section of the hydroelectric station.

Assuming $y = 1 + \eta$, $x_1 = 1 + \zeta$, $x_2 = 1 + \xi$ we will investigate stability. Let us start with the main operating conditions /point O_1/. Equations /5.8/ for the perturbed case are as follows ($\sigma = \xi - \zeta$, $c = \beta/(1-\beta)$):

$$\frac{d\eta}{dt} = -\varepsilon(2\eta + \sigma + \xi + \eta^2),$$

$$\frac{d\xi}{dt} = \kappa\sqrt{|\sigma|}\,\mathrm{sign}\,\sigma,$$

$$\frac{d\sigma}{dt} = \eta + \frac{c(\sigma+\xi)}{1+c(\sigma+\xi)} - (1+\alpha)\kappa\sqrt{|\sigma|}\,\mathrm{sign}\,\sigma.$$

On the right hand sides of the differential equations /5.9/ we find a non-linearizable non-linearity $\kappa\sqrt{|\sigma|}\,\mathrm{sign}\,\sigma$. Therefore, the usual method for appraising the stability of

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a singularity by means of the roots to the characteristic equation is not applicable here. Introducing $\sigma = \sigma_1/\sigma_1/$ and also a new time scale according to the relationship $d\tau = \frac{dt}{2/\sigma_1(t)/}$ we can write /5.9/ as follows $(x \equiv \xi, y \equiv \eta, x_1 \equiv \sigma_1)$

$$\frac{dx}{d\tau} = 2k x_1 x_1/ ,$$

$$\frac{dy}{d\tau} = -2\varepsilon/x_1/(2y + x + y^2 + x_1/x_1/) ,$$

$$\frac{dx_1}{d\tau} = -k(1+\alpha)x_1 + cx + y + cx_1/x_1/ - c^2x^2 - 2c^2xx_1/x_1/ + c^3x^3. \qquad /5.10/$$

The characteristic equation for the set /5.10/ has two zero and one negative real root. On the basis of Malkin's theorem /Ref. 50/ the nature of unperturbed motion in the set /5.10/ is determined entirely by the behaviour of the solution to the "contracted" set consisting of two equations:

$$\frac{dx}{dt} = k\left[\frac{cx+y}{k(1+\alpha)} + X(x,y)\right],$$

$$\frac{dy}{dt} = -\varepsilon\left[x + 2y + Y(x,y)\right], \qquad /5.11/$$

where $X(x,y)$, $Y(x,y)$ contain terms of the second and higher degree with respect to x, y. Moreover, since set /5.10/ is pieced together, $X(x,y)$ and $Y(x,y)$ will de different for $x_1 > 0$ and $x_1 < 0$.

The unperturbed motion $x = y = 0$ in equations /5.11/ is stable for $c < \frac{1}{2}$ and $\varepsilon > \frac{c}{2(1+\alpha)}$ and unstable for $c < \frac{1}{2}$ and $\varepsilon < \frac{c}{2(1+\alpha)}$. The stability region boundary $\varepsilon = \frac{c}{2(1+\alpha)}$ is dangerous. On the basis of Malkin's theorem mentioned above unperturbed motion in equation /5.10/ will also be stable for $c < \frac{1}{2}$ and $\varepsilon > \frac{c}{2(1+\alpha)}$, and unstable for $c < \frac{1}{2}$ and $\varepsilon < \frac{c}{2(1+\alpha)}$. The stability conditions $\varepsilon > \frac{c}{2(1+\alpha)}$ and $c < \frac{1}{2}$ are the same as those by Tom.

Liapunov's direct method is used for determining the sufficient boundaries for the regions of stability and unstability in the large sense of the main steady-state operating conditions. By means of two functions, which are [illegible] forms, it is once more proved that the conditions $\varepsilon > \frac{c}{2(1+\alpha)}$ and $c < \frac{1}{2}$ are necessary and sufficient for stability of main steady-state operating and, moreover, the required regions in phase space are selected.

The perturbed equations for the second steady-state operating conditions are as follows /y_0, x_{10}, x_{20} being the coordinates of the singularity C_2/:

$$\frac{du}{dt} = -\varepsilon(2y_0 u + \sigma + w + u^2),$$

$$\frac{dw}{dt} = k\sqrt{|\sigma|}\,\mathrm{sign}\,\sigma,$$

$$\frac{d\sigma}{dt} = u + \frac{1-\beta}{1+\beta x_{20}} - \frac{\beta_1(\sigma+w)}{1+\beta_1(\sigma+w)} - k(1+\alpha)\sqrt{|\sigma|}\,\mathrm{sign}\,\sigma, \qquad /5.12/$$

where $\sigma = v - w$ and $\beta_1 = \frac{\beta}{1+\beta x_{20}}$.

Transformations similar to those above show that the unperturbed motion $u = v = w = 0$ is unstable in the small sense with $c < \frac{1}{2}$ and any values of the other parameters.

Using Liapunov's direct method it is shown that these conditions are sufficient for unstability of the second steady-state operating condition in the large [illegible].

Then studying back flow $y < 0$, it is expedient to employ a system of surface without contact:

$$V(y, x_1, x_2) = \frac{\beta_2}{2\varepsilon}y^2 + \frac{1}{2}(1+\beta x_1)^2 + \frac{\alpha}{2}(1+\beta x_2)^2 = C \qquad /5.13/$$

which subtends to a point for $C = 0$ having the coordinates $y = 0, x_1 = x_2 = \frac{1}{\beta}$. The moving point in the region moves so that it crosses over from the surface with a larger C to the surface with a smaller C and falls either into the region $y = 0$ or leaves the region for which [illegible] are given $(1+\beta x_1 > 0, 1+\beta x_2 > 0)$.

[illegible], back flow is unstable. Surfaces [illegible] C also enable us to show that "infinity" is [illegible] single out in the phase space the region [illegible]

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unstability. If the moving point should fall into this latter region, it will leave the region for which equations /5.8/ are given no matter what the values of the parameters are. The existance of an "absolutely " unstable region permits us to appraise the largest possible region for stability of main steady-state operation when Tom's conditions are satisfied.

d. A hydroelectric station with n reservoirs in series each /except for the last/ having a lumped resistance fig. 13/: The differential equations for the perturbed state describing processes in such a system with direct flow can be written as follows in dimensionless form after uniformization

$$\dot{U}_1 = \varepsilon B_1 \left[U_{n+1} + 2C_1 Y_{10} U_1 + C_1 U_1^2 \pm K_1' (C_{n+1} U_1 - C_{n+2} U_2)^2 \right],$$

$$\dot{U}_i = \varepsilon B_i \left[U_{n+i} - U_{n+i-1} + 2C_i Y_{i0} U_i + C_i U_i^2 \pm K_i' (C_{n+i} U_i - C_{n+i+1} U_{i+1})^2 \mp K_{i-1}' (C_{n+i-1} U_{i-1} - C_{n+i} U_i)^2 \right] \quad i = 2, 3, \ldots, n,$$

$$\dot{U}_i = B_i \left(C_i U_{i-n} - C_{i+1} U_{i-n+1} \right) \quad i = n+1, n+2, \ldots, 2n-1$$

$$U_{2n} = B_{2n} \left(C_{2n} U_n + \frac{C U_{2n}}{1 + C U_{2n}} \right), \qquad /5.14/$$

where $\varepsilon = \frac{g h_0^2 F_1}{L_1 f_1 U_0^2}$, $B_i = \frac{L_1'}{L_i}$, $B_{n+i} = \frac{F_1'}{F_i}$, $C_i = \frac{h_i}{h_2}$, $C_{n+i} = \frac{f_i}{f_1}$,

$C = \frac{h_0}{H + h_0 x_{n0}}$, $Y_{i0} = \frac{1}{C_{n+i}}$, $x_{i0} = -\sum_{j=1}^{n} C_j Y_{j0}^2$ $(i = 1, 2, \ldots, n)$, $K_n' = 0$,

K_i depends on the resistance of the ith diaphragm.

The top sign in the equations is taken for $C_{n+i} U_i - C_{n+i+1} U_{i+1} > 0$ and the bottom sign for $C_{n+i} U_i - C_{n+i+1} U_{i+1} < 0$.

The small sense stability conditions for unperturbed motion may be found much more simply in this case because the characteristic equation for /5.14/ can be expressed as a continuous fraction, which easily lends itself to the method of $\mathcal{D}$- division /Ref. 51/.

Note that the problem at hand was studied by Vogt /Ref. 52/ with $n = 2$ when the section of the first reservoir was assumed small. In Ref. 53 this case with $n = 2$ was considered when the sections of both reservoirs were accounted for.

In order to single out the large sense stability region for steady-state operation /Ref. 54/, let us take Liapunov's function as a quadratic form V of variables U_i. When the n restrictions imposed on the constant A_{ij} by Silvester's theorem are satisfied, the function V will have a positive sign. Selecting the constants A_{ij} accordingly /they must satisfy $n(2n-1)$ relationships/, we can express the derivative $\frac{dV}{dt}$ as follows:

$$\frac{dV}{dt} = -U_1^2 \left\{ \varepsilon C_1 B_1 \sum_{j=1}^{2n} (A_{1j} + A_{j1}) U_j + 4\varepsilon B_1 C_1 Y_{10} A_{11} - B_{n+1} C_{n+1} (A_{n+1,1} + A_{1,n+1}) \right\} -$$

$$- \sum_{i=2}^{n} U_i^2 \left\{ \varepsilon C_i B_i \sum_{j=1}^{2n} (A_{ij} + A_{ji}) U_j + 4\varepsilon B_i C_i Y_{i0} A_{ii} - B_{n+i} C_{n+i} (A_{n+i,i} + A_{i,n+i}) + B_{n+i} C_{n+i} (A_{n+i,i} + A_{i,n+i}) + B_{n+i-1} C_{n+i} (A_{n+i-1,i} + A_{i,n+i-1}) \right\} -$$

$$- \varepsilon \sum_{i=n+1}^{2n-1} U_i^2 \left[B_{i-n} (A_{i-n,i} + A_{i,i-n}) - B_{i+1-n} (A_{i-n+1,i} + A_{i,i-n+1}) \right] -$$

$$- \sum_{i=n+1}^{2n-1} \sum_{j=n+1}^{2n} \left[B_{i-n} (A_{i-n,j} + A_{j,i-n}) - B_{i-n+1} (A_{i-n+1,j} + A_{j,i-n+1}) \right] U_i U_j -$$

$$- \frac{U_{2n}^2}{1 + C U_{2n}} \left\{ C \left[C B_{2n} \sum_{j=1}^{2n} (A_{2n,j} + A_{j,2n}) + \varepsilon B_n (A_{n,2n} + A_{2n,n}) U_{2n} \right] + \right.$$

$$\left. + \varepsilon B_{2n} (A_{n,2n} + A_{2n,n}) 2C B_{2n} A_{2n,2n} \right\} \qquad /6.3/$$

V will be a function of negative sign in some region g including the origin if the following condition is satisfied

$$4\varepsilon B_1 C_1 Y_{10} A_{11} - B_{n+1} C_{n+1} (A_{n+1,1} + A_{1,n+1}) > 0,$$

$$4\varepsilon B_i C_i Y_{i0} A_{ii} - B_{n+i} C_{n+i} (A_{n+i,i} + A_{i,n+i}) +$$

$$+ B_{n+i-1} C_{n+i} (A_{n+i-1,i} + A_{i,n+i-1}) > 0, \quad i = 2, 3, \ldots, n,$$

$$\varepsilon B_n (A_{n,2n} + A_{2n,n}) - 2C B_{2n} A_{2n,2n} > 0. \qquad /6.4/$$

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for a negative sign quadratic form:

$$B=-\sum_{i=n+1}^{2n-1}[\beta_{i-n}(A_{i-n,i}+A_{i,i-n})-\beta_{i-n+1}(A_{i-n+1,i}+A_{i,i-n+1})]u_i^2-$$

$$-\sum_{i=n+1}^{2n-1}\sum_{\substack{j=n+1\\ j\neq i}}^{2n-1}[\beta_{i-n}(A_{i-n,j}+A_{j,i-n})-\beta_{i-n+1}(A_{i-n+1,j}+$$

$$+A_{j,i-n+1})]u_i u_j \qquad /6.5/$$

This gives a total of $2n$ conditions.

Thus, when the $2n^2+3n$ imposed on the arbitrary [illegible] A_{ij} and the system parameters are satisfied [illegible] a total number of $2n^2+5n-2$ /, the function v will have a positive sign, and its derivative a negative sign in the region g. In order to determine the large sense stability region for steady-state operation, a family of hypersurfaces $v=C$ and the hypersurface $v=0$ are constructed with the latter tangent to the hypersurface $C=C_0$ of the family. The region of the $2n$-dimensional phase space inside the hypersurface $v=C_0$ will be included entirely in the large sense stability region for steady-state operation. Modern high-speed electronic digital computers may be used effectively for calculating C_0 by using Liapunov's theorem on the unstability of unperturbed motion, we can find in a similar way the region of a priori unstability for steady-state conditions.

The direct method of Liapunov can be used to show that when there is no additional captation through the equilibrating reservoir, back flow in the problem considered is possible.

When there is additional flow of fluid into the first $n-1$ equilibrating reservoirs the stability conditions for unperturbed motion do not change even though the stability region itself in the phase space becomes smaller. With additional captation at some initial sections of the headwork system stable back flow conditions can exist.

It should be noted in conclusion that the stability problem for steady-state operation of a hydroelectric station with a differential equalizing reservoir is a special case of the stability problem for steady-state operation with n equalizing reservoirs when $n=2, \beta_2\to\infty$ and $C_2\to 0$. Here, the first reservoir must have a lumped resistance /this explains why the first $n-1$ reservoirs were taken to have lumped resistances/.

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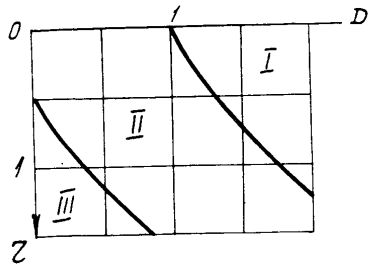

Fig. 1

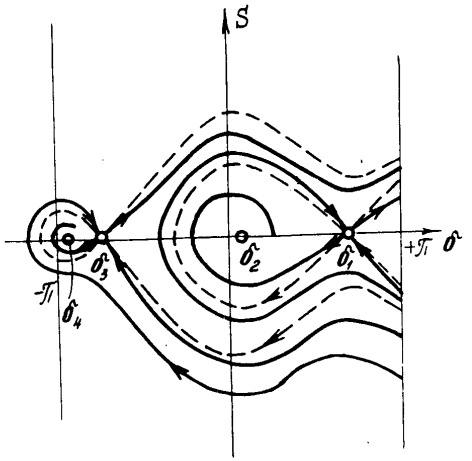

Fig. 2

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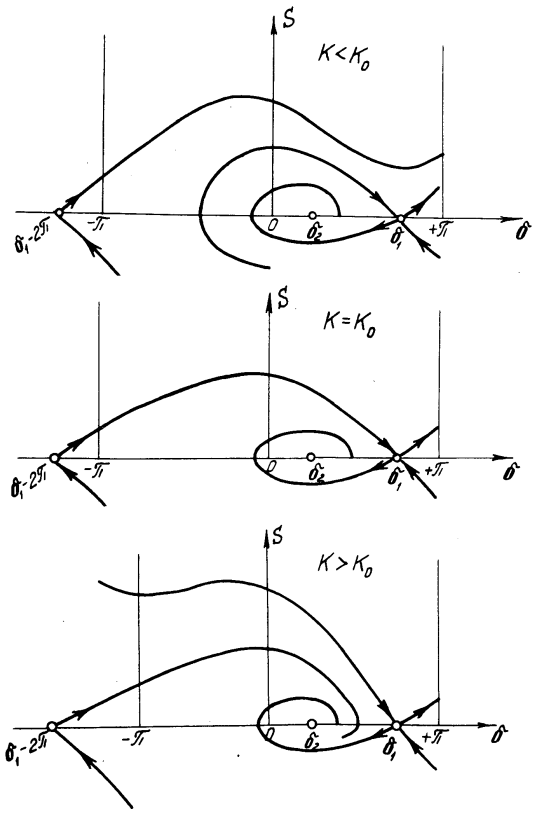

Fig. 3

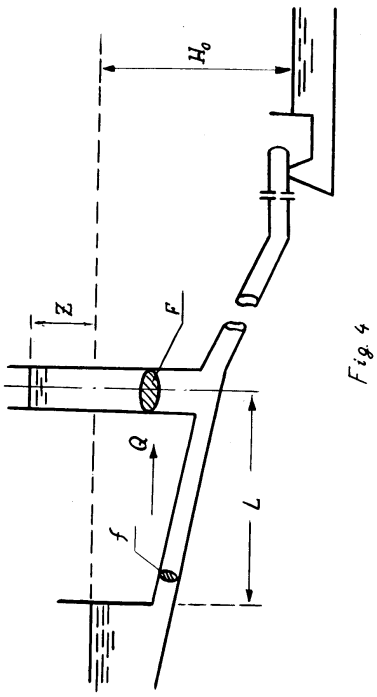

Fig. 4

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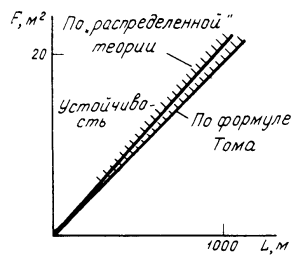

Fig. 5

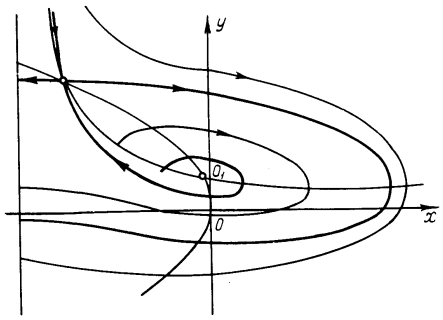

Fig. 7

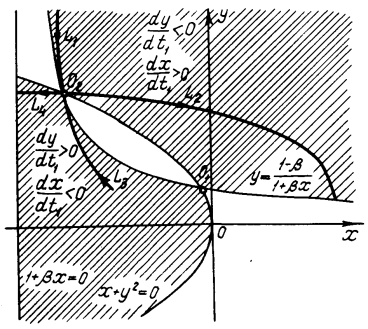

Fig. 6

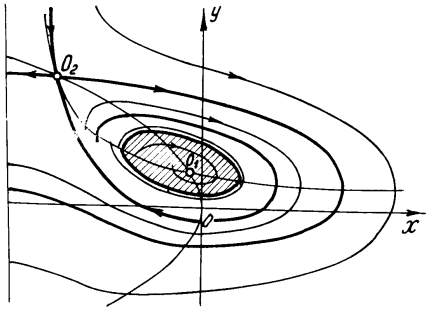

Fig. 8

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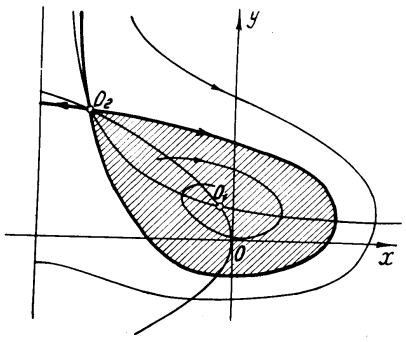

Fig. 9

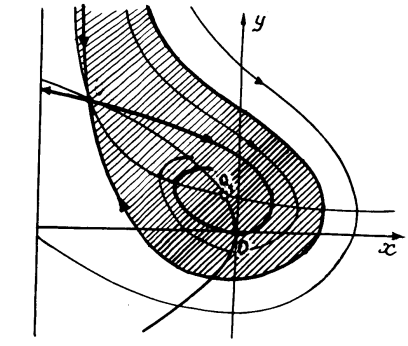

Fig. 10

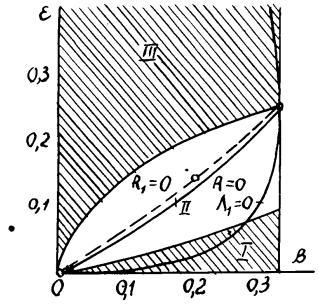

Fig. 11

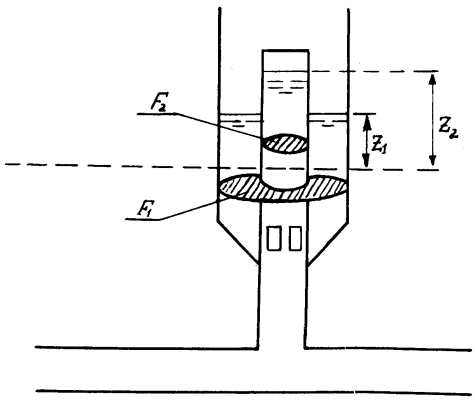

Fig. 12

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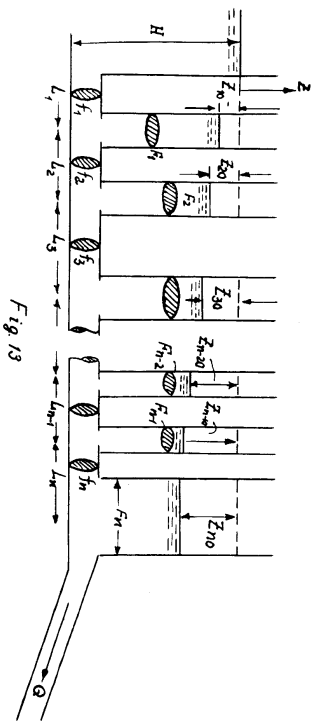

Fig 13

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J. S. Arshanykh – Academy of Sciences of the Uzbek SSR, Institute of Mathematics named after V. I. Romanovsky, Tashkent

И. С. АРЖАНЫХ

О ЦЕПНЫХ СИСТЕМАХ ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ

Non-linear oscillations are defined by the solutions of the system of differential equations arising from the corresponding physical laws. Here we consider for the sake of difinitness the systems connected with mechanics /the universality of kinetic and canonical systems of differential equations considered in the present paper will be seen from the note to the theorem 5 § 2/.

We call the chain system such system, the coordinates of which can be decomposed into groups

$$\begin{array}{ll} q_{11}, q_{12}, \cdots q_{1n_1} & \text{/ I group/} \\ q_{21}, q_{22}, \cdots q_{2n_2} & \text{/2 group/} \\ \cdots\cdots\cdots\cdots & \\ q_{k1}, q_{k2}, \cdots q_{kn_k} & \text{/ k group /} \end{array} \qquad /1/$$

so that its equations of motion should have the form

$$\frac{d}{dt}\frac{\partial L^{(1)}}{\partial \dot{q}_{1i_1}} - \frac{\partial L^{(1)}}{\partial q_{1i_1}} = 0, \quad i_1 = 1,2,\ldots n_1$$

$$\frac{d}{dt}\frac{\partial L^{(2)}}{\partial \dot{q}_{2i_2}} - \frac{\partial L^{(2)}}{\partial q_{2i_2}} = 0, \quad i_2 = 1,2,\ldots n_2 \qquad /2/$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d}{dt}\frac{\partial L^{(k)}}{\partial \dot{q}_{ki_k}} - \frac{\partial L^{(k)}}{\partial q_{ki_k}} = 0 \quad i_k = 1,2,\ldots n_k$$

where the functions $L^{(j)}$ $(j = 1,2,\ldots k)$ in the general case depend on time, on all coordinates and on all velocities:

$$L^{(j)} = \Phi_j(t, q_{11}, \cdots, q_{1n_1}, q_{21}, \cdots q_{2n_2} \cdots q_{k1} \cdots q_{kn_k},$$

$$\dot{q}_{11}, \cdots, \dot{q}_{1n_1}, \dot{q}_{21}, \cdots, \dot{q}_{2n_2} \cdots \dot{q}_{k1} \cdots \dot{q}_{kn_k}) \qquad /3/$$

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We shall call these functions link kinetic potentials and the number of links K a test of the chain system. If it is possible to reduce the number of links of the chain system without introduction of new variables /coordinates/, then the number K will be considered as a conditional test, but if such reduction of the number K is impossible, this test will be the real test of the considered system. The chain systems have a number of characteristic properties. The present paper is devoted to the investigation of this properties.

I. THE EXAMPLES OF THE CHAIN SYSTEMS

For the sake of most clearness of the concept of the chain systems and their tacts we shall consider some examples.

1. A conservative holonome system

Its equations of motion, as it is known, have the form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\lambda} - \frac{\partial L}{\partial q_\lambda} = 0 \qquad \lambda = 1,2,\dots \ell$$

This system is one-tacted, but in some cases it may be represented as a conditionally poly-tacted one. For example:

2/ A triple mathematical pendulum with masses of points m, n, p and lengths of threads connecting them, a, b, c is determined in the plane motion by three angles α, β, γ of deviation of the threads from the vertical. The conditional tact of this system is three, since the equations of motion may be represented in the form

$$\frac{d}{dt}\frac{\partial A}{\partial \dot{\alpha}} - \frac{\partial A}{\partial \alpha} = 0$$

$$\frac{d}{dt}\frac{\partial B}{\partial \dot{\beta}} - \frac{\partial B}{\partial \beta} = 0$$

$$\frac{d}{dt}\frac{\partial C}{\partial \dot{\gamma}} - \frac{\partial C}{\partial \gamma} = 0$$

where $A = \frac{1}{2}(m+n+p)a^2\dot{\alpha}^2 + (n+p)ab\cos(\alpha-\beta)\dot{\alpha}\dot{\beta} + p a c \cos(\alpha-\gamma)\dot{\alpha}\dot{\gamma} + (m+n+p)ga\cos\alpha$,

$B = \frac{1}{2}(n+p)b^2\dot{\beta}^2 + (n+p)ab\cos(\alpha-\beta)\dot{\alpha}\dot{\beta} + p b c \cos(\beta-\gamma)\dot{\beta}\dot{\gamma} + (n+p)gb\cos\beta$,

$C = \frac{1}{2}pc^2\dot{\gamma}^2 + pac\cos(\alpha-\gamma)\dot{\alpha}\dot{\gamma} + pbc\cos(\beta-\gamma)\dot{\beta}\dot{\gamma} + pgc\cos\gamma$.

2/ the pendulum with length ℓ and mass m at a body with mass M sliding along the inklined plane is determined by two coordinates: the length s of the path passed by the body and the angle φ of deviation from the vertical. The real tact is one, and the conditional one is two, since the equations of motion can be represented in the form:

$$\frac{d}{dt}\frac{\partial S}{\partial \dot{s}} - \frac{\partial S}{\partial s} = 0$$

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\varphi}} - \frac{\partial \Phi}{\partial \varphi} = 0,$$

where

$S = \frac{1}{2}(M+m)\dot{s}^2 + m\ell\dot{s}\dot{\varphi}\cos(\varphi+\alpha) + (M+m)gs\sin\alpha$

$\Phi = \frac{1}{2}m\ell\dot{s}\dot{\varphi}\cos(\varphi+\alpha) + \frac{1}{2}m\ell^2\dot{\varphi}^2 + mg\ell\cos\varphi$

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*/ A free rigid body may be considered as a conditionally two-tacted system with the following link kinetic potentials:

$$L^{(c)} = \frac{1}{2} M v_c^2 + U(x_c, y_c, z_c, \varphi, \psi, \theta)$$

$$L^{(\omega)} = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + U.$$

2. NON-CONSERVATIVE HOLONOME SYSTEM

The equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\lambda} - \frac{\partial L}{\partial q_\lambda} = Q_\lambda \ ,$$

after the work of generalized forces along the virtual displacements has been reduced to the canonical form:

$$\sum_{\lambda=1}^{\ell} Q_\lambda \delta q_\lambda = \delta U + \sum_{\rho=1}^{z} V_\rho \delta U_\rho \ ,$$

may be written in the form

$$\frac{d}{dt}\frac{\partial L^{(1)}}{\partial \dot{q}_\lambda} - \frac{\partial L^{(1)}}{\partial q_\lambda} = 0 \ , \quad \lambda = 1, \ldots, \ell,$$

$$\frac{d}{dt}\frac{\partial L^{(2)}}{\partial \dot{q}_{e+\rho}} - \frac{\partial L^{(2)}}{\partial q_{e+\rho}} = 0 \ , \quad \rho = 1, \ldots, z,$$

where

$$L^{(1)} = L + U + \sum_{\rho=1}^{z} \dot{q}_{e+\rho}\left(\sum_{\lambda=1}^{\ell} \frac{\partial U_\rho}{\partial q_\lambda}\dot{q}_\lambda + \frac{\partial U_\rho}{\partial t}\right),$$

$$L^{(2)} = \frac{1}{2}\sum_{\rho=1}^{z} \dot{q}_{e+\rho}^2 - \sum_{\rho=1}^{z} V_\rho q_{e+\rho} \ .$$

Hence, non-conservative, holonome system is a two-tacted one. The tact two is a real one here, if the field of generalized forces is a rotational one, i.e. under the condition

$$\frac{\partial Q_\lambda}{\partial \dot{q}_\mu} \neq \frac{\partial Q_\mu}{\partial \dot{q}_\lambda}$$

As it was shown /I/ a holonome conservative system with connections imposed has the same real tact as the considered one. A conditional tact, the connections being imposed, is equal to three.

3. NON-HOLONOME, NON-CONSERVATIVE SYSTEM

The connections

$$\sum_{\lambda=1}^{\ell} A_{\mu\lambda}\dot{q}_\lambda + A_\mu = 0, \quad \mu = 1, \ldots, m$$

being imposed, the equations of motion will have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\lambda} - \frac{\partial L}{\partial q_\lambda} = Q_\lambda + \sum_{\mu=1}^{m} \Lambda_\mu A_{\mu\lambda} \ .$$

If one reduces the elementary work of generalized forces to the canonical form and represents the equations of connections in the canonical form:

$$\frac{dR_\mu}{dt} + \sum_{\rho=1}^{z_\mu} Q_{\rho\mu}\frac{dR_{\rho\mu}}{dt} = 0 \ , \quad \mu = 1, \ldots, m,$$

then, as it is easy to see, the equations of motion will have the form

$$\frac{d}{dt}\frac{\partial L^{(g)}}{\partial \dot{q}_\rho} - \frac{\partial L^{(g)}}{\partial q_\rho} = 0 \ , \quad \rho = 1, \ldots, z,$$

$$\frac{d}{dt}\frac{\partial L^{(f)}}{\partial \dot{f}_\mu} - \frac{\partial L^{(f)}}{\partial f_\mu} = 0 \quad \mu = 1, \ldots, m,$$

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$$\frac{d}{dt}\frac{\partial L^{(h)}}{\partial \dot{h}_{\rho\mu}} - \frac{\partial L^{(h)}}{\partial h_{\rho\mu}} = 0 ,$$

$$\frac{d}{dt}\frac{\partial L^{(q)}}{\partial \dot{q}_{\lambda}} - \frac{\partial L^{(q)}}{\partial q_{\lambda}} = 0 ,$$

where

$$L^{(q)} = \frac{1}{2}\sum_{\rho} \dot{q}_{\rho}^{2} - \sum_{\rho} q_{\rho} V_{\rho} ,$$

$$L^{(f)} = \frac{1}{2}\sum_{\mu} \dot{f}_{\mu}^{2} - \sum_{\mu} f_{\mu} \Lambda_{\mu} ,$$

$$L^{(h)} = \frac{1}{2}\sum_{\rho\mu} \dot{h}_{\rho\mu}^{2} - \sum_{\rho\mu} h_{\rho\mu} \Lambda_{\mu} Q_{\rho\mu}$$

$$L^{(q)} = T + U + \sum_{\rho=1}^{z} \dot{q}_{\rho} \dot{u}_{\rho} + \sum_{\mu=1}^{m} \dot{f}_{\mu} \dot{\rho}_{\mu} + \sum_{\mu=1}^{m}\sum_{\rho=1}^{z_{\mu}} \dot{h}_{\rho\mu} R_{\rho\mu}$$

Consequently we have reduced non-holonome, non-conservative system to a chain system with four links.

§ 2. Properties of the chain system

The investigation of the process of oscillations is connected finally with the integrating of differential equations defining these oscillations. Therefore we shall examine certain problems of the theory of integrating of the chain systems /2/.

Theorem I.

Let a kinetic potential $L^{(h)}$ be invariant with respect to Lie group with the operator

$$\sum_{i_h=1}^{n_h} u_{hi_h} \frac{\partial}{\partial q_{hi_h}} \qquad /4/$$

Then the chain system has an integral

$$\sum_{i_h=1}^{n_h} u_{hi_h} \frac{\partial L^{(h)}}{\partial \dot{q}_{hi_h}} = const.$$

Indeed, we have, according to the invariancy of $L^{(h)}$ /5/

$$\sum_{i_h=1}^{n_h} \left(\frac{\partial L^{(h)}}{\partial q_{hi_h}} u_{hi_h} + \frac{\partial L^{(h)}}{\partial \dot{q}_{hi_h}} \dot{u}_{hi_h} \right) = 0$$

Therefore the time derivative of the left-hand side of equality /5/, which is equal to

$$\sum_{i_h=1}^{n_h} \left(\dot{u}_{hi_h} \frac{\partial L^{(h)}}{\partial \dot{q}_{hi_h}} + \frac{d}{dt}\frac{\partial L^{(h)}}{\partial \dot{q}_{hi_h}} u_{hi_h} \right),$$

does vanish having taken into account the equations

$$\frac{d}{dt}\frac{\partial L^{(h)}}{\partial \dot{q}_{hi_h}} = \frac{\partial L^{(h)}}{\partial q_{hi_h}} .$$

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For more convenience we designate in the following the coordinates of system by q_ν, $\nu = 1,2,\dots, N = n_1+n_2+\dots n_k$, accepting:

$$q_{11} = q_1,\ q_{12} = q_2, \dots, q_{1n_1} = q_{n_1}, \dots$$

$$q_{k1} = q_{N-n_k+1}, \dots, q_{kn_k} = q_N .$$

Theorem 2.

The chain system may be reduced without introducing new coordinates to the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\nu} + \sum_{\rho=1}^{\tau} F_\rho \frac{\partial L_\rho}{\partial \dot q_\nu}\right) = \frac{\partial L}{\partial q_\nu} + \sum_{\rho=1}^{\tau} F_\rho \frac{\partial L_\rho}{\partial q_\nu} \qquad /6/$$

$\nu = 1,2,\dots N.$

Indeed, we construct the differential form

$$\Omega(\delta) \equiv \sum_{i_1=1}^{n_1}\left(\frac{\partial L^{(1)}}{\partial q_{i_1}}\delta q_{i_1} + \frac{\partial L_\rho^{(1)}}{\partial q_{i_1}}\delta q_{i_1}\right) +$$

$$+ \sum_{i_2=n_1+1}^{n_1+n_2}\left(\frac{\partial L^{(2)}}{\partial q_{i_2}}\delta q_{i_2} + \frac{\partial L_\rho^{(2)}}{\partial q_{i_2}}\delta q_{i_2}\right) + \dots +$$

$$+ \sum_{i_k=N-n_k+1}^{N}\left(\frac{\partial L^{(k)}}{\partial q_{i_k}}\delta q_{i_k} + \frac{\partial L_\rho^{(k)}}{\partial q_{i_k}}\delta q_{i_k}\right) \qquad /7/$$

... reduce it to the canonical form

$$\Omega(\delta) = \delta L + \sum_{\rho=1}^{\tau} F_\rho \delta L_\rho \equiv$$

$$\equiv \sum_{\nu=1}^{N}\left[\left(\frac{\partial L}{\partial q_\nu} + \sum_{\rho=1}^{\tau} F_\rho \frac{\partial L_\rho}{\partial q_\nu}\right)\delta q_\nu + \right.$$

$$\left. + \left(\frac{\partial L}{\partial \dot q_\nu} + \sum_{\rho=1}^{\tau} F_\rho \frac{\partial L_\rho}{\partial \dot q_\nu}\right)\delta \dot q_\nu\right]. \qquad /8/$$

The equations /6/ follow by the comparison /7/ and /8/.

Let us call the system of equations /6/ a kinetic system of rank τ. If $\tau = 0$, we have the Lagrange equations. Thus, the kinetic systems of the rank more than zero are the natural generalization of the Lagrange systems. It follows from the examples exposed in [1], that non-conservative holonome systems /example 2/, and non-holonome systems too /example 3/ are dynamic systems of chain type, which can be reduced to the systems of rank more than zero. The importance of the problem of integrating systems /6/ follows from the above-mentioned relation between the chain systems and the kinetic systems of rank more than zero. The connection of the problem of integrating with the theory of Lie groups is the most essential.

Let us call the function L of system /6/ main kinetic potential, the functions L_ρ complementary kinetic potentials, F_ρ - multipliers. It turns out that the existence condition of the integral in connection with the invariancy with respect to Lie group does not depend on the multipliers /the integral itself depends on them/.

Theorem 3.

If the main kinetic potential and the complementary ones are invariant with respect to the group with the operator

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$$\sum_{\nu=1}^{N} u_\nu \frac{\partial}{\partial q_\nu} \qquad /9/$$

then the kinetic system with rank more than zero have an integral

$$\sum_{\nu=1}^{N} u_\nu \left(\frac{\partial L}{\partial \dot q_\nu} + \sum_{\rho=1}^{z} F_\rho \frac{\partial L_\rho}{\partial \dot q_\nu} \right) = const \qquad /10/$$

Indeed, determine the time derivative of the left-hand side of the equality /10/, taking into account the equations /7/. Then we get an expression

$$\sum_{\nu=1}^{N} \left[\dot u_\nu \left(\frac{\partial L}{\partial \dot q_\nu} + \sum_{\rho=1}^{z} F_\rho \frac{\partial L_\rho}{\partial \dot q_\nu} \right) + u_\nu \left(\frac{\partial L}{\partial q_\nu} + \sum_{\rho=1}^{z} F_\rho \frac{\partial L_\rho}{\partial q_\nu} \right) \right]. \qquad /11/$$

But this expression vanishes according to the invariancy equations

$$\sum_{\nu=1}^{N} \left(u_\nu \frac{\partial L}{\partial q_\nu} + \dot u_\nu \frac{\partial L}{\partial \dot q_\nu} \right) = 0$$

$$\sum_{\nu=1}^{N} \left(u_\nu \frac{\partial L_\rho}{\partial q_\nu} + \dot u_\nu \frac{\partial L_\rho}{\partial \dot q_\nu} \right) = 0 \qquad \rho = 1,2,\dots,z \qquad /12/$$

Theorem 4.

Let the equations

$$P_{m+\lambda} = \frac{\partial L}{\partial \dot q_\lambda} + \sum_\rho F_\rho \frac{\partial L_\rho}{\partial \dot q_{m+\lambda}}, \quad \lambda = \overline{1,N-m} \qquad /13/$$

may be solved for $\dot q_{m+\lambda}$. Then the kinetic system may be replaced by an equivalent one

$$\dot q_{m+\lambda} = \frac{\partial R}{\partial p_{m+\lambda}} + \sum_\rho F_\rho \frac{\partial R_\rho}{\partial p_{m+\lambda}},$$

$$\dot p_{m+\lambda} = -\frac{\partial R}{\partial q_{m+\lambda}} - \sum_\rho F_\rho \frac{\partial R_\rho}{\partial q_{m+\lambda}}, \qquad /14/$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q_\mu} + \sum_\rho F_\rho \frac{\partial R_\rho}{\partial \dot q_\mu} \right) = \frac{\partial R}{\partial q_\mu} + \sum_\rho F_\rho \frac{\partial R_\rho}{\partial q_\mu}, \qquad /15/$$

$$\mu = \overline{1,m}$$

where

$$R(t, q_1, \dots, q_N, \dot q_1, \dots, \dot q_m, p_{m+1}, \dots, p_N) =$$

$$= -L + \sum_\lambda p_{m+\lambda} \dot q_{m+\lambda}, \quad R_\rho = -L_\rho. \qquad /16/$$

Indeed we evaluate

$$\sum(\delta) \equiv \delta R + \sum_\rho F_\rho \delta R_\rho$$

in two assumptions: 1/ R, R_ρ and F_ρ being functions of $t, q_\nu, \dot q_\nu$ ($\nu = \overline{1,N}$) and being functions of $t, q_\nu, \dot q_\mu, p_{m+\lambda}$ ($\mu = \overline{1,m}$, $\lambda = \overline{1,N-m}$) taking into account the fundamental equations /1/, 2/ as a result of equating of coefficients of δq_μ, $\delta q_{m+\lambda}$, $\delta \dot q_\mu$ and $\delta p_{m+\lambda}$ we shall get a set of equalities, which according to /13/ and the initial kinetic system /6/ give a system of equations /14/ and /15/. The

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system [11-/15/ is evidently a generalization of the south systems of the theory of equations of holonome conservative systems.

It is easy to prove, that if the functions $\mathcal{R}$ and R_ρ are invariant with respect to a group, with the operator

$$\sum_\mu U_\mu \frac{\partial}{\partial q_\mu} ,$$

then the equations /15/ have an integral

$$\sum_\mu U_\mu \left(\frac{\partial \mathcal{R}}{\partial q'_\mu} + \sum_\rho F_\rho \frac{\partial R_\rho}{\partial q'_\mu} \right) = const .$$

In particular, if we have $N-m$ cyclic coordinates,

$$\frac{\partial \mathcal{R}}{\partial q'_{m+\lambda}} = 0, \quad \frac{\partial R_\rho}{\partial q_{m+\lambda}} = 0, \quad \lambda = \overline{1, N-m} \qquad /17/$$

the cyclic integrales are

$$\frac{\partial L}{\partial \dot q_{m+\lambda}} + \sum_\rho F_\rho \frac{\partial L_\rho}{\partial \dot q_{m+\lambda}} = C_\lambda = const \qquad /18/$$

and it is necessary to accept $p_\lambda = C_\lambda$ in the equations /15/ /the cyclic coordinates being disregarded/.

Theorem 5.

If the equations

$$p_\nu = \frac{\partial L}{\partial \dot q_\nu} + \sum_\rho F_\rho \frac{\partial L_\rho}{\partial \dot q_\nu} , \quad \nu = \overline{1, N} \qquad /19/$$

are solvable for velocities, the kinetic system is equivalent to the canonical system of rank τ

$$\dot q_\nu = \frac{\partial H}{\partial p_\nu} + \sum_\rho F_\rho \frac{\partial H_\rho}{\partial p_\nu} ,$$

$$\dot p_\nu = -\frac{\partial H}{\partial q_\nu} - \sum_\rho F_\rho \frac{\partial H_\rho}{\partial q_\nu} \qquad /20/$$

where the main canonical potential H and the complementary ones H_ρ are expressed in terms of variables t, q_ν, p_ν by formulae

$$H = -L + \sum_\nu p_\nu \dot q_\nu , \quad H_\rho = -L_\rho . \qquad /21/$$

The proof of this theorem is the same as of the preceding one. We'll make one remark connected with the universality of canonical system of rank more than zero.

Let some system of even order be given as

$$\frac{dx_i}{dt} = X_i (t, x_1, \dots, x_{2N}), \quad i = \overline{1, 2N},$$

which describes, for instance, the process of non-linear oscillations in the electrical circuit. Separate the variables x_i into two groups q_ν and p_ν and consider the system

$$\dot q_\nu = Q_\nu (t, q_1, \dots q_N, p_1, \dots, p_N), \quad \dot p_\nu = P_\nu .$$

Build the differential form

$$\omega(\delta) \equiv \sum_\nu (Q_\nu \delta p_\nu - P_\nu \delta q_\nu)$$

and reduce it to the canonical form

$$\omega(\delta) = \delta H + \sum_{\rho=1}^{\tau} F_\rho \delta H_\rho .$$

Then the given system takes the canonical form /20/. If then we pass from p_ν to $\dot q_\nu$, the canonical system will be

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transformed to a kinetic one with the rank more than zero. The canonical system of rank more than zero has a number of properties /see [5]/, and we will consider two of them being of interest for the problem of integration.

Theorem 6.

A canonical system with the rank more than zero is invariant with respect to the contact transformations.

Indeed let us perform the transformation

$$q_\nu = \varphi_\nu(t, \xi_1, \dots \xi_N, \eta_1, \dots \eta_N), \qquad /22/$$
$$p_\nu = \psi_\nu(t, \xi_1, \dots, \xi_N, \eta_1, \dots \eta_N),$$

according to which the equality

$$\sum_\nu (p_\nu \delta q_\nu - \eta_\nu \delta \xi_\nu) = \delta W, \qquad /23/$$

holds, i.e. suppose, that the transformation /22/ has the form

$$p_\nu = \frac{\partial W}{\partial q_\nu}, \qquad \eta_\nu = -\frac{\partial W}{\partial \xi_\nu} \qquad /24/$$

The system /20/ may be written briefly in the form

$$\sum_\nu (\dot{q}_\nu \delta p_\nu - \dot{p}_\nu \delta q_\nu) = \delta H + \sum_\rho F_\rho \delta H_\rho. \qquad /25/$$

On the other hand, /23/ gives us

$$d\delta W - \delta\left(dW - \frac{\partial W}{\partial t} dt\right) = \sum_\nu \left[(dp_\nu \delta q_\nu - dq_\nu \delta p_\nu) - (d\eta_\nu \delta \xi_\nu - \delta \eta_\nu d\xi_\nu)\right]$$

$$\sum_\nu (\dot{q}_\nu \delta p_\nu - \dot{p}_\nu \delta q_\nu) = \sum_\nu (\dot{\xi}_\nu \delta \eta_\nu - \dot{\eta}_\nu \delta \xi_\nu) - \delta \frac{\partial W}{\partial t}. \qquad /26/$$

Replacing the left-hand side of the equality /25/ by the right-hand side of /26/, one obtains

$$\sum_\nu (\dot{\xi}_\nu \delta \eta_\nu - \dot{\eta}_\nu \delta \xi_\nu) = \delta K + \sum_\rho F_\rho \delta K_\rho$$
$$K = H + \frac{\partial W}{\partial t}, \qquad K_\rho = H_\rho$$

that is equivalent to a system

$$\dot{\xi}_\nu = \frac{\partial K}{\partial \eta_\nu} + \sum_\rho F_\rho \frac{\partial K_\rho}{\partial \eta_\nu},$$
$$\dot{\eta}_\nu = -\frac{\partial K}{\partial \xi_\nu} - \sum_\rho F_\rho \frac{\partial K_\rho}{\partial \xi_\nu}$$

We give now one of the numerous examples for using the proved theorem, on example of obvious interest for the problem of integration.

Theorem 7.

Let the main canonical potential H and the complementary ones H_ρ not depend on time and be in involution, i.e. the equalities

$$(H, H_\rho) = 0, \ (H_{\rho_1}, H_{\rho_2}) = 0, \ \rho, \rho_1, \rho_2 = \overline{1,r} \qquad /30/$$

hold.

To integrate such an involutive system it is sufficient to obtain a total integral

$$W = W(q_1, \dots q_N, h, h_1, \dots h_r, c_1, \dots, c_{N-r-1}) \qquad /31/$$

of the compatible system of partial differential equations

$$H\left(q_1, \dots, q_N, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_N}\right) = h,$$
$$H_\rho\left(q_1, \dots q_N, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_N}\right) = h_\rho.$$

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Then the integrals of system will be

$$\frac{\partial W}{\partial h} = t - t_0 , \quad \frac{\partial W}{\partial c_\lambda} = d_\lambda = const , \quad \lambda = 1, \overline{N-r-1} \qquad /33/$$

and in order to obtain all the rest it will be necessary to integrate the system

$$\frac{d}{dt}\frac{\partial W}{\partial h_\rho} = F_\rho , \qquad \rho = \overline{1, r} . \qquad /34/$$

In order to prove this, perform a transformation /24/, having obtained W from the equations

$$H = \xi_1 , \quad H_\rho = \xi_{1+\rho} , \quad \rho = \overline{1, r} . \qquad /35/$$

Then the first group of the equations gives

$$\dot{\xi}_\nu = 0 , \quad \xi_\nu = c_\nu = const \quad \nu = \overline{1, N} \qquad /36/$$

and the second group of the equations splits into three:

1/ for $\nu = 1$, $\dot{\eta}_1 = -1$, $\eta_1 = -t + t_0$

2/ for $\nu = 2, \ldots r+1$, $\dot{\eta}_{1+\rho} = -F_\rho$

3/ for $\nu > 1+r$, $\eta_\nu = 0$, $\eta_\nu = const$.

Then change the notations, putting

$$\xi_1 = h , \quad \xi_{1+\rho} = h_\rho , \quad \rho = \overline{1, r}$$

$$\xi_{1+r+\lambda} = c_\lambda \quad \lambda = 1, \overline{N-r-1} \quad \eta_{1+r+\lambda} = -d_\lambda .$$

Now from the transformation formulae /24/ we have

$$p_\nu = \frac{\partial W}{\partial q_\nu} , \quad \nu = \overline{1, N} ,$$

$$-t + t_0 = -\frac{\partial W}{\partial h} , \quad -F_\rho = \frac{d}{dt}\left(-\frac{\partial W}{\partial h_\rho}\right)$$

$$-d_\lambda = -\frac{\partial W}{\partial c_\lambda} ,$$

where

$$W = W(q_1, \ldots, q_N, h, h_1, \ldots, h_r c_1, c_2, \ldots c_{N-r-1}).$$

§ 3 AUTONOMOUS CHAIN SYSTEMS

The theorem of invariancy of canonical system of rank more than zero,

$$\dot{q}_\nu = \frac{\partial H}{\partial p_\nu} + \sum_\rho F_\rho \frac{\partial H_\rho}{\partial p_\nu} ,$$

$$\dot{p}_\nu = -\frac{\partial H}{\partial q_\nu} - \sum_\rho F_\rho \frac{\partial H_\rho}{\partial p_\nu} \qquad /36/$$

with respect to the contact transformations give us the means to reduce such system to a new chain structure, when this system is autonomous, i.e. when the right-hand sides of /36/ do not depend explicitly on time. As the autonomous systems are of often occurrence to consider such system is of certain interest.

We perform the transformation

$$\xi_\nu = \frac{\partial \Omega}{\partial q_\nu} , \qquad \eta_\nu = \frac{\partial \Omega}{\partial \xi_\nu} .$$

Then the system /36/ is transformed into the [illegible] system

$$\frac{d\xi_\nu}{dt} = \frac{\partial H}{\partial \eta_\nu} + \sum_\rho F_\rho \frac{\partial H_\rho}{\partial \eta_\nu} ,$$

$$\frac{d\eta_\nu}{dt} = -\frac{\partial H}{\partial \xi_\nu} - \sum_\rho F_\rho \frac{\partial H_\rho}{\partial \xi_\nu}$$

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We determine the function $\Omega(q_1, \dots, q_N, \xi_1, \dots \xi_N)$ from the partial differential equation

$$H\left(q_1, \dots, q_N, \frac{\partial \Omega}{\partial q_1}, \dots, \frac{\partial \Omega}{\partial q_N}\right) = \xi_N \quad /41/$$

Then the system /40/ is represented in the form

$$\frac{d\xi_\mu}{dt} = \sum_{\rho=1}^{2} F_\rho \frac{\partial H}{\partial \eta_\mu}, \quad \frac{d\eta_\mu}{dt} = -\sum_{\rho=1}^{2} F_\rho \frac{\partial H_\rho}{\partial \xi_\mu}, \quad \mu = \overline{1, N-1} \quad /42/$$

$$\frac{d\xi_N}{dt} = \sum_{\rho=1}^{2} F_\rho \frac{\partial H}{\partial \eta_N}, \quad \frac{d\eta_N}{dt} = -1 - \sum_{\rho=1}^{2} F_\rho \frac{\partial H_\rho}{\partial \xi_N}. \quad /43/$$

We replace the indepenient variable by the new one according to the formulae

$$F_1 dt = dt_1 \quad /44/$$

and suppose

$$\frac{F_\rho}{F_1} = F_{\rho_1}^{(1)}, \quad \rho_1 = \overline{2, 2}, \quad \xi_\mu = q_\mu^{(1)}, \quad \eta_\mu = p_\mu^{(1)}.$$

Then the system /42/ may be written down in the form

$$\frac{dq_\mu^{(1)}}{dt_1} = \frac{\partial H_1}{\partial p_\mu^{(1)}} + \sum_{\rho_1=2}^{2} F_{\rho_1}^{(1)} \frac{\partial H_{\rho_1}}{\partial p_\mu^{(1)}},$$

$$\frac{\partial p_\mu^{(1)}}{\partial t_1} = -\frac{\partial H_1}{\partial q_\mu^{(1)}} - \sum_{\rho_1=2}^{2} F_{\rho_1}^{(1)} \frac{\partial H_{\rho_1}}{\partial q_\mu^{(1)}} \quad \mu = \overline{1, N-1} \quad /45/$$

Now we perform the contact transformation

$$p_\mu^{(1)} = \frac{\partial \Omega_1}{\partial q_\mu^{(1)}}, \quad \eta_\mu^{(1)} = -\frac{\partial \Omega}{\partial \xi_\mu^{(1)}} \quad /46/$$

and determine a function $\Omega_1(q_1^{(1)}, \dots, q_{N-1}^{(1)}, \xi_1^{(1)}, \dots, \xi_{N-1}^{(1)})$ from the equation

$$H_1\left(q_1^{(1)}, \dots, q_{N-1}^{(1)}, \xi_N, \frac{\partial \Omega_1}{\partial q_1^{(1)}}, \dots, \frac{\partial \Omega_1}{\partial q_{N-1}^{(1)}}\right) = \xi_{N-1}^{(1)}, \quad /47/$$

considering ξ_n as a parameter.

Then the system /45/ is represented in the form

$$\frac{d\xi_\lambda^{(1)}}{dt_1} = \sum_{\rho_1=2}^{2} F_{\rho_1}^{(1)} \frac{\partial H_{\rho_1}}{\partial \eta_\lambda^{(1)}}, \quad \frac{\partial \eta_\lambda^{(1)}}{dt_1} = -\sum_{\rho_1=2}^{2} F_\rho^{(1)} \frac{\partial H_{\rho_1}}{\partial \xi_\lambda^{(1)}}, \quad /48/$$

$$\frac{d\xi_{N-1}^{(1)}}{dt_1} = \sum_{\rho_1=2}^{2} F_{\rho_1}^{(1)} \frac{\partial H_{\rho_1}}{\partial \eta_{N-1}^{(1)}}, \quad \frac{\partial \eta_{N-1}^{(1)}}{dt_1} = -1 - \sum_{\rho_1=2}^{2} F_{\rho_1}^{(1)} \frac{\partial H_{\rho_1}}{\partial \xi_{N-1}^{(1)}}$$

/49/

Replace the independent variable by the new one according to the formula

$$F_2^{(1)} dt_1 = dt_2 \quad /50/$$

and suppose

$$\frac{F_{\rho_2}^{(1)}}{F_2^{(1)}} = \frac{F_{\rho_2}}{F_2} = F_{\rho_2}^{(2)}, \quad \rho_2 = \overline{3, 2}, \quad \xi_\lambda^{(1)} = q_\lambda^{(2)}$$

$$\eta_\lambda^{(1)} = p_\lambda^{(2)}.$$

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Then the system /48/ can be written down in the form

$$\frac{d\alpha_\lambda^{(2)}}{d\tau_2} = \frac{\partial H_2}{\partial \beta_\lambda^{(2)}} + \sum_{\rho_2=3}^{\tau} F_{\rho_2}^{(2)} \frac{\partial H_{\rho_2}}{\partial \beta_\lambda^{(2)}} ,$$

$$\frac{d\beta_\lambda^{(2)}}{d\tau_2} = -\frac{\partial H_2}{\partial \alpha_\lambda^{(2)}} - \sum_{\rho_2=3}^{\tau} F_{\rho_2}^{(2)} \frac{\partial H_{\rho_2}}{\partial \alpha_\lambda^{(2)}} ,$$

$$\lambda = \overline{1, N-2} . \qquad /51/$$

Now we introduce the new variables according to the formulae

$$P_\lambda^{(2)} = \frac{\partial \Omega_2}{\partial q_\lambda^{(2)}} , \qquad \zeta_\lambda^{(2)} = -\frac{\partial \Omega_2}{\partial \xi_\lambda^{(2)}} \qquad /52/$$

and determine Ω_2 from the equation $H_2 = \xi_{N-2}^{(2)}$ considering ξ_N and $\xi_{N-1}^{(1)}$ as parametres, introduce a new independent variable $d/t_3 = F_3^{(2)} d/t$ and so on.

After τ steps we obtain a canonical system:

$$\frac{dq_i^{(2)}}{dt_\tau} = \frac{\partial H_\tau}{\partial p_i^{(2)}} , \quad \frac{dp_i^{(2)}}{dt_\tau} = -\frac{\partial H_\tau}{\partial q_i^{(2)}} , \quad i = \overline{1, N-\tau} \qquad /53/$$

and two complementary equations.

We transform all the complentary equations /43/, /49/, ... to an independent variable t and denote the dependent variables x_j $j = \overline{1, 2\tau}$. Then the rest of equations will be

$$\frac{dx_j}{dt} = Z_j ; \quad j = \overline{1, 2\tau} , \qquad /54/$$

and we reduce these equations to the canonical form

$$\frac{dx_\rho}{dt} = \frac{\partial E}{\partial y_\rho} + \sum_{\sigma=1}^{s} G_\sigma \frac{\partial E_\sigma}{\partial y_\rho} , \qquad /55/$$

$$\frac{dy_\rho}{dt} = -\frac{\partial E}{\partial x_\rho} - \sum_{\sigma=1}^{s} G_\sigma \frac{\partial E_\sigma}{\partial x_\rho} ,$$

$$\rho = \overline{1, \tau} \qquad (s < \tau)$$

and perform the same transformations.

Then we obtain after S steps

$$\frac{dx_\kappa^{(s)}}{dt_{\tau+s}} = \frac{\partial E_s}{\partial y_\kappa^{(s)}} , \qquad \frac{dy_\kappa^{(s)}}{dt_{\tau+s}} = -\frac{\partial E_s}{\partial x_\kappa^{(s)}} , \quad \kappa = \overline{1, \tau-s} \qquad /56/$$

and a complementary system of order 2 s.

Let us carry out the same transformations with respect to this complementary system.

As the result we obtain a true canonical system and complementary one, and we transforme the last in the same manner. After a finite number of steps all the systems will be represented in the Hamiltonian canonical form.

Hence, a system of ordinary differential equations of even order , not depending explicitly on time, may be represented without introducing new variables in the form of a chain canonical system of zero rank:

$$\frac{d\alpha_{\kappa_1}}{d\tau_1} = \frac{\partial H^{(1)}}{\partial \beta_{\kappa_1}} , \quad \frac{d\beta_{\kappa_1}}{d\tau_1} = -\frac{\partial H^{(1)}}{\partial \alpha_{\kappa_1}} , \quad \kappa = \overline{1, N} ,$$

$$\frac{d\alpha_{\kappa_2}}{d\tau_2} = \frac{\partial H^{(2)}}{\partial \beta_{\kappa_2}} , \quad \frac{d\beta_{\kappa_2}}{d\tau_2} = -\frac{\partial H^{(2)}}{\partial \alpha_{\kappa_2}} , \quad \kappa_2 = \overline{N+1, N_2}$$

$$\frac{d\alpha_{\kappa_m}}{d\tau_m} = \frac{\partial H^{(m)}}{\partial \beta_{\kappa_m}} , \quad \frac{d\beta_{\kappa_m}}{d\tau_m} = -\frac{\partial H^{(m)}}{\partial \alpha_{\kappa_m}} , \quad \kappa_m = \overline{N_{m-1}+1, N} .$$

$$H^{(i)} = H^{(i)}(\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N) \quad i = \overline{1, m} \qquad /58/$$

The main property of this system consists of its invariancy

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with respect to the separated contact transformations: if the transformations

$$\beta_{\alpha_1} = \frac{\partial \omega_1}{\partial \alpha_{\alpha_1}}, \quad \beta_{\alpha_1} = -\frac{\partial \omega_1}{\partial \alpha_{\alpha_1}}, \omega_1 = \omega_1(\alpha_1, \ldots, \alpha_{\alpha_1}, a_1, \ldots, a_{\alpha_1})$$

$$\beta_{\alpha_m} = \frac{\partial \omega_m}{\partial \alpha_{\alpha_m}}, \quad \beta_{\alpha_m} = -\frac{\partial \omega_m}{\partial \alpha_{\alpha_m}}, \omega_m = \omega_m(\alpha_{m+1}, \ldots, \alpha_{\alpha_m}, a_{\alpha_{m-1}+1}, \ldots, a_{\alpha_m}) \quad /59/$$

are performed, the chain system will not be changed and will have in new variables the form

$$\frac{d\alpha_{\alpha_i}}{d\tau_i} = \frac{\partial H^{(i)}}{\partial \beta_{\alpha_i}}, \quad \frac{d\beta_{\alpha_i}}{d\tau_i} = -\frac{\partial H^{(i)}}{\partial \alpha_{\alpha_i}}, \quad i = \overline{1, m} \qquad /60/$$

This property allows to extend the method of integration of Hamilton-Jacobi's type to some systems of ordinary differential equations.

Let the system of partial differential equations

$$H^{(i)}\left(\alpha_1, \ldots, \alpha_N, \frac{\partial \omega_1}{\partial \alpha_1}, \ldots, \frac{\partial \omega_1}{\partial \alpha_{N_1}}, \frac{\partial \omega_2}{\partial \alpha_{N_1+1}}, \ldots, \frac{\partial \omega_2}{\partial \alpha_{N_2}}, \ldots, \frac{\partial \omega_m}{\partial \alpha_{N_{m-1}}}, \ldots, \frac{\partial \omega_m}{\partial \alpha_N}\right) = \alpha_N, \quad i = \overline{1, m}$$

$$(N_m = N) \qquad /61/$$

be compatible and to admit the complete integrals

$$\omega_1(\alpha_1, \ldots, \alpha_{N_1}, a_1, \ldots, a_{N_1}),$$

$$\omega_{i+1}(\alpha_{N_i+1}, \ldots, \alpha_{N_{i+1}}, a_{N_i+1}, \ldots, a_{N_{i+1}}), \qquad /62/$$

$$i = \overline{1, m-1}$$

$$\beta_{\alpha_i} = \frac{\partial \omega_i}{\partial \alpha_{\alpha_i}}, \quad \frac{\partial \omega_i}{\partial \alpha_{\alpha_i - 1}} = C_{\alpha_i - i} = \text{const}, \quad$$

$$\frac{\partial \omega_i}{\partial \alpha_{\alpha_i}} = \tilde{C}_{\alpha_i} + C_{\alpha_i},$$

$$\alpha_i = \overline{1, N} \qquad \alpha_2 = \overline{N_1 + 1, N_2}, \quad \ldots \quad \alpha_m = \overline{N_{m-1} + 1, N}$$

will be the solutions of the chain system.

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Е. А. БАРБАШИН

ПОСТРОЕНИЕ ПЕРИОДИЧЕСКОГО ДВИЖЕНИЯ КАК ОДНА ИЗ ЗАДАЧ ТЕОРИИ ПРОГРАММНОГО РЕГУЛИРОВАНИЯ

In this report conditions under which it is possible to select the input signal of system to approximately realize the predetermined periodic process are given.

The problem being investigated concerns the thery of programming control [I] /p.231/ since it deals with the possibility of finding programming functions ensuring stable periodic programmed conditions. Under real conditions programming functions may be given only approximately.

Estimations of permissible errors in setting the programming functions are given. From purely mathematical point of view the problem envolves setting conditions for maintenance and stability of periodic motion with permanent disturbances limited in norm. Absolute value, mean absolute value, and mean square value of the above permissible error are given. The case when the periodic motion being constructed is discontinuous is investigated. Some of main results of investigation are considered for the nonperiodic approximated motion.

§ I. Let us consider the system of differential equations:

$$\frac{dx_i}{dt} = f_i(x_1, \dots, x_n, t) + \varphi_i(t), \quad i = 1, 2, \dots, n. \qquad /I.I/$$

Assuming all the functions $f_i(x_1, \dots, x_n, t)$ to be periodic functions of time t with period ω, let us find such periodic functions $\varphi_i(t)$ that the given system of periodic functions $x_i = \psi_i(t)$, $i = 1, 2, \dots, n$ with period ω would be solution of system /I.I/.

Solution of this simple problem is:

$$\varphi_i(t) = \psi_i'(t) - f_i(\psi_1(t), \dots, \psi_n(t), t). \qquad /I.2/$$

Practically however this solution may be absolutely unacceptable. Indeed if system of functions $x_i = \psi_i(t)$ defines certain programmed conditions these conditions will

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be real only if they are stable with respect to initial disturbances as well as to permanent disturbances. Besides under real conditions the programming functions $\varphi_i(t)$ are selected from a certain narrowly limited class of functions as for example from algerbaic polinomials, trigonometric polinomials, piecewise constant discontinuous functions etc. That is why equations /2/ may be satisfied only approximately with certain error. It is easiest to calculate the mean square error of approximation in case when the programming functions $\varphi_i(t)$ are linear combinations of functions of a certain system of mutually orthogonal functions. It is well known that knowing the absolute value of error makes possible to evaluate the mean value and mean square value of error, while evaluation of its mean value is possible when one knows its mean square value. Thus the study of problems of maintenance and stability of periodic motion of system /I.I/ with permanent disturbances limited in the mean square is of certain interest. Investigation of disturbances limited in the mean may be, as is shown below, transferred to the shock and δ - like disturbances as well.

In the general case stability of motion with permanent disturbances limited in the mean was considered in the paper by V.E.Germaidze and N.N.Krassovsky /2/. Stability of periodic motion with permanent disturbances limited in absolute value are considered in papers /3,4/, while existence, maintenance, and stability of periodic motion with external disturbances limited in modulus are considered in papers /5, 6, 7/ using the method of Lyapunov's functions.

In the present paper we use a method differing from the above mentioned ones and get simple evaluation of absolute, and mean square values of permissible error for the approximation of programming functions with which the periodic motion Γ being approximated possesses the following properties:

I/ All trajectories originating with $\varepsilon = 0$ within sufficiently small neighbourhood Γ remain within the ε- neighbourhood of Γ.

2/ Within the ε- neighbourhood of Γ there exists asymptotically stable periodic motion whose region of attraction includes certain neighbourhood of Γ.

Thus if the error of approximation is in permissible limits even a sufficiently small error in selecting the initial conditions will not prevent us approximately realize the set periodic conditions since the actually obtained conditions will asymptotically approximate the periodic ones only slightly differing from those which were set.

§ 2. Let us assume at first that functions $\psi_i(t)$ which define the periodic motion Γ are continuous and piecewise differentiable.

Substituting the variables in system /I.I/

$$z_i = x_i - \psi_i(t), \quad i = 1, 2, \ldots, n,$$

we get:

$$\frac{dz_i}{dt} = f_i(z_1+\psi_1(t), \ldots z_n+\psi_n(t), t) + \psi_i(t) - \psi_i'(t), \quad i = 1, 2, \ldots, n.$$

Denoting

$$Z_i(z_1, \ldots, z_n, t) = f_i(z_1+\psi_1(t), \ldots, z_n+\psi_n(t), t) - f_i(\psi_1(t), \ldots, \psi_n(t), t),$$

$$\mathcal{Z}_i(t) = \varphi_i(t) - \psi_i'(t) + f_i(\psi_1(t), \ldots, \psi_n(t), t), \quad i = 1, 2, \ldots, n$$

the system /I.I/ may be written as:

$$\frac{dz_i}{dt} = Z_i(z_1, \ldots, z_n, t) + \mathcal{Z}_i(t), \quad i = 1, 2, \ldots, n. \qquad /2.I/$$

Equations /2.I/ evidently are the system of equations for disturbed motion, while functions $\mathcal{Z}_i(t)$ define the error of approximation of programming functions, and the de-

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viation of the solution of system /2.I/ from zero coincides with the deviation of solution $x_i(t)$ of system /I.I/ from the set periodic motion.

Isolating the linear part from functions $Z_i(z_1,\dots,z_n,t)$ According to some law, we shal have /2.I/ as:

$$\frac{dz_i}{dt}=\sum_{k=1}^{n}a_{ik}(t)z_k+R_i(z_1,\dots,z_n,t)+r_i(t) \qquad /2.2/$$

or in matrix vector form:

$$\frac{dz}{dt}=A(t)z+R(z,t)+r(t). \qquad /2.3/$$

Let us define the norm of vector z and norm of matrix A with the following relations*) $\|z\|=\max_{1\le i\le n}|z_i|$,

$$\|A\|=\max_{1\le i\le n}\sum_{k=1}^{n}|a_{ik}|.$$

Let D - be the region given by inequality

$$\|z\|\le\varepsilon,\quad 0\le t<\infty.$$

Let us assume for the system /2.3/:

a/ $A(t), R(z,t), r(t)$ are periodic in t with period ω.

b/ In region D $R(z,t)$ satisfies condition

$$\|R(z,t)-R(y,t)\|\le L\|z-y\|,\quad z\in D,\ y\in D.$$

c/ Functions $a_{ik}(t)$ and $R_i(z,t)$ with fixed z are absolutely integrable /according to Lebesgue/ in the interval $[0,\omega]$.

d/ Functions $r_i(t)$ are integrable /according to Lebesgue/ in the interval $[0,\omega]$.

e/ There exists a fundamental matrix $W(t,\tau)$ of the system $\frac{dz}{dt}=A(t)z$ which satisfies the conditions

$$W(\tau,\tau)=E \quad (E - \text{unit matrix}):$$

$$\|W(t,\tau)\|\le B e^{-\alpha(t-\tau)},\quad \alpha>1,\ \alpha>0$$

[illegible]

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*) [illegible] norm of vector [illegible] may be defined by some other of the known methods, then norm of matrix [illegible] will be defined [illegible]

It should be noted that conditions /b,c,d/ according to theorem of Caratheodory [9] /p. I20/ ensure existence and uniqueness of solutions of system /2.3/ in region D.

Let us denote $\|r(t)\|=\rho(t)$.

Theorem I.

Let all the conditions $(a-f)$ as well as one of the following conditions be satisfied:

A/ $\rho_0=\sup_{0\le t\le\omega}\rho(t)<\frac{\varepsilon}{2B^2}\lambda$,

B/ $\rho_1=\int_0^{\omega}\rho(t)\,dt<\frac{\varepsilon}{2B^2}e^{-\lambda\omega}(1-e^{-\lambda\omega})$,

C/ $\rho_2=\left(\int_0^{\omega}\rho^2(t)\,dt\right)^{1/2}<\frac{\varepsilon}{2B^2}\left(\frac{2\lambda}{e^{2\lambda\omega}-1}\right)^{1/2}(1-e^{-\lambda\omega})$.

Let $\delta=\varepsilon/2B$. The following claims are true:

I/ Any solution $z(t)$ of system /2.3/ is within the limits of region D when $\|z(t_0)\|\le\delta$.

2/ In the region D there exists an asymtotically stable trajectory to wich all the other trajectories emerging from the region D tend to approach.

§ 3. Let us now consider the case when the periodic motion $x_i=\psi_i(t)$, $i=1,2,\dots,n$ which is being constructed may possess a finite number of points of discontinuity of the first kind. In that case the programming functions being approximated should have the following form:

$$\varphi_i(t)=\psi_i'(t)-f_i(\psi_1(t),\dots,\psi_n(t),t)\quad i=1,2,\dots,n \qquad /3.1/$$

in points where there exists a derivative $\psi_i'(t)$ and

$$\varphi_i(t)=\eta_{ik}\,\delta(t-t_k) \qquad /3.2/$$

in points of discontinuity $t_1,t_2,\dots,t_m$. Here η_{ik} is the value of discontinuity of function $\psi_i(t)$ with $t=t_k$, while $\delta(t-t_k)$ is the Dirac function.

Assuming that we shall approximate the function $\psi_i(t)$ with functions of the same type we shall get the error of approximation as

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$$z_i(t) = z_i^0(t) + \sum_{k=1}^{m} z_{ik}\,\delta(t-t_k) \quad i = 1, 2, \dots, n, \qquad /3.3/$$

where $z_i^0(t)$ are absolutely integrable functions in $[0,\omega]$.

Substituting the variable $z_i = x_i - \psi_i(t)$ in system /I.I/ we again get the system /2.I/, and isolating the linear part according to some rule we get the system /2.3/.

One should note however, that even in the case when functions $f(x_1, \dots, x_n, t)$ of system /I.I/ are continuously differentiable by $x_1, \dots, x_n$, the matrix $A(t)$ and function $R(z,t)$ may happen to be discontinuous functions of time.

In a more general case when isolating the system of first approximation and solving the problem of stability of this system one should keep in mind the results obtained by M.A.Eiserman and F.R.Gantmacher /I0/.

Let
$$\rho(t) = \rho^0(t) + \sum_{k=1}^{m} \gamma_k\,\delta(t-t_k), \qquad /3.4/$$
where $\rho^0(t) = \max|z_i^0(t)|$ with $t = t_k$, $\gamma_k = \max_{i \le n} |z_{ik}|$.

Theorem 2.

Let the conditions /a, b, c, e, f be satisfied and functions $z_i(t)$ are given as /3.3/

Let $\delta = \frac{\varepsilon}{2B}$, and there exists inequality:
$$\rho_1 = \int_0^{\omega} \rho(t)\,dt = \int_0^{\omega} \rho^0(t)\,dt + \sum_{k=1}^{m} \gamma_k < \frac{\varepsilon}{2B^2} e^{\lambda\omega}(1 - e^{-\lambda\omega}).$$

Under these conditions both claims of Theorem I hold true.

It should be noted that Theorem 2 may be formulated for a more general case when the programming functions $\varphi_i(t)$ have limited variation.

In the above case one should consider the equation:
$$z(t) = W(t,t_0)z_0 + \int_{t_0}^{t} W(t,\tau)R(z,\tau)\,d\tau + \int_{t_0}^{t} W(t,\tau)\,dG,$$
where the second integral is the Stieltjes integral with integrating function $G(t) = (G_1(t), \dots, G_n(t))$, possessing limited variation. This case is similar to the above if we introduce generalized functions $z_i(t)$ defining them by condition $z_i(t) = z_{ik} G_i(t)$ in points where there exists a derivative, and by $z_i(t) = z_{ik}\,\delta(t-t_k)$ in points of discontinuity $G_i(t)$ /z_{ik} - value of discontinuity of function may be infinite /though denumerable/.

A still more general approach to the problem is possible considering the generalized differential equation which was introduced after generalization of integral of Perron by J.Kurzweil /II/.

§ 4. Using the method of proof of Theorem I and a certain idea given in article /I2/ we may apply part of the results of the above Theorem for the case when system /I.I/ as well as conditions being programmed $x_i = \psi_i(t)$ are not periodic.

Let ω be an arbitrary positive number, and let:
$$\rho(t) = \|z(t)\|,$$
$$h_0 = \sup_{0 \le t \le \infty} \rho(t), \quad h_1 = \sup_{0 \le t \le \infty} \int_t^{t+\omega} \rho(t)\,dt, \quad h_2 = \sup_{0 \le t \le \infty} \left(\int_t^{t+\omega} \rho^2(t)\,dt \right)^{1/2}.$$

Theorem 3.

Let the conditions / b, c, d, e, f / be satisfied /additional condition being that integrability of corresponding functions should take place is any interval $[t, t+\omega]$, $t \ge 0$/ while there exists at least one of the following inequalities:

A') $h_0 < \frac{\varepsilon}{2B}\lambda$,

B') $h_1 < \frac{\varepsilon}{2B} e^{\lambda\omega}(1 - e^{-\lambda\omega})$,

C') $h_2 < \frac{\varepsilon}{2B}\left(\frac{2\lambda}{e^{2\lambda\omega}}\right)^{1/2}(1 - e^{-\lambda\omega})$.

Then any solution $z(t)$ of system /2.I/ defined by condition $\|z(t_0)\| < \varepsilon/2B$ will remain within the limits of region δ with $t \ge t_0$.

In the above case the [illegible] of points of discontinuity

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Finally let us consider the case when the approximated motion $x_i = \psi_i(t)$ /non-periodic/ possesses isolated discontinuities of the first kind. Let us assume for the above case that once again we have the form /3.3/, and let us define the generalized function $\rho(t)$ according to /3.4/.

Let

$$h_1 = \sup_{0 \le t < \infty} \int_t^{t+\omega} \rho(t)\, dt .$$

Reasoning in a similar way we see that with $h_1 < \frac{\varepsilon}{2B} e^{-\lambda\omega}(1 - e^{-\lambda\omega})$ the solution of $z(t)$ system /2.3/, defined by condition $\| z(t_0) \| < \varepsilon / 2B$ will remain within the region D when $t \ge t_0$.

Thus in this latter case as well we may realize the desired programmed conditions within the limits of error ε .

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N. N. Bautin — Gorky State University named after N. I. Lobachevsky

Н. Н. БАУТИН

ТЕОРИЯ ТОЧЕЧНЫХ ПРЕОБРАЗОВАНИЙ И ДИНАМИЧЕСКАЯ ТЕОРИЯ ЧАСОВ

I. Escapements employed in watch making and in the construction of measuring devices consist of the two main parts:

- a balance or an oscillating regulator /either with the proper period or without it/, which keeps oscillatory motion, and
- a scape wheel rotated by a constant movement accurate to the forces of friction.

A scape wheel working either directly or by means of some intermediate link in the stationary regime, periodically interacts with a balance imparting to the latter the impulses which compensate friction losses. The other parts of a watch mechanism: the wheels system, an oscillation counter /hands/ etc.... together with an escapement define certain specific features of the mechanism corresponding to its purpose.

The main details of the escapement - a balance oscillating under the influence of both the forces of friction and the scape wheel impulses as well as the scape wheel itself with its moment that interacts with the balance - form a closed dynamic system.

A considerable number of works are devoted to the investigation of the separate parts of a watch mechanism. These works discuss the methods of stabilization of

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the characteristics of these parts when some parameters are varied.

The approach to the investigation of a watch mechanism as a set of parts working independently is peculiar to the clocks and watches theory as it was developed last century. The manuals on watch making based on this theory usually involve only the methods for designing different parts and the prescriptions for constructing the kinematics of the mechanism of a few widely spread designs.

The investigation of the separate blocks of a watch mechanism cannot however answer a considerable number of basic problems of the clocks and watches theory:

- on possible regimes of a watch operation,
- on the dependence of the selfoscillation period on the parameters of a system,
- on the role of the various dynamic and kinematic factors /the proper period,friction, the moments of inertia of details,pallet forms,etc .../ for the stabilization of the auto-oscillation period,
- on the estimation of various kinematic models of an escapement from the point of view of dynamics.

2. The treatment of the escapement mechanism as an integral dynamic system requires such mathematical means, which the science of the last century did not possess. During the whole of the last century chronometry as the science of exact time measuring and precision time keeping was based on the theory of the free pendulum and on the work of an English astronomer G.B.Airy. The work was dedicated to the investigation of the influence of the outer impulses, imparted to the oscillating pendulum,on the period of its oscillation. The work gave the leading ideas for designing and constructing various escapements and no doubt played an important role in chronometry.

Airy studied the pendulum which was swinging without attenuation under the influence of an instantaneous impulse. The result is know in chronometry as Airy's theorem. Airy established a certain relation between the direction and phase of the instantaneous impulse and the duration of a half oscillation of a pendulum. In chronometry this theorem was interpreted in the sense that a definite impulse distributed symmetrically on the both sides of the position of equilibrium does not influence in any way the period. Such impulse can be treated as a superposition of two equal impulses acting in the opposite direction on the period. Such point of view meets with approval of modern authors. For example in the book by A.L.Rawlings "The Science of Clocks and Watches", New-York,London,1948. Airy's considerations are employed for calculating "the escapement error" and comparative estimation of a series of escapements.

From the stand point of the modern theory, however,these views are to be seriously corrected. It turns out to be that in spite of the habitual notions /originated from Airy's work/ the change of the angle of an impulse in the process of operation does not still appear to be a sufficient condition for changing the period; on the other hand, the constancy of an impulse angle, even when its position in relation to that of the balance equilibrim is symmetrical, does not yet guarantee the independence of a period on its amplitude.

The requirements of the clocks and watches science of the 19th century to the mechanisms designed for time measuring /isochronism of the swinging of an oscillator and the

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fulfillment of Airy's theorem conditions - not to change the period while imparting the impulse/ played no doubt an essential part in the process of creation and choice of the construction of the chronometric and free anchor escapements. These requirements, however, do not appear to be necessary for the ideal design. The purpose of the escapement is the stabilization of the period and the problem will be solved if the period of autooscillation irrespective of its value is stabilized as to the changes of the system parameters. It is not essential here whether the autooscillation period coincides with the proper oscillation period of the system or not. The requirement of the coincidence of the autooscillation period of a watch with the proper oscillation period substitutes the problem of the period stabilization by another problem which in fact has no relation to the stabilization problem.

The existing mechanisms of the free anchor escapements /that took their form more than 100 years ago/ have as a characteristic feature some tendency to the symmetrization of the escapement and to the creation of the same conditions of imparting the impulse during the forward and backward balance movement. It appears, however, that the best conditions for stabilizing the period in the mechanism with a free anchor escapement are provided when we give up seemingly so natural symmetry in the anchor mechanism and use the nonsymmetrical mechanism at the expense of unequal width of pallets or of the unequal length of the anchor arms. When the construction is non-symmetrical it is possible to change /to straighten/ the dynamic characteristics of the escapement by displacing the position of equilibrium.

3. The mathematical apparatus that may be used for the investigation of the escapement as an integral dynamic system first appeared at the very end of the last century and was developed only in the recent decade. It is the apparatus of the theory on nonlinear oscillations including the qualitative theory of the differential equations and the theory of point transformations.

The autooscillation approach to the theory of watches first appeared in "The Vibration Theory" by A.A.Andronov and S.E.Chaikin published in 1937. The book discusses the simplest dynamic models of an escapement as the systems with one degree of freedom assuming that an impulse is imparted instantenously. The important point which should be especially stressed here is that "The Vibration Theory" presented the mathematical apparatus adequate to the character of the treated problems. The investigation is carried through without the assumption that any terms in equations of movement are small and thus without the assumption that the period of an oscillation is proximate/close/ to that of an autooscillation. The whole investigation of the structure of the decomporition of the phase-space into trajectories took the form of constructing the point transformation of a straight line into itself and by the investigation of the invariant points of the transformation.

The mathematical apparatus of the point transformations theory proved to be adequate to the problems arising in the dynamic theory of watches and proved to be effective for getting both the qualitative and quantitative results without necessity of imposing the requirements of the small non-linearity for the application. In its application the apparatus does not require any preliminary knowledge about the

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[illegible] character of the movements in the system as for instance the proximity to the conservative systems as it is required [illegible] of small parameters is [illegible]. [illegible] transformations gives the possibility to [illegible] the new [illegible], the very existence of which had not been [illegible] with [illegible].

[illegible] method of [illegible] transformations [illegible] to [illegible] [illegible]

[illegible] the foreground problem thereat is the global [illegible] of the general picture of all the processes without the aim to get an exact quantitative coincidence with the experiment. Nonlinearity in the equations of the movement of an escapement dynamic model either appears, [illegible] take into account the forces acting in the system, or is introduced by the kinematics of the system.

For minimizing the difficulties of the calculations, [illegible] very serious if one tries to take into account the nonlinearities in the equations of movement, as the [illegible] of both the kinematics and a mechanism dynamics one is bound to come over to the consideration of a simplified [illegible] the investigation of all the [illegible] and the investigation of the [illegible] properties of the escapements one can consider not the separate constructions but their simplified models with changed ("linearized") kinematics. [illegible] gives the possibility [illegible] comparatively simple analytical expressions characterizing the system dynamics (the relationship of the [illegible] oscillation period with such parameters as the stiffness of a balance spring, the moments of inertia of the [illegible] of an escapement, the torque on the pivot of [illegible] wheel, [illegible] and [illegible] friction in a system, etc.). [illegible] proximity between the [illegible] and the model [illegible] and under the ordinary simplifying assumptions allows to recalculate the construction parameters for those of the model. Each of the models, however, will reflect the dynamics of some certain type of real constructions, which can greatly differ by kinematics within each type. The calculations made for models can be applied therefore for a great number of constructions differing by kinematics.

5. The [illegible]

The investigation of the [illegible] [illegible]

in 1657 is known to be a very important advance. Huygens [illegible] the [illegible]

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advantage of the cycloidal pendulum over the ordinary one.

In 1944 L.I.Mandelstam set forward the problem to compare the derivatives of the period to the mode-parameters for the clocks with a pendulum and for the similar pre-Hallilean clock but without a restoring force. Mandelstam was sure that the comparison would allow to exactly formulate the dynamic peculiarities of the Hallileo-Huygens clock in relation to the pre-Hallilean clock and thus would allow to characterize exactly the role of the restoring force i.e. the role of the pendulum and the spring as the stabilizer of the autooscillation period of the clock.

The comparison of the regimes on meeting strokes in direction opposite to the movement of a pendulum and on pushing strokes in the direction of the moving pendulum allowed to solve Mandelstam's problem for the model under consideration and to find out the special role that the regimes on pushing strokes play for the stabilization of the autooscillation period. By this way we can also find out the role of friction for stabilizing the autooscillation period. It appears that with the proper period of the oscillation present, friction contributes to the stabilization and sometimes just the substantial friction but not a small one.

The investigation of a dynamic problem is brought to the construction and investigation of the point transformation of a certain three-dimensional phase space into itself, just the space of the postimpulse states that is generated by the equations of the model movement.

Description of the Dynamic Model

Let the scape wheel be realized as an infinite ribbon equipped with teeth and a balance moves progressively between the ribbons of the scape wheel and is equipped with the pallets f and g subjected to the strokes. We shall assume that in the intervals between the strokes the scape wheel has only one tooth put forward, which instantly goes away after a stroke and a new one comes according to the law given below. Let the number of all the scape wheel teeth be N and the teeth be situated at the equal distances $2a$ and successively enumerated. Let us assume that the strokes are delivered successively by the teeth having the numbers

$$j, j+n(\mathrm{mod}\,N), j-1, j-1+n(\mathrm{mod}\,N) \quad \text{etc...}$$

and that the balance is linked with the equilibrium point by the spring and the equilibrium position of the balance is constructively chosen so that at the moment when j-tooth is above the equilibrium position on the upper ribbon, teeth $j+n$ and $j+n+1$ on the lower ribbon are situated symmetrically in relation to equilibrium position and are at the equal distances from it.

The equations of the movement for the time intervals between the strokes can be written in their usual terms as following:

$$m\ddot{x} + b\dot{x} + \varkappa_0 x = 0, \quad M\ddot{y} = \pm P_0. \qquad /1/$$

where "+" is taken for the teeth of the lower ribbon and "-" for the teeth of the upper one of the scape wheel. Let us assume that the strokes are instantaneous and that prestroke velocities $\bar{x}, \bar{y}$ are related with the poststroke velocities $\dot{x}, \dot{y}$ by means of the following rules usual for not viscoelastic strokes

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[illegible] the restoring coefficient.

Equations /1/ and /2/ allow to write the point transformation Ωfg and Ωgf for the transition between the poststroke states

$f(x_0, y_0 = x_0/\lambda, \dot{y}_0 = \dot{x}_0/\lambda)$ and $f(x_1, y_1 = x_1/\lambda, \dot{y}_1 = \dot{x}_1/\lambda)$

which are generated by the successive strokes upon the pallets f and g or vice versa g and f. Because of the symmetry of the phase space the system /1/ is not changed when $x, y, \dot{x}, \dot{y}$ are replaced by $-x, -y, -\dot{x}, -\dot{y}$. It is convenient to identify the symmetrical states and to consider only the type of the passages /e.g. the one generated by the strokes first for the pallet f and then g/. The consideration of the sequence of the poststroke states is now brought to the consideration of the space transformation of the poststroke states of $x, \dot{x}, y/(x=y=0)$ into itself at the unlimited repetition of the transformation $\overline{T} = \overline{\Omega fg}$ where the dash line above denotes the passage after the transformation of Ωfg into the symmetrical point. The model will approximately describe the movement of the construction if we take

$$x = \varphi, \quad y = \left(\frac{r}{l} tg\gamma\right)\psi, \quad m = J, \quad M = \frac{\pi l^2}{r^2} ctg^2\gamma,$$

$$P_0 = \frac{ql}{r} ctg\gamma.$$

The point transformation Ωfg :

$$x_1 = -\alpha\left\{\frac{\lambda(\mu x_0 - \dot{x}_0)}{\mu - \lambda} e^{\lambda\tau} + \frac{\mu(\lambda x_0 - \dot{x}_0)}{\lambda - \mu} e^{\mu\tau}\right\} - (1-\alpha)(\dot{y}_0 - 2\rho\tau),$$

$$\dot{y}_1 = -\beta\left\{\frac{\lambda(\mu x_0 - \dot{x}_0)}{\mu - \lambda} e^{\lambda\tau} + \frac{\mu(\lambda x_0 - \dot{x}_0)}{\lambda - \mu} e^{\mu\tau}\right\} + (1-\beta)(\dot{y}_0 - 2\rho\tau),$$

$$x_1 = x_0 + \rho\tau^2 + \dot{y}_0\tau - a,$$

where λ and μ are the roots of the characteristic equation $\zeta^2 - 2h\zeta + \varkappa = 0$ /real or complex/, $h = \frac{\delta}{2m}$, $\varkappa = \frac{\varkappa_0}{m}$, $\rho = \frac{P_0}{2M}$

and τ is the least positive root of equation

$$\frac{\mu x_0 - \dot{x}_0}{\mu - \lambda} e^{\lambda\tau} + \frac{\lambda x_0 - \dot{x}_0}{\lambda - \mu} e^{\mu\tau} + x_0 + \rho\tau^2 + \dot{y}_0\tau - a = 0.$$

The simple fixed points of the transformation defined by conditions

$$\dot{x}_1 = \dot{x}_0, \quad \dot{y}_1 = \dot{y}_0, \quad x_1 = x_0$$

correspond to the symmetrical periodic solution. The period 2τ is connected with the parameters of the model by equation

$$\rho\tau^2[2\alpha - \beta + 2(1-\alpha)U(h, \varkappa; \tau)] = a\beta, \quad /3/$$

where

$$U(h, \varkappa; \tau) = \frac{e^{-h\tau}(\cos\omega\tau + \frac{h}{\omega}\sin\omega\tau) + 1}{e^{-2h\tau} + 2e^{-h\tau}\cos\omega\tau + 1} \quad /4/ \qquad (\omega = \sqrt{\varkappa - h^2}).$$

The fixed point will be stable if the following conditions are fulfilled:

$$A \equiv \beta(2\alpha - \beta)\frac{1}{g} + \beta(1-\alpha)[2(1-\bar{\alpha}_1) - g\tau\Phi_1] > 0,$$

$$B \equiv 2[1 + (1-\beta)^2] + [\alpha^2 + (\alpha-\beta)^2](b_1 - a_1) + [2(1-\alpha)^2 - (1-\beta)^2 - 1](1-\bar{\alpha}_1) - 2g(1-\bar{\alpha}_1)^2(1-\alpha)^2 - \beta(1-\alpha)[2\bar{\alpha}_0 + g\tau\Phi_2] > 0,$$

where $C \equiv 4[1 + (\alpha-\beta)^2 b_1] - B > 0,$

$$\frac{1}{g} = (e^{\mu\tau} + 1)/(e^{\lambda\tau} + 1), \quad \bar{\alpha}_0 = \frac{e^{\mu\tau} - e^{\lambda\tau}}{(\mu - \lambda)\tau}, \quad /5/$$

$$\bar{\alpha}_1 = \frac{\mu e^{\lambda\tau} - \lambda e^{\mu\tau}}{\mu - \lambda}, \quad b_1 = e^{(\mu+\lambda)\tau}, \quad a_1 = \frac{\mu e^{\mu\tau} - \lambda e^{\lambda\tau}}{\mu - \lambda},$$

$$\Phi_1 = \frac{1}{\mu - \lambda}[\mu^2 e^{\mu\tau}(e^{\lambda\tau} + 1)^2 - \lambda^2 e^{\lambda\tau}(e^{\mu\tau} + 1)^2],$$

$$\Phi_2 = \frac{1}{\mu - \lambda}[\mu^2 e^{\mu\tau}(e^{2\lambda\tau} - 1) - \lambda^2 e^{\lambda\tau}(e^{2\mu\tau} - 1)].$$

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In the case of the model of the watch without the proper period $\alpha = 0$. From equation /4/ we get

$$U(h, 0, \tau) = \vartheta(h\tau) = \frac{1}{1+e^{-2h\tau}} .$$

The difference between the dynamic properties of the clocks and watches with the proper period and those without one is closely connected with the difference in the behaviour of the sequences of the functions $U_\nu(h_\nu, \alpha, \tau)$ and $\vartheta_\nu(h_\nu\tau)$ when $h_\nu \to 0$. It is not difficult to show that for any $\chi > 0$ and any positive $\tau \neq \frac{\pi}{\sqrt{\alpha}}(2n+1)\,(n = 0,1,\ldots)$ the limit values of the functions U_ν and ϑ_ν for the Hallileo-Huygens clock and for the pre-Hallilean clock coincide:

$$\lim_{h_\nu \to 0} U_\nu(h_\nu, \alpha, \tau) = \lim_{h_\nu \to 0} \vartheta_\nu(h_\nu\tau) = \frac{1}{2} .$$

However, the sequence of the functions $U_\nu(h_\nu, \alpha, \tau)$ for the clocks with the proper period /contrary to the sequence of thr functions $\vartheta_\nu(h_\nu\tau)$ for the clocks without the proper period/ converges with $h_\nu \to 0$ to the limit function nonuniformity and the limit function has discontinuity when

$$\tau = \frac{\pi}{\sqrt{\alpha}}(2n+1).$$

When $h_\nu \to 0$ the sequence of the curves $U_\nu(h_\nu, \alpha, \tau)$ converges to the limit curve that contains "the segment" $U \geq \frac{1}{2}$ when τ is equal to $\frac{\pi}{\sqrt{\alpha}}(2n+1)$.

Fig.3 shows the curves of the dependence of a halfperiod τ on ρ value characterizing the torque on the scape wheel pivot. The curves are constructed according to equations /3/ and /4/ for the different values of h and fixed valies of the rest parameters. The presence of a horizontal segment AB of the limit curve is a remarkable

fact. The presence of such a segment is a characteristic feature of the proper period watch and it allows to find the parts of the curves with [illegible] however small they be for small enough h and for the values of τ close to the value $\frac{\pi}{\sqrt{\alpha}}$ /the latter characterizing a half-period of the proper oscillation of the balance/ . Heavy solid lines on Fig.3 show the parts of the curves which correspond to the stable regimes, for which conditions /5/ are fulfilled. One can see from Fig.3 that with the domain of the stable regimes varies in such a way that just the parts of the curves with little slope adjacent to AB segment turn to be unstable /since the expression for coefficient β in /5/ because of the term $g\Phi_2$ becomes negative when $h \to 0$ and $\omega\tau \to \pi$ /. One cannot therefore vary only one parameter h to satisfy both the requirements for the given smallness $\frac{\partial\tau}{\partial\rho}$ and for the stability conditions.

It is possible to show, however, that for the proper period watches we can simultaneously fulfil both conditions varying together with the parameters of the system. Fig.4 shows the dependence curves of a half-period τ on the value of ρ which characterizes the torque on the scape wheel pivot. These curves are constructed for the fixed values of parameters $\alpha, \varkappa$ and k $(\alpha = 1, \varkappa = 1, k = 0{,}5)$ and various values of h and τ satisfying the relation

$$\varkappa = \frac{1-k}{1+f} = 1 - 2h .$$

Heavy solid lines show the parts of the curves corresponding to the stable regimes. The points M and N on each curve show the parts corresponding to the regimes on pushing strokes. It is easy to see that M and N lay on the "separating" curve, which passes through the end point

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B of the segment AE and is tangent in this point to the branch CB of the limit curve. The figure illustrates the role of pushing strokes for stabilizing the autooscillation period. Fig. 4 shows how the parts of the curves, which correspond to pushing strokes, draw to the segment of the horizontal limit curve with $h \to 0$ and $\alpha \to 1$.

The Role of Friction for Stabilizing an Autooscillation Period

Let us compare the limit curves, when $h = 0$ /i.e. assuming the absense of friction/ for the watches with the proper period of oscillations and for the similar watches without the proper period. Since a balance energy dissipation is due only to the losses at not quite tough stroke /$\kappa \neq 1$/ when $h = 0$,then both of the watches will work in the meeting strokes regimes. The limit curves will coincide wherever there is the coincidence of the limit values of the functions $U(h, \mathcal{E}, \tau)$ and $\vartheta(h, \tau)$, therefore the autooscillation periods for the watches of both types will be exactly the same if friction is absent and all the other parameters will be equal /though the amplitudes and the places of the mutual co-strokes are different/.

Let us compare now the dynamic characteristics for the watch having the proper period of oscillation and for the similar watch without it but with considerable friction in system. The steepness of the curves for models with $\mathcal{E} = 0$ and $\mathcal{E} \neq 0$ with $h \neq 0$ will substantially differ at the interval of the values of ρ , which

corresponds to a rectilinear segment of the limit curve. However, as it has already been found out, the maintaining of the stability on the curve parts - however close they be to the segment mentioned - requires special selection of the model parameters / J must be great/. If the values of the parameter J are arbitrary /for example when $J < 1$ / the maintaining of the stability at the interval of ρ values corresponding to the segment becomes possible when we introduce the considerable friction into the system. The role of friction here is to suppress the increase of the value

$$g = \left[e^{-2h\tau} + 2e^{-h\tau}\cos\omega\tau + 1\right]^{-1}$$

under the conditions of stability /5/, when τ tends to the value of $\frac{\pi}{\sqrt{\kappa\mathcal{E}}}$,and by means of that to secure stability. Fig. 5 presents the dynamic characteristics on the parameters ρ constructed for different h /when $\mathcal{E} = 1$, $J = 0{,}5$, $\kappa = 0{,}5$, $\alpha = 1$ /.

Heavy solid lines show the parts of the curves corresponding to the stable regime. For $h = 0{,}2$ in the model with the proper period, stability is secured along the whole length of the curve part adjacent to the segment AB of the limit curve "from below" /for $h = 0{,}1$ just the part adjacent to AE turns to be unstable/. Let us note that in real constructions the introduction of the proper period into the system at the same time increases friction on account of the internal friction in the spring material.

[illegible]'s Regulator with an Elastic Plate Spring.

The theory of the triggering regulator with a plate spring is also brought to constructive end investigation the transformation [illegible] of the three-dimensional phase space of the postimpulse states. The investigation reveals

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in particular that for the model under consideration for some fixed values of paremeters various stable periodic regimes are possible. Up till now such a possibility has not been taken into account while designing regulators but the author had some occasion to observe the presence of the two stable regimes on the industrial set during the operation and he caused the jumps from one regime to the other by either slowing down or pushing up /and thus accelerating/ the escape wheel with the hand. The investigation of the point transformation also reveals some special and peculiar role of friction substantially influencing the system behaviour. Only the presence of friction causes the appearance of such domains of the parameters values for which the various stable regimes are possible depending on the initial conditions.

Description of the Dynamic Model

Let the escape wheel be realized as an infinite ribbon equipped with teeth situated at the equal distances a and driven to move by some constant force and let the spring fixed to a balance moving progressively along the direction lines at the angle of φ to the direction of the escape wheel ribbon movement, the balance being linked with the equilibrium position of the spring (Fig. 7). We shall assume, that at the intervals between the strokes the escape wheel has one prominent tooth that instantly goes away after stroke and the following tooth comes in succession.

The movement equations for the time intervals between the strokes will be:

$$m\ddot{x} + b\dot{x} + \varkappa_0 x = 0, \quad M\ddot{y} = P. \qquad /6/$$

Let us assume the strokes to be instantaneous: then the prestroke velocities $\bar{\dot{x}}, \bar{\dot{y}}$ will be connected with the poststroke velocities $\dot{x}, \dot{y}$ by the relations:

$$\dot{x} = \alpha\bar{\dot{x}} + \sigma(1-\alpha)\bar{\dot{y}}, \quad \dot{y} = \beta\bar{\dot{x}} + (1-\sigma\beta)\bar{\dot{y}}, \qquad /7/$$

where $\sigma = \cos\varphi$, $\beta = \frac{m\sigma(1+k)}{M+m\sigma^2}$, $\alpha = \sigma\beta - k$

and k - coefficient of restoration.

The Point Transformation

Equations /6/ and /7/ allow to write the point transformation Ω for transitions between the successive post-stroke states:

$$\{x_0, y_0 (\equiv \tfrac{x_0}{\sigma}), \dot{x}_0, \dot{y}_0\} \text{ and } \{x_1, y_1 (\equiv \tfrac{x_1}{\sigma}), \dot{x}_1, \dot{y}_1\}$$

generated by the successive strokes of the escape wheel teeth upon a balance.

Let: $\frac{x_0}{m} = \varkappa$, $\frac{b}{2m} = h$, $\sqrt{\varkappa^2 - h^2} = \omega$, $\frac{x}{a} = \xi$,

$\frac{\dot{x}}{a\omega} = \dot{\xi}$, $\frac{\dot{y}}{a\omega} = \dot{\varrho}$, $\frac{h}{\omega} = H$, $\frac{P}{2aM\omega^2} = \rho$, $\omega t = \tau$.

The point transformation Ω will then assume the form

$$\dot{\xi}_1 = \alpha e^{-H\tau}\left[\dot{\xi}_0(\cos\tau - H\sin\tau) - \xi_0(1+H^2)\sin\tau\right] + \sigma(1-\alpha)\left[2\rho\tau + \dot{\varrho}_0\right],$$

$$\dot{\varrho}_1 = \beta e^{-H\tau}\left[\xi_0(\cos\tau - H\sin\tau) - \dot{\varrho}_0\sin\tau\right] + (1-\sigma\beta)\left[2\rho\tau + \dot{\varrho}_0\right],$$

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$$\xi_1 = e^{-H\tau}\left[\xi_0 \sin\tau + \dot{\xi}_0(\cos\tau + H\sin\tau)\right],$$

where τ is the least positive root of the equation:

$$e^{-H\tau}\left[\xi_0 \sin\tau + \dot{\xi}_0(\cos\tau + H\sin\tau) - (\rho\tau^2 - 1)\sigma - \dot{\xi}_0\tau - \xi_0\right] = 0.$$

The fixed point and the corresponding periodic solution will be stable /i.e. all the three roots of the corresponding equation in the finite differences will be less than a unit by module / if the following conditions are fulfilled:

$$A \equiv \sigma\beta(2\alpha - \sigma\beta)\frac{1}{g} + \sigma\beta(1-\alpha)[2(1+\bar{a}_1) + g\Phi_1] > 0,$$

$$B = [\alpha^2 + (\alpha - \sigma\beta)^2](a_1 + b_1) + 2\left[1 + (1 - \sigma\beta)^2\right] - 2g(1-\alpha)^2(1+\bar{a}_1) + [2(1-\alpha)^2 - (1-\sigma\beta)^2 - 1](1+\bar{a}_1) + \sigma\beta(1-\alpha)[2a_0 + g\Phi_2\cdot\Theta] > 0,$$

$$C = 4[1 + (\alpha - \sigma\beta)^2 b_1] - B > 0,$$

where

$$\frac{1}{g} = e^{2H\Theta} - 2e^{H\Theta}\cos\Theta + 1, \quad b_1 = e^{-2H\Theta}, \quad \bar{a}_0 = \frac{e^{-H\Theta}\sin\Theta}{\Theta},$$

$$1 + \bar{a}_1 = 1 - e^{-H\Theta}(\cos\Theta + H\sin\Theta), \quad a_1 + b_1 = e^{-2H\Theta} + e^{-H\Theta}(\cos\Theta - H\sin\Theta),$$

$$\Phi_1 = e^{-H\Theta}\left[4He^{-H\Theta} + (1-H^2)(1-e^{-2H\Theta})\sin\Theta - 2H(1+e^{-2H\Theta})\cos\Theta\right],$$

$$\Phi_2 = e^{-H\Theta}\left[(e^{-2H\Theta} + 1)(1-H^2)\sin\Theta + 2H(1-e^{-2H\Theta})\cos\Theta\right].$$

For a simple fixed point of the transformation the non-dimensional period Θ is connected with the model parameters by equation

$$\rho\Theta^2\left[2\alpha - \sigma\beta + (1-\alpha)\frac{e^{H\Theta} - H\sin\Theta - \cos\Theta}{ch\,H\Theta - \cos\Theta}\right] = \sigma\beta. \quad /10/$$

Nondimensional parameters ρ and H characterize correspondingly the torque on the pivot of the scape wheel and tough friction.

The point transformation Ω will have as many fixed points as there are positive roots of equation /10/.

Let us compare dynamic characteristics of the regulator when friction is absent and is present in system.

1.° When friction is absent from the system /$H = 0$ / and the values of all the other parameters are fixed there exists a single fixed point of transformation /8/, which can be stable and unstable. The presence of friction /$H \neq 0$ / substantially influences the system behaviour.

2.° When $H > 0$ and α/ $0 < \alpha < 1$ / are fixed we can chose the fixed J, σ and ρ in such a way, that the transformation will have more than one stable fixed point.

The regime in which a regulator will work, when there are several stable periodic solutions, will depend on the initial conditios. With an accidental change of the initial conditions /e.g. as a result of some push or commotion/ the regulator can jump from one regime to some other.

3.° When $H > 0$ the interaction of the scape wheel and the plate in a stable regime can be either on meeting or on pushing strokes. For fixed $H > 0$ the condition $\cos\Theta - H\sin\Theta - e^{-H\Theta} > 0$ distinguishes the intervals of the values Θ /including either a period of the proper oscillation of the plate or its whole multiple values / inside of which the system can work in the pushing strokes regime, when the conditions of /9/ are satisfied.

4.° When $H = 0$ the moments on the scape wheel calculated from equation /10/ are universaly proportional to the squares of the periods. When $H > 0$ the deviation from this law will be the more the closer the autooscillation period to that of the plate proper oscillation or to its multiple. The curves of the dependence of a period on the torque are presented on the upper part of Fig. 8. These curves are plotted for fixed values of the parameters

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$J = \frac{1}{75}$, $\varphi = 45°$, $k = 0{,}5$, $H = 0$, $H = 0{,}001$

according to the equation /IO/.

The curves showing the dependence of the prestroke velocity of the plate and the place of a stroke on the period Θ are given in the lower part of Fig. 8 in the same scale along the axis Θ for $H = 0{,}001$. The parts AB and CD on the stable section of the curve /ρ(2)/ correspond to the pushing strokes /$\bar{\eta} > 0$/. Increasing ρ one can pass into such a domain of the values of parameters where there exists only one stable periodic movement.

Chronometric and Free Anchor Escapement.

Close by their dynamics the models of the chronometric and free anchor escapement are treated in the idealization of the impulse as a protracted one. The investigation of the dynamic problem - because of the presence of the constructive step of the scape wheel - is brought to the construction and investigation of the point transformation of a straight line into itself.

Among the dynamic parameters friction plays more a special part. Due to many accidental reasons friction can vary greatly, this variation - contrary to the other dynamic parameters - being not related to any individual part of the mechanism. The investigation of the friction influence and the means of the stabilization period relative to the variation of friction requires therefore the consideration of the escapement as a closed dynamic system.

It should be noted that the influence of friction on

the work of a real mechanism is brought to the variation of the two parameters : the friction for a balance and the torque on the axis of the scape wheel. The variation of the friction in the wheels system or in the turns of the winding of the spring is brought to changing the torque. Friction can sharply variate not only due to the lubricant quality changes but with a mere change of the balance pivots position or wheels system as well. Therefore to find out the character of dependence of the autooscillation period upon the two parameters mentioned as well as to find the factors, capable to change this character, is of considerable practical interest. On the chronometric escapement model /which is simple in design/ one can find out the factors defining the properties of the escapement dynamic characteristics, i.e. the curves of the dependence of an autooscillation period upon the parameters /on the friction for a balance and on the torque of the scape wheel pivot/. These factors can be seen when one investigates the "equal period curves" in respect of the parameters ρ and r on the phase plane of the balance states. The character of the dependence of a period on an amplitude /changing when the torque and friction changes/ is defined by the position of "the impulse start curve" relative to the corresponding curves of an equal period.

It appears to be possible to influence the relative location of the impulse start curve and the family of the equilibrium position by means of changing the balance arm curves by changing the latter inside the impulse angle we can turn an increasing dynamic characteristics into a decreasing one and vice versa.

The comparison of the models with the chronometric and free anchor escapements allows to understand the cause

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of the difference between the chronometric escapement dynamic characteristics and those of existing anchor escapement designs. The difference is not caused by ana deep contrasts in the dynamic properties of the models but is due only to the peculiarities of the existing designs formed historically. The investigation of the free anchor escapement allows therefore to show the ways for perfection of the existing designs of the free anchor escapement / the methods of the rectification of the escapement dynamic characteristics/.

A. Chronometric Escapement

While investigating the dynamic of an escapement one can neglect those kinematics peculiarities of the construction which are the result of non-coincidence of the traj - ectories of the scape wheel tooth and a balance impulse stone during an impulse.

This assumption allows to change a rotating movement of the scape wheel and of the balance for a rectilinear one.

The equations for the model will be:

$m\ddot{x} + kx - \alpha x^3 = -R\frac{\dot{x}}{|\dot{x}|}$, /for the balance/

$m\ddot{x} + kx - \alpha x^3 = -R - M_0 - uP$, /with releasing/

$(m+M)\ddot{x} + kx - \alpha x^3 = P - R$, /during impulse/

$M\ddot{y} = P$, /for a scape wheel /

where α is a coefficient, characterizing nonlinearity of the spring, M_0 is the mean value of the resistance of the spring moment, the spring being of the quiescence /rest/ stone, u is a coefficient characterizing the triggering mechanism, P is a constant force imparting an impulse and making the scape wheel to rotate.

The phase space of the model and bringing up the problem to the point transformations.

The phase space of the model with two degrees of freedom is a fourdimensional one. However, due to the assumed idealization of the impulse the scape wheel and the balance are connected kinematically during the impulse and this brings to the degeneration of the phase space. The coincidence of the coordinates and the velocities of the scape wheel with the balance /at the impulse/ during the work of the escapement - which is a peculiar feature for the wide class of the models of the clock escapements, in which the impulse is idealized as a protracted one - allows one to bring the investigation of the limit points of the trajectories as well as of the structure of the phase portrait to the investigation of the limit points and the phase portrait "projections" of the trajectories on the plane of the coordinater, which characterize the balance position /on the plane $x, \dot{x}$ /.

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Model with Linear Restoring Force and One Degree of Freedom.

The factors defining the dynamic characteristics of an escapement are most clearly seen on the simplest models with one degree of freedom if we neglect the mass of the escapement wheel and assume that after releasing the scape wheel it instantaneously overtakes the balance. Then the impulse start will coincide with the balance coordinate at which the releasing of the scape wheel takes place and the impulse "angle" /interval/ will be constant during the escapement work /the end of the impulse - the coordinate of the outlet of the scape wheel out of gear/.

In order to simplify the model still more we shall not take into account for the present the energy losses for releasing the scape wheel assuming that the interval for the releasing of the scape wheel is equal to zero.

The equations for the motion in the nondimensional varieties: $\ddot{\xi} + \xi = -r\frac{\dot{\xi}}{|\dot{\xi}|}$; $\ddot{\xi} + \xi = \rho - r$,

Let us denote by ϑ, τ and ϑ' the time of the motion of the dimensional point along the segments of the trajectories $\mathcal{SE}$ and $\mathcal{ES}'$ /Fig.II/ and denote by $\dot{a}$ and $\dot{b}$ the $\dot{\xi}$ values corresponding to $\xi = a$ and $\xi = b$. The set of equations

$$\sin\vartheta = \frac{\dot{a}}{\xi_o - r}, \quad \cos\vartheta = -\frac{a + r}{\xi_o - r}, \qquad /11/$$

$$\left.\begin{aligned} \sin\tau &= \frac{\dot{a}(b - r - \rho) - \dot{b}(a + r - \rho)}{\dot{a}^2 + (a + r - \rho)^2} \\ \cos\tau &= \frac{\dot{a}\dot{b} + (a + r - \rho)(b + r - \rho)}{\dot{a}^2 + (a + r - \rho)^2} \end{aligned}\right\} \qquad /12/$$

$$\sin\vartheta' = \frac{\dot{b}}{\xi_1 + 3r}, \quad \cos\vartheta' = \frac{(b + r)}{\xi_1 + 3r} \qquad /13/$$

allows starting from the value of $\xi = -\xi_o$ on ξ axis successively to find the values of $\vartheta, \dot{a}, \tau, \dot{b}, \vartheta'$ and at last $\xi = -\xi$, and thus defines point transformation $T = SES'$ of the segment G into itself. The transformations S, E and S' are given correspondingly by equations /11/, /12/ and /13/. From equation /12/ and /13/ it follows:

$$(\xi_1 + 3r)^2 = (\xi_o - r)^2 + 2\rho(b - a).$$

The single fixed point of the transformation: $\xi_r = \xi_o = \frac{\rho(b-a)}{4r} - r \equiv \xi^*$ is always stable.

The Dynamic Characteristics of an Escapement and Factors Defining Their Character.

The curves expressing the relation of the autooscillation period to some of the system parameters /when all the other parameters are fixed/ will be called dynamic characteristics of an escapement of these parameters.

Let us investigate the factors which define the dynamic characteristics of an escapement of the parameter /of the torque on the scape wheel pivot/.

Assuming $\xi_1 = \xi_o$ and fixing the period value

$$T = \pi + \vartheta + \tau + \vartheta' = 2\pi + \lambda$$

/λ - constant/, one can consider equations /11/, /12/, /13/ - when b and r are fixed - as the parametric equations /with parameter ρ / of the equal period curves of the parameter ρ , these curves representing the locus of the impulse starts on $\xi, \dot{\xi}$ plane /when the fixed coordinate of the end of the impulse $\xi = b$ / at the moment of which the period maintains a constant value irrespective of the torque ρ .

The equations of the curves with equal period in the plane $\xi, \dot{\xi} / a, \dot{a}$ / can be written as follows:

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$$a=\left(4r-\frac{b+r}{\cos\theta'}\right)\cos\theta-r,\quad \dot{a}=-\left(4r-\frac{b+r}{\cos\theta'}\right)\sin\theta\,\dot{\theta}\quad /14/$$

where one can take for the parameters the values θ, τ or θ', which are connected among themselves by the relations:

$$tg\frac{\tau}{2}=\frac{(b+r)(\cos\theta'+\cos\theta)-4r\cos\theta'\cos\theta}{(b+r)(\sin\theta'+\sin\theta)-4r\cos\theta'\sin\theta},$$

$$\theta+\tau+\theta'=\pi+\lambda \qquad /15/$$

and with the parameter ρ by the relation

$$(\rho-r-a)=(a+r)tg\theta\cos\tau+(b+r)tg\theta'.$$

Let us invertigate this curve. From equation /15/ we find that when $\lambda=0$ we shall have $\theta+\frac{\tau}{2}=\frac{\tau}{2}+\theta'=\frac{\pi}{2}$ and consequently

$$a=\left(4r-\frac{b+r}{\sin\frac{\tau}{2}}\right)\sin\frac{\tau}{2}-r,\quad \dot{a}=-\left(4r-\frac{b+r}{\sin\frac{\tau}{2}}\right)\cos\frac{\tau}{2}\dot{\tau}>0 \quad /16/$$

Curve /16/ has vertical asymptote $a=-b-2r$ /when $\tau\to 0$ / and lies entirely inside the band of the maximum with $4r$. Curves /14/ which correspond to the fixed $\lambda<0$ or $\lambda>0$, are situated on the phase plane correspondingly to the left or to the right from curve /16/. Contrary to curve /16/ the other curves have no vertical asymptote situated like "a fan" from the both sides of curve /16/. Fig.12 presents the curves of the equal period, constructed for the series of the values of λ and with fixed b and r.

It is not difficult to see that the character of the dependence of the period upon the amplitude is defined by the position of the impulse start curve relative to the curves of an equal period. Period T will not depend upon amplitude / $\frac{dT}{d\xi^*}=0$ /, if the impulse start curve coincides with one of the curves of an equal period. This period will be increasing / $\frac{dT}{d\xi^*}>0$ / or decreasing / $\frac{dT}{d\xi^*}<0$ / function of an amplitude, if the impulse start curve is <u>the curve without contact</u> for the family of the curves of an equal period. Finally, the curve of the dependence of a period upon an amplitude will have in some point a horizontal tangent line / $\frac{dT}{d\xi^*}=0$ / if the impulse start curve is tangent to one of the curves of an equal period.

In the model under consideration /with one degree of freedom/ the curve of the impulse start is the straight line $\xi=a$. If then $b+r>0$, depending upon the position of the curve $\xi=a$ in relation to the equal period curve for $\lambda=0$, the demonstration point on the straight line may intersect the equal period curves /while moving upward the line/ in the direction of the increasin as well as decreasing λ. Therefore, changing value a one can arbitrarily realize the <u>increazing</u> as well as <u>decreasing</u> dynamic characteristics under the parameters ρ. Changing the position of the straight line $\xi=a$ we change the quality character of the dynamic characteristics /we come over from an increasing characteristics to a decreasing one and vice versa/ on account of an impulse angle change /the length of the integral $b-a$ /. However, we can obtain some changes of the dynamic characteristics in a desired direction without changing the impulse angle by means of mere change of the balance equilibrium. In fact, the straight line symmetrical to the axis ξ with the line of the end of an impulse $\xi=b$, lies inside the band /not more than 4 r width/ containing the curve of an equal period for $\lambda<0$. Therefore the displacement [illegible]

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with the unchanged impulse angle displaces the band together with the equal period curve / $\lambda = 0$ / in this or that direction.

On account of the change of the balance equilibrium position this curve can take an arbitrary position relative to the line $\xi = a$.

Fig.13 shows the dynamic characteristics change of the parameter ρ , when the angle of the impulse $b - a$ is unchanged, on account of some change of a balance equilibrium position.

Analytical character of the dependence of the period $T = \theta + \tau + \theta' + \pi$ upon the position of an impulse angle is defined by the value $\frac{dT}{da}$. One can show that

$$\frac{dT}{da} = \rho \int_a^b \left[\xi_0^2 - \xi^2 - 2r(\xi + \xi_0) + 2\rho(\xi - a) \right]^{-\frac{3}{2}} d\xi > 0.$$

and therefore by changing the impulse angle position relative to the position of the balance equilibrium one can change the period /for displacement of the dynamic characteristics/ in an arbitrary direction.

The consideration analogous to that made above, shows that by the same way one can change as well the qualitative character of the dynamic characteristics of the parameter by changing the position of the balance equilibrium.

Model with the Linear Restoring Force and Two Degrees of Freedom.

Nondimensional equations of motion :

$$\ddot{\xi} + \xi = -r - \frac{\dot{\xi}}{|\dot{\xi}|} \qquad \text{/for free motion of the balance/}$$

$$\ddot{\xi} + \xi = -r - r_0 - \mu\rho \qquad \text{/with releasing/}$$

$$\ddot{\xi} + \omega^2 \xi = \omega^2 (\rho - r) \qquad \text{/during an impulse/}$$

$$\ddot{\zeta} = J\rho \qquad \text{/ for the scape wheel/}$$

The point transformation /Fig.14/

$$(\xi_1 + 3r)^2 = (\xi_0 - r)^2 + \omega^2 (\sigma + r - b)^2 - \omega^2 (b + r - \rho)^2 \qquad /17/$$

$$+ (b + r)^2 - (\sigma - r)^2 - 2(a - c)(r_0 + \mu\rho).$$

$$(\xi_0 - r)^2 = (a + r)^2 + \frac{(\sigma + r - (a + r)\cos\sqrt{2(\sigma - s)/J\rho})^2}{\sin^2 \sqrt{2(\sigma - s)/J\rho}} +$$

$$+ 2(a - c)(r_0 + \mu\rho).$$

Here σ and s are correspondingly the coordinates of the beginning and the end of "the scape wheel angle" of releasing and σ and b are the coordinates of the beginning and the end of "the impulse angle".

Equations /17/ can be considered as the equations of the curves in the plane ζ, ξ.

In a model with two degrees of freedom the inertia of the wheels system is a factor due to which the impulse start does not coincide with the coordinate of releasing. Moreover, the start impulse coordinate will change with the system parameters changes. We shall call it an impulse start curve a locus at the impulse starts /in the stationary regime/ on the phase plane $\xi, \dot{\xi}$. The impulse start curve in a model with two degrees of freedom will have a form differ[illegible] to the line ξ = const, and this circumstance gives the possibility to find it close to the equal period curves when

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$\lambda \neq 0$. The stabilization of the period can take place for the values of the period T considerably differing from the period of the balance autooscillations.

Model with the Nonlinear Restoring Force and One Degree of Freedom.

Nondimensional equations of motion:

$$\ddot{\xi} + \xi - 2\beta\xi^3 = -r\frac{\dot{\xi}}{|\dot{\xi}|}, \quad \ddot{\xi} + \xi - 2\beta\xi^3 = \rho - r. \quad /18/$$

Phase curves belonging now to one of the families

$$\xi^2 + \eta^2 = \beta\xi^4 + 2\rho\xi + h,$$

where $\rho = -r$, $\rho = r$ or $\rho = \rho - r$ near the origin as before will be the segments of the closed curves. The band of the phase plane $\xi^2 < \frac{1}{6\beta}$ corresponds to the operation part of the characteristics of the restoring force $\xi - 2\beta\xi^3$.

We shall call by the transformation S - the passage of the point $/-\xi_0, 0/$ along the trajectories of the first of equations /18/ to the point $/a, u/$ on the ray $\xi = a$, $\eta > 0$ by the transformation E - the passage of the point $/a, u/$ along the trajectories of the second of equations /18/ to the point $/b, v/$ on the ray $\xi = b$, $\eta > 0$ /at the impulse/; by the transformation S' and S'' - the passages along the trajectories of the first of equations /18/ correspondingly from the point $/b, v/$ to the point $/\xi_1, 0/$ and from the point $/\xi_1, 0/$ to the point $/-\xi_2, 0/$ /the latter along the trajectories in the lower half-plane/. The transformation T = SES'S'' maps some segment on the axis ξ into itself.

The set of equations

$$(\xi_1 + \xi_2)\{(\xi_2 - \xi_1)[\beta(\xi_1^2 + \xi_2^2) - 1] - 2r\} = 0,$$
$$\beta(\xi_0^4 - \xi_1^4) + \xi_1^2 - \xi_0^2 + 2r(\xi_0 + \xi_1) + 2\rho(a - b) = 0$$

defines the point transformation T = SES'S'' given in the parametric form /ξ_1 - the parameter/. The structure of the transformation /the number of the fixed points and the character of their stability is defined by the relative position of the curves on the coincident planes ξ, ξ_2 and ξ, ξ_0.

The Fixed Point of the Transformation and the Periodical Solution.

For the fixed point of the transformation /for periodical solution/ $\xi_2 = \xi_0$ and therefore

$$(\xi_0 - \xi_1)[\beta(\xi_0^2 + \xi_1^2) - 1] = 2r, \quad (\xi_1 + \xi_2 \neq 0)$$
$$(\xi_0 + \xi_1)\{(\xi_0 - \xi_1)[\beta(\xi_0^2 + \xi_1^2) - 1 + 2r\} - 2\rho(b - a) = 0$$

or

$$(\xi_0 - \xi_1)[\beta(\xi_0^2 + \xi_1^2) - 1] = 2r, \quad \xi_0 + \xi_1 = \frac{\rho(b-a)}{2r}. \quad /19/$$

When $\beta < 0$ the curves /19/ have the single common point. When $\beta > 0$ the curves can have three common real points, only one of which, however, will correspond to the fixed point of the transformation considered. The fixed point and the corresponding periodic motion will be stable because condition

$$\left(\frac{d\xi_2}{d\xi_0}\right)_{\xi_2 = \xi_0} = \frac{(2\beta\xi_1^3 - \xi_1 + r)(2\beta\xi_0^3 - \xi_0 + r)}{(2\beta\xi_0^3 - \xi_0 - r)(2\beta\xi_1^3 - \xi_1 - r)} < 1$$

for the coordinates of the fixed point /$\xi_0 > 0$, $\xi_1 > 0$, $\xi_0^2 + \xi_1^2 < \beta^{-1}$/ is always fulfilled.

The autooscillation period T is expressed by the elliptical integrals of the first kind. Fig.15 shows the dynamic characteristics in respect to the parameter ρ, the characteristics being constructed with $r = 0{,}02$, $b - a = 0{,}75$ and $\beta = 0{,}0005$ or $\beta = -0{,}0005$

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These curves are the result of the common influence upon the period of the two following factors: of the position of the curve of the impulse start and of the nonlinearity of the restoring force. The influence of the first factor is stronger when ρ and consequently an amplitude are small and the influence of the second factor is stronger when the amplitudes are, on the contrary, large. The curves show the possibility to compensate to a certain degree the deviation of the characteristics from the horizontal line arising on account of the nonlinearity of the restoring force by changing the position of the impulse angle in relation to the balance equilibrium position.

Free Anchor Escapement

In the dynamic model the scape wheel is realized as an infinite ribbon equipped with teeth and urged to move rectilinear by some constant force Q. The balance moves along the scape wheel ribbon and is linked with the spring equilibrium position. The anchor-fork in the model is realized by means of the rods of variable length in such a way that during the turn of the anchor fork pallets, which interact with the teeth of the scape wheel, and the notch of the fork, which interacts with the impulse stone of the balance, not only the rectilinear displacements along the directrixes. The parameters of the construction are easily recalculated for the model parameters. For the model with two degrees of freedom the fixed points of the transformation can be found as the common points of the curves

$$\xi_0 = f_0(C),\ \xi_1 = f_1(C) \qquad \text{and} \qquad \bar{\xi}_2 = f_2(\bar{C}),\ \bar{\xi}_1 = f_1(\bar{C})$$

/the parameters C and $\bar{C}$ are the coordinates of the encounter of the scape wheel and the anchor fork taken in the coincident planes $\xi_0\xi_1$ and $\bar{\xi}_2\bar{\xi}_1$. In the model with one degree of freedom - correspondigly as the common points of the curves $\xi_0 = f_0(\xi_1)$ and $\bar{\xi}_2 = f_2(\xi_1)$.

Let us compare the dynamic characteristics of the free anchor escapement with the symmetrical and nonsymmetrical anchor-fork for the simplest model with one degree of freedom. It is not difficult to show that the change of an impulse angle with respect to the equilibrium position of the balance in the model with symmetrical fork cannot change the dynamic characteristics within the [illegible] limits as it takes place in the chronometric escapement model.

For the symmetrical escapements we shall have $-\frac{3}{2}$

$$\frac{dT}{d\beta} = \rho Q \int_{\beta_0}^{\beta_1} \left[\xi_0^2 - \xi^2 + 2(\rho Q - r)\xi - \rho Q(\beta_0 + \beta_1) \right]$$

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$$-\left[\xi_0^2-\xi^2-2(PQ-\Gamma)\xi+PQ(b_0+b_1)\right]^{-\frac{3}{2}}\Big\}\,d\xi,$$

where T is the period, ξ_0 is the autooscillation amplitude, b_0 and b_1 are the coordinates of the start and the end of the impulse /for $\dot{\xi}>0$ and $\dot{\xi}<0$ the start and the end exchange their places/, PQ and Γ are the constants.

Let us denote $\frac{1}{2}(b_0+b_1)=\mu$. Since b_1-b_0 = const. and ξ_0 does not depend upon b_0 it is not difficult to establish from the last expression that $\frac{dT}{db}$ as a function of μ is an odd function identically not equal to zero.

For the fixed values of the parameters PQ and Γ with the symmetrical position of the impulse angle in relation to the balance equilibrium position /for $\mu=0$/ the value of the period will be - as one can easily see- minimal and hence by giving up the symmetry of the impulse angle one can attain only the increasing of the period.

The peculiarities of the dynamics of the considered model are connected - as it is not difficult to show - only with those restrictions which the symmetry of the anchor fork imposes on the dynamics of the model. The close relationship between the models of the chronometric and free anchor escapements are revealed if we give up the symmetry in the fork-anchor design. If we neglect the mass of the anchor fork and the friction of a tooth with the pallets in the equations describing the motion of the anchor escapement model, we can treat the equations of the motion of the chronometric escapement model /in some suitable idealization/ as the result of limit process in the equations of the anchor escapement motion.

[illegible]

one

[illegible]

[illegible] PQ [illegible] tain model with the non-symmetrical fork /with the [illegible] the different width/.

The diagrams show the changing of escape[illegible] [illegible] ract [illegible] on the parameter b_0 which [illegible] zes the position impulse angle b_1-b_0 in relation to the bal[illegible] ce equilibrium position. The dependence of the auto[illegible]cillation amplitude ξ_1 /independent of the impulse angle position/ upon the parameters PQ and Γ is shown on the diagram in fig[illegible].

Model of the Electromechanical Watch

I. The Description of the Design

The elementary diagram of the Hipp's escapement [illegible] electromechanical clock is shown in Fig. [illegible]. A [illegible] pendulum /M/ performs free attenuating [illegible] tactor /A/ shortens the electrowatch circuit [illegible] a pushing impulse to the pendulum thus enlarging its amplitude, After the impulse the pendulum [illegible] oscillates with the decreasing amplitude, until the [illegible] on. The contactor /A/ is [illegible] essential.

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part of the escapement. It is shown separately in two positions in Fig.2I. In the first position the pendulum is shown at the moment of changing the velocity sign with the amplitude decreased to such a degree that the reed /a/ suspended of the pivot is detained between the teeth of the comb /d/ linked with the pendulum rod. During the backward movement of the pendulum the reed /a/ bends the contact spring /C/ and shortens the electromagnet circuit /the second portion in Fig.2I /.

2. The Simplifying Assumptions.

The Escapement Model.

Considering the construction one can see two its peculiarities:

a/ The impulse place constancy.

The impulse is transmitted at a certain angle of the pendulum deflection, this angle depending only upon the construction of the trigering mechanism.

b/ The contactor works up at some interval of the amplitudes.

The minimum amplitude of an oscillation during which the electromagnet circuit shortens, can be different. Let us consider the simplified model of the escapement having retained the main peculiarities of the conduct arrangement.

Let us assume that in the model:

1/ The energy is dissipated only on the account of the tough friction /the energy losses through the contactor are neglected/.

2/ The restoring force is linear. Therefore the oscillator motion between the impulses will be expressed by the equation

$$m\ddot{\varphi} + f\dot{\varphi} + \varkappa\varphi = 0 .$$

3/ An instantaneous impulse. With the fixed coordinate $\varphi = \alpha$ near the equilibrium position the constant energy is transmitted according to the law

$$W^2 - V^2 = h , \qquad /20/$$

where V and W are correspondingly the preimpulse and postimpulse velocities and h is a constant characterizing e.m.f. of the power supply feeding the electromagnet.

Let $A_1 > \ldots > A_j$ be a series of the successive "left amplitudes" with the free attenuating pendulum motion between two successive impulses.

Let us assume that in this model:

4/ The conductor works up every time as the amplitude A_j gets into the interval

$$|m| < A_j \leq |M| ,$$

where $|M|$ is the maxim amplitude value with which the contactor works up and

$$m = M \exp\left(-\frac{2\pi}{m\omega}\right) ,$$

$$\omega = \frac{\sqrt{4m\varkappa - f^2}}{2m} .$$

3. The Phase Space of the Model.

The Point Transformation.

Let the segment P of the line $\varphi = \alpha$, on which $q_0 < \varphi \leq Q$ is the mapping of the segment q of the line $\varphi = C$, under

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g is a segment on which $M_1\varphi < -\pi$,

when the representation point is moving along the trajectories in the upper half-plane $\varphi, \dot{\varphi}$. The point of the segment S corresponds to the states at the moments immediately after the impacts. Here the representation point enters the segment S into the point with the ordinate $\dot{\varphi}$ with increment - according to equation /20/.

Let σ be the mapping of S according to the motion of the phase trace of the model in the upper half-plane. Segments Δ_i are such that $\beta^2 - 4m_1\alpha < 0$, where α and β are the points of the segment Δ at their initial and the points of the segment $\sigma\Delta$ at their terminal ordinates. The trajectories are joined together at the ends of the segments in such a way that the end point W_i of one segment and the initial point W_{i+1} of the following segment would satisfy relationship /20/. According to the definition segment S is the one cut off on the straight line $\varphi = \alpha$ by the spiral segment passing through its "upper" point /the distance between the two adjoining turns of the spiral/.

Therefore each trajectory, having the point on segment $\sigma\Delta$ as its initial point will necessarily intersect segment Δ. The representation point in this case goes over from segment Δ again to segment $\sigma\Delta$. The image point of the phase point [illegible] on segment $\sigma\Delta$ itself.

[illegible] a certain nondimensional value $\omega\tau = \beta_i \Delta_i$ [illegible] equations /20/ and /21/

$$\omega^{\ell} = \omega^{\ell} \frac{\beta_{\ell+1} \sin\theta_{\ell+1}}{\beta_{\ell} \sin\theta_{\ell}} \omega_{\ell}, \quad \ell = 1, 2, \ldots, \rho_1 - 1$$

[illegible]

$$\omega^{\ell} = \omega^{\ell} \frac{\beta_{\ell+1} \sin\theta_{\ell+1}}{\beta_{\ell} \sin\theta_{\ell}} \omega_{\ell}, \quad \ell = \rho_1 + 1, \ldots, \rho_1 + \rho_2 - 1$$

[illegible]

$$\omega^{\ell} = \omega^{\ell} \frac{\beta_{\ell+1} \sin\theta_{\ell+1}}{\beta_{\ell} \sin\theta_{\ell}} \omega_{\ell}, \quad \ell = \rho_1 + \rho_2 + \ldots + \rho_{n-1} + 1, \ldots, \rho_1 + \rho_2 + \ldots + \rho_n - 1$$

[illegible] Ω

The simplest motion corresponds to the stable fixed point of transformation Ω.

The corresponding periodical solution will be presented on the phase plane $\varphi, \dot{\varphi}$ by the spiral having ρ_1 turns at the short-circuiting by an instantaneous impulse.

In the general case a [illegible] periodical motion of transformation Ω will be fixed either of the two forms

$$\Omega^{\rho_1} \Omega_1 \Omega_2 \Omega^{\rho_2} \Omega_1 \cdots \Omega^{\rho_n} \Omega_1 \Omega_2 \Omega^{\ell} \qquad (23)$$

$$\rho_1 + \rho_2 + \ldots + \rho_n + n = k, \quad \ell = 1, 2, \ldots, \rho_1 - 1$$

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and Ω^{o} - identical transformation/ or of the transformation

$$\Omega_{N}^{p_1-\ell}\Omega_{N+1}\Omega_{N}^{p_2}\Omega_{N+1}\cdots\Omega_{N}^{p_n}\Omega_{N+1}\Omega_{N}^{\ell}$$

and will correspond to the steady periodical motion in which the alternation with the impulses takes place - these impulses being of the series of N and N + I oscillator vibrations.

For transformation /22/, for example, the series of N + I oscillations with the succeeding impulse go one by one in groups: at first $p_1-\ell$, then $p_2, \ldots, p_n$ and at last ℓ times. Each of these groups is separated from the other one by the series succession of N oscillations with the following impulse.

The structure of the phase portrait of the parameters /along h axis/ in the intervals, which correspond to the various types of motion, is rather complicated. However, the most interesting point for practice is the selection of the simplest motions and thus of those intervals of the changes of value h, for which there is simple fixed transformation point. Value h can be approximately recalculated for the mean value of the e.m.f. for an electric power supply.

For fixed M = 0.05 and H = 0.005 and for N = I,2,3... IO Fig.23 shows the intervals along axis h_1, these intervals corresponding to the periodic solutions with the impulse transmission in one, two etc. oscillator vibrations.

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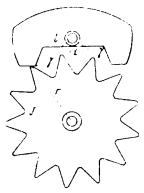

Fig. 1

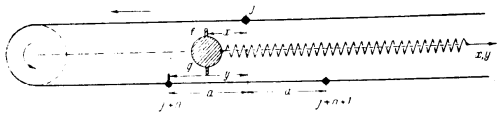

Fig. 2

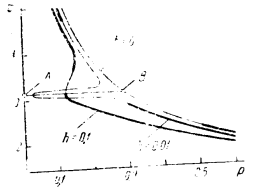

Fig. 3

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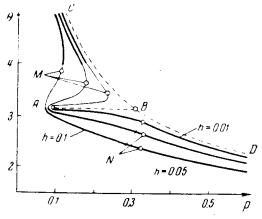

Fig. 4.

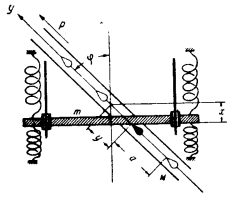

Fig. 7.

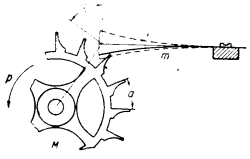

Fig. 6

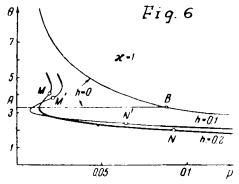

Fig. 5.

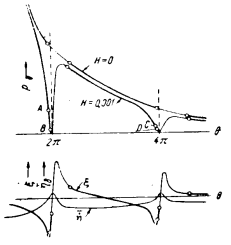

Fig. 8.

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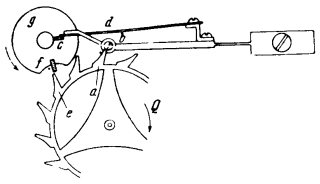


Fig. 9.

Fig. 10.

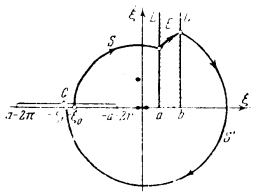


Fig. 11.

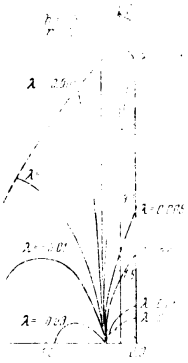


Fig 12

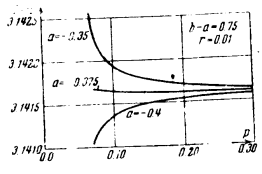
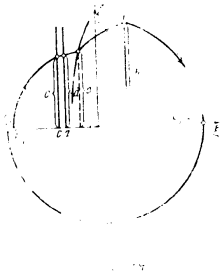


Fig 13.



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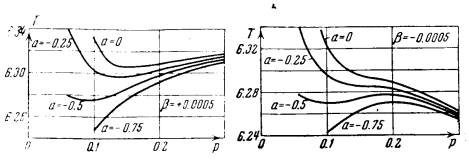


Fig. 15.

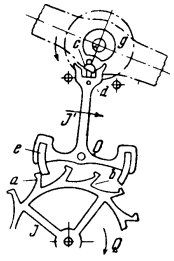


Fig. 16.

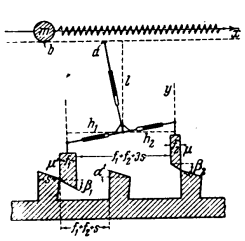
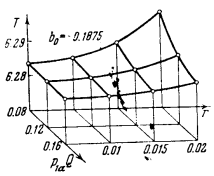
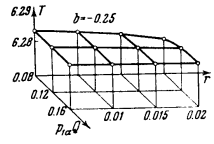


Fig. 17.





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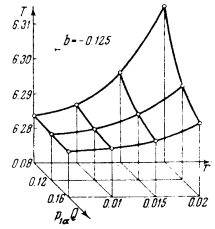

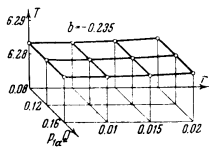

Fig. 18a

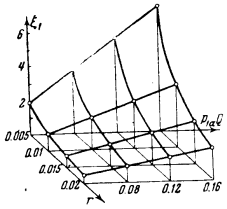

Fig. 19.

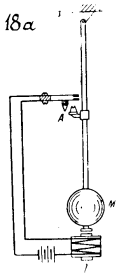

Fig. 20.

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Fig 21.

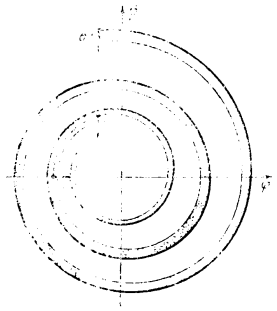

Fig 22.

Fig 23

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И. И. БЛЕХМАН

ИНТЕГРАЛЬНЫЙ КРИТЕРИЙ УСТОЙЧИВОСТИ ПЕРИОДИЧЕСКИХ ДВИЖЕНИЙ НЕКОТОРЫХ НЕЛИНЕЙНЫХ СИСТЕМ И ЕГО ПРИЛОЖЕНИЯ

INTEGRAL CRITERION OF STABILITY PERIODICAL MOTIONS OF SOME NON-LINEAR SYSTEMS AND ITS APPLICATION

I. I. Blechman
(Leningrad, USSR)

Synopsis

This paper shows that the investigation of stability periodical solutions of a non-linear differential equations system containing a small parameter under determined conditions comes to the stationary points character study of some function $\mathcal{D}$, depending on $\alpha_1, \ldots, \alpha_\kappa$ parameters of a "generating solution".

The conditions of the minimum of this function in the stationary point $(\alpha_1^*, \ldots, \alpha_\kappa^*)$ are indispensable (and in a number of cases sufficient) conditions of stability a periodical solution corresponding to the $\alpha_1 = \alpha_1^*, \ldots, \alpha_\kappa = \alpha_\kappa^*$ parameters values. For a real plotting of function $\mathcal{D}$ only a knowledge of the generating system periodical solutions set, depending on the mentioned above parameters, is required.

In the simplest case the function $\mathcal{D}$ is equal to the mean, for period value of Lagrange function taken with an inverse sign and calculated for a generate solution.

If we consider the system equilibrium positions a

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degenerate periodical motions, then function $\mathcal{D}$ is turning into system potential energy Π. Thus in this particular case the integral stability criterion comes to Lagrange. (Lejeune-Dirichlet) theorem of equilibr stability of the conservative system with holonomic st ionary constraints.

Here are given [illegible] of vibrators' rotation synchronization [illegible] particular case of [illegible] problem) the criterion was initially suggested by S.P. Lavrov and the author in the work [illegible] and was later proved by the author in article [3].

§ 1. Theorem on periodical solutions of some non-linear differential equations systems.

Let us consider the differential equations system

$$\frac{dx_s}{dt} = X_s(t, x_1, \dots, x_n) + \mu f_s(t, x_1, \dots, x_n, \mu) \quad (s=1,\dots,k)$$

where X_s and f_s analytical funtions of $x_1, \dots, x_n$ closed region G, to which as it is assumed, belo all periodical solutions considered below.

In relation to time t functions X_s and f_s, assumed to be continuous periodical functions of peri

$T = 2\pi$ [illegible]

[illegible]

μ [illegible]

[illegible]

$$\frac{dx_s^0}{dt} = X_s(t, x_1^0, \dots, x_n^0) \quad (s=1,\dots,k)$$

allow a set of periodical solutions

$$x_s^0 = x_s^0(t, \alpha_1, \dots, \alpha_k) \quad (s=1,\dots,k)$$

[illegible]

[illegible] near [illegible] below "generating" values [illegible]

[illegible]

[illegible] system periodi[illegible]

[illegible] values μ [illegible]

[illegible] of some [illegible] equation [illegible]

[illegible]

mentioned is written as a [illegible]

[illegible]

x) The requirements listed may be reduced.

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[illegible] of which [illegible] expressed by derivatives [illegible] left parts of equations for the determination of [illegible] parameters [illegible].

article [6] the corresponding theorem was proved [illegible] quasi-linear nonautonomous systems.

The proof of series of analogical theorems for periodical and nearly-periodical [illegible] nonlinear systems of a more general form is given in the monograph of I.G. Malkin [4].

In our article [7] a corresponding result was obtained for autonomous quasi-linear systems.

Finally, recently an identical theorem was proved by S.N. Shimanov for periodical oscillations of quasi-linear systems with retardation.

Let us bring the theorem for the non-autonomous nonlinear system (I.1), considered by I.G.Malkin [4]. Let us consider the system of the first variation corresponding to equations (1.2) and to solution (I.3)

$$\frac{dy_s}{dt} = p_{s1} y_1 + \dots + p_{sn} y_n \quad , \quad (s=1,\dots,n) \tag{I.4}$$

where

$$p_{sj} = \left[\frac{\partial X_s}{\partial x_j}\right]_0 = \frac{\partial X_s(t, x_1^\circ, \dots, x_n^\circ)}{\partial x_j^\circ} \tag{I.5}$$

are periodical functions t, while the rectangular parentheses and the sign "0" here and further indicate

that the derivatives are calculated for some of the generating solutions (I.3).

In conformance with the well known A. Poincare theorem the system (I.4) admits at least k periodical solutions of form

$$y_{sj}(t) = \frac{\partial x_s^\circ}{\partial \alpha_j} \quad , \quad (s=1,\dots,n;\ j=1,\dots,k), \tag{I.6}$$

obtained by differentiation of solutions (I.3) according to parameters. At least k periodical solutions $z_{sj}(t)$ have the linear equations system

$$\frac{dz_s}{dt} + p_{1s} z_1 + \dots + p_{ns} z_n = 0, \quad (s=1,\dots,n), \tag{I.7}$$

conjugated with (I.4).

Let us assume first that the equations (I.4) (I.7) have exactly k periodical solutions and let us understand under $z_{sj}(t)$ the periodical solutions of conjugated system (I.7) satisfying relationships

$$\sum_{s=1}^{n} y_{si} z_{sj} = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \tag{I.8}$$

Such a choice of conjugated [illegible] system solutions in the case considered is always possible [4].

As the sytem (I.4) has exactly k independent periodical solutions with a period $2\pi/\omega$, the moduli of roots of a characteric equation of this system are equal to one. But all other roots of the characteristic have moduli less than one.

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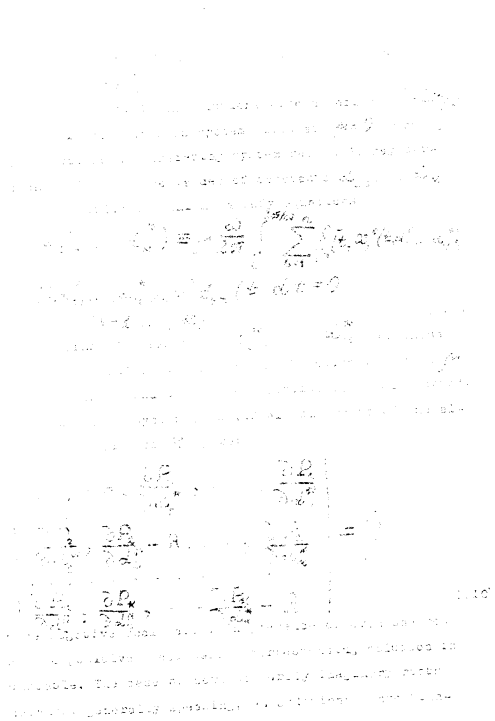

tion.[x]

Let us add that if the system of the first variation (I.4), besides (I.6), has periodical solutions with frequencies $\omega_j = 2\pi/T_j$, non-multiple to frequency $\omega = 2\pi/T$, then the conditions of negativity roots real parts equation (I.10) will be necessary but not sufficient conditions of stability.

§ 2. The integral criterion of periodical solutions stability.

Let us assume the existence of function $\mathcal{D} = \mathcal{D}(\alpha_1, \dots, \alpha_n)$, such as

$$\frac{\partial \mathcal{D}}{\partial \alpha_r} = -P_r(\alpha_1, \dots, \alpha_n), \quad (r = 1, \dots, n) \tag{2.1}$$

Further we shall assume functions P_r and $\mathcal{D}$ as real.

Decomposing function $\mathcal{D}$ in the Taylor series near the generating values of parameters $\alpha_1 = \alpha_1^*, \dots, \alpha_n = \alpha_n^*$, in conformance with (I.9) and (2.1) we obtain

$$\mathcal{D}(\alpha_1, \dots, \alpha_n) - \mathcal{D}(\alpha_1^*, \dots, \alpha_n^*) =$$
$$= \sum_{r=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 \mathcal{D}}{\partial \alpha_r \partial \alpha_j}\right)_{\alpha=\alpha^*} (\alpha_r - \alpha_r^*)(\alpha_j - \alpha_j^*) + \dots =$$
$$= \sum_{r=1}^{n} \sum_{j=1}^{n} \left[-\left(\frac{\partial P_r}{\partial \alpha_j}\right)_{\alpha=\alpha^*}\right](\alpha_r - \alpha_r^*)(\alpha_j - \alpha_j^*) + \dots \tag{2.2}$$

x) In I.G. Malkin's [4] monograph formulated theorem is accompanied by requirements of unequal roots of equation (I.10). This requirement is yet not obligatory and is only needed in the process of the theorem demonstration by the method chosen by I.G. Malkin (see, for instance [6] and [7]).

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[illegible] items higher than of second order in relation [illegible] are not written.

[illegible] function $\mathcal{D}$ in point $(\alpha_1^*, \dots, \alpha_k^*)$ [illegible], it is enough that the quadratic form in equation (I.8) should be a positive with a determined sign. In turn, in order to achieve this, it is necessary and enough [8] that all the roots of k order algebraic equation coinciding with equation (I.10) should be negative. (It is to be added that all the roots of this equation are real in so far as conforming to (2.I)

$$\partial P_r / \partial \alpha_j^* = \partial P_j / \partial \alpha_r^* = -\partial^2 \mathcal{D} / \partial \alpha_r \partial \alpha_j .$$

If even only one root of equation (I.10) is positive, the quadratic form is not with a determined sign and the minimum is absent. The case when together with negative roots there are roots equal to zero is doubtfull and, generally speaking, needs investigation of decomposition (2.2) [redacted] of a higher order. Thus the conditions of the minimum function $\mathcal{D}$ presence in point $(\alpha_1^*, \dots, \alpha_k^*)$ based on the analysis of second order members in decomposition (2.2), coincide with stability conditions of a corresponding periodical solution of the equations system (I.I).

Thus in the problem of the existence and stability periodical solutions considered, the function $\mathcal{D}$ plays the same role as the potential energy of system Π in the problem of seeking and investigation of stability equilibrium positions of a conservative system with holonomic

stationary constraints. In other words, for the periodical solutions of equations of the studied type theorems identical to the well known theorems of Lagrange (Lejeune-Dirichlet) and A.M. Ljapunov are true.

The stability criterion formulated above is called integral in so far as function $\mathcal{D}$, according to (2.I) and (I.9), depends on the mean for the period characteristics of the system motion. Function $\mathcal{D}$ may be called the potential of the mean [redacted] forces.

§ 3. The use of the integral criterion in the problem of mechanical vibrators synchronization.

Inspite of the particular character of the non-linear equations system considered above, it may be pointed out to a series of mechanical systems the motion of which is described by means of exactly this type of equations.

The following problem [1] is of interest for the dynamics of multi-vibrator vibrating machines drive as well as for other applications.

Let a certain number k of mechanical vibrators (unbalanced rotors, brought to rotation by means of electric or other motors), be mounted on one or several solid bodies connected between them and to a foundation by means of an arbitrary system of elastic elements, having ν degrees of freedom. The hard bodies deviations from a posi-

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tion of static equilibrium are characterised by generalised coordinates $x_1, \ldots, x_\nu$ (let us call them oscillation coordinates), and the positions of rotor [illegible] rotor's rotation $\varphi_1, \ldots, \varphi_\kappa$ [illegible] coordinates).

[illegible] consists in clearing up the existence and stability conditions of the so called syncronous motions of a system, i.e. motions where all vibrators, inspite of possible difference of their parameters and the difference in torques acting on their rotors are rotating with an equal on its absolute value average angle velocity ω and the solid bodies are performing oscillations with a frequency ω.

[illegible] may be electrical or mechanical constraints between rotors (forced synchronization). However these constraints may lack and (self)synchronization).

In other words, the question concerns investigations of system motions of the form

$$\varphi_s = \pm\omega t + \alpha_s + \mu\psi_s^*(\omega t), \quad (s=1,\ldots,\kappa)$$
$$x_r = \mu\chi_r(\omega t), \quad (r=1,\ldots,\nu) \tag{3.1}$$

where ± 1 (the vibrators' rotation may take place in different directions), α_s - constants, and $\psi_s^*(\omega t)$ and $\chi_r(\omega t)$ - periodical time t functions with a period $2\pi/\omega$, while it may be considered that the μ value is a small parameter and

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\psi_s^*(\omega t)\,dt = 0 \tag{3.2}$$

Often it is interesting to calculate the phases α_s in stabil syncronous vibrator's motions or, on the contrary, to chose such system parameters which should secure the motion with in advance given values of α_s and thus the definite law of solid bodies oscillations would be determined.

The motion of the system considered is described by a system of differential equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_s} - \frac{\partial L}{\partial \varphi_s} = Q_s, \quad (s=1,\ldots,\kappa)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot x_r} - \frac{\partial L}{\partial x_r} = Q_r, \quad (r=1,\ldots,\nu) \tag{3.3}$$

where $L = T - \Pi$ - Lagrange function of system (T - kinetic, Π - potential energy) obtained in the absence of generalized forces Q_i, which, generally speaking are assumed to be non-conservative.

The forces Q_s ($s=1,\ldots,\kappa$) in equations (3.3) corresponding to rotation coordinates in problems concerning of synchronization by their physical meaning represent a sum of rotation torques and resistance forces torques brought to the vibrator shaft; and functions Q_r ($r=1,\ldots,\nu$) in equations for oscillations coordinates - the resistance forces and sometimes - reactive forces transmitted from the vibrators to the bodies. The presence of the latter forces is bound with the presence of forces Q_s, ($s=1,\ldots,\kappa$).

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Functions L and $Q_i\,(i=1,\dots,k+\nu)$ may depend on generalized coordinates and velocities as well as on time t and must after sucstition of φ_s by their expressions according to (3.4) these functions t with a period $2\pi/\omega$. The last condition is surely effected if L and Q_i in relation to φ_s are periodical functions with a period 2π, in relation to t - periodical functions with a period $2\pi/\omega$ and the function of differences $\sigma_i\varphi_i-\sigma_j\varphi_j$ and $\sigma_i\varphi_i-\omega t$. It is to notice that the formulated condition would be superfluous and not solutions of type (3.I) but simply periodical solutions of system (3.3) would be sought. Then it should be enough to ask only the periodicity of functions L and Q_i in time with the same period.

Concerning the character of functions L and Q_i assumptions are made identical to assumptions concerning functions X_s and Y_s of system (2.I).

It is to notice that not only problems of self-synchronization, forced electrical and mechanic synchronization of vibrators come to the investigation of solutions of type (3.I) equations type (3.3) but also the problems of multidisc not wholly balanced rotors oscillations, as well as problems of a special self-phase control stability work [10]. Identical equations are also obtained in the problem of electric generator's synchronization.

Taking into account the character of solutions (3.I)

interesting for us and the specific character of the problem let us introduce into the system (3.3) small parameter μ, representing the system in form

$$I_s\ddot{\varphi}_s=-\kappa_s\dot{\varphi}_s+\kappa_s\sigma_s\omega+\mu\,\mathcal{P}_s,\quad (s=1,\dots,k),$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_r}-\frac{\partial L}{\partial x_r}=Q_r^{(0)}+\mu\,Q_r^{(1)},\quad (r=1,\dots,\nu)$$

where

$$\mu\,\mathcal{P}_s=I_s\ddot{\varphi}_s+\kappa_s\dot{\varphi}_s-\kappa_s\sigma_s\omega-\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}_s}+\frac{\partial L}{\partial\varphi_s},$$

$$\mu\,Q_r^{(1)}=Q_r-Q_r^{(0)}$$

and under I_s and $\kappa_s>0$ we may understand respectively the inertia moment and viscous resistance coefficient of the rotor of vibrator.

Let us use the results of §§ I and 2 for obtaining conditions of existence and stability solutions type (3.I) of system (3.4), (3.5). The generating system corresponding to these equations

$$I_s\ddot{\varphi}_s^{\circ}=-\kappa_s\dot{\varphi}_s^{\circ}+\kappa_s\sigma_s\omega,\quad (s=1,\dots,k),\tag{3.7}$$

$$\left[\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_r}-\frac{\partial L}{\partial x_r}=Q_r^{(0)}\right]_0,\quad (r=1,\dots,\nu)\tag{3.8}$$

admits a solution's system for rotation coordinates

$$\varphi_s^{\circ}=\sigma_s(\omega t+\alpha_s),\quad (s=1,\dots,k),\tag{3.9}$$

depending on k of arbitrary parameters $\alpha_1,\dots,\alpha_k$

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Let

$$x_r^* = x_r^*(\omega t, \alpha_1, \dots, \alpha_k), \quad (r = 1, \dots, \nu) \tag{3.10}$$

be the respective periodical solution for oscillations coordinates found from equations (3.8) after substitution (3.9). It is to notice that in vibrator's synchronization problems (3.8) there are equations of solid bodies small oscillations under the action of rotating with constant velocity vibrator's according to law (3.9).

Let us pass in equations (3.9) from variables φ_s to variables $\mu\psi_s^* = \psi_s$ according to formula (3.I) and then to new variables $\gamma_1, \dots, \gamma_{2(k+\nu)}$ according to formulas

$$\psi_s = \gamma_s - \gamma_{k+s}, \quad \dot{\psi}_s = \frac{\kappa_s}{I_s}\gamma_{k+s}, \quad (s = 1, \dots, k);$$

$$x_r = \gamma_{2k+r}, \quad \dot{x}_r = \gamma_{2k+\nu+r}, \quad (r = 1, \dots, \nu). \tag{3.11}$$

Then equations (3.4) will be written in form of the following first order system

$$\dot{\gamma}_s = \frac{\mu}{\kappa_s\sigma_s}\Phi_s, \quad \dot{\gamma}_{k+s} = -\frac{\kappa_s}{I_s}\gamma_{k+s} + \frac{\mu}{\kappa_s\sigma_s}\Phi_s, \quad (s = 1, \dots, k);$$

$$\dot{\gamma}_{2k+r} = \gamma_{2k+\nu+r}, \tag{3.12}$$

$$\dot{\gamma}_{2k+\nu+r} = F^{(0)}_{2k+\nu+r}(\gamma_1, \dots, \gamma_{2(k+\nu)}, t) + \mu F^{(1)}_{2k+\nu+r}(\gamma_1, \dots, \gamma_{2(k+\nu)})$$

Here $F^{(0)}_{2k+\nu+r}$ $(r = 1, \dots, \nu)$ and $F^{(1)}_{2k+\nu+r}$ are functions the concrete shape of which does not play further any substantial role.

The generating solution (3.9) and (3.10) in the variables γ will be written in form

$$\gamma_s^\circ = \alpha_s, \quad \gamma_{k+s}^\circ = 0, \quad (s = 1, \dots, k);$$

$$\gamma_{2k+r}^\circ = \gamma_{2k+r}^\circ(\omega t, \alpha_1, \dots, \alpha_k),$$

$$\gamma_{2k+\nu+r}^\circ = \dot{\gamma}_{2k+r}^\circ = \dot{\gamma}_{2k+r}^\circ(\omega t, \alpha_1, \dots, \alpha_k) \quad (r = 1, \dots, \nu) \tag{3.13}$$

The corresponding to equations (3.12) system of the first variation corresponding to the generating solution takes the shape

$$\dot{y}_s = 0, \quad \dot{y}_{k+s} = -\frac{\kappa_s}{I_s}y_{k+s}, \quad (s = 1, \dots, k);$$

$$\dot{y}_{2k+r} = \sum_{j=1}^{2(k+\nu)} b_{2k+r,j}(t)\, y_j, \quad (r = 1, \dots, 2\nu);$$

$$b_{2k+r,j}\left(t + \frac{2\pi}{\omega}\right) = b_{2k+r,j}(t) \tag{3.14}$$

and admits the system of periodical solutions

$$y_{si} = \frac{\partial \gamma_s^\circ}{\partial \alpha_i} = \delta_{si}, \quad y_{k+s,i} = 0;$$

$$y_{2k+r,i} = \frac{\partial \gamma_{2k+r}^\circ}{\partial \alpha_i}, \quad y_{2k+\nu+r,i} = \frac{\partial \gamma_{2k+\nu+r}^\circ}{\partial \alpha_i}$$

$$(s, i = 1, \dots, k; \ r = 1, \dots, \nu) \tag{3.15}$$

The system of equations conjugated with (3.14)

$$\dot{z}_s = -\sum_{j=1}^{2\nu} b_{2k+j,s}(t)\, z_{2k+j},$$

$$\dot{z}_{k+s} = \frac{\kappa_s}{I_s}z_{k+s} - \sum_{j=1}^{2\nu} b_{2k+j,k+s}(t)\, z_{2k+j},$$

$$(s = 1, \dots, k); \tag{3.16}$$

$$\dot{z}_{2k+r} = -\sum_{j=1}^{2\nu} b_{2k+j,2k+r}(t)\, z_{2k+j}, \quad (r = 1, \dots, 2\nu)$$

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has a system of periodical solutions

$$x_{si} = \delta_{si}, \quad x_{k+s,i} = 0; \quad x_{2k+r,i} = 0;$$
$$(s,i = 1,\dots,k;\ r = 1,\dots,2\nu) \tag{3.17}$$

At this the functions $y_{\mu i}$ and $x_{\mu i}$ satisfy to conditions (I.8):

$$\sum_{\mu=1}^{2(k+\nu)} y_{\mu i}\, x_{\mu j} = \sum_{\mu=1}^{k} \delta_{\mu i}\,\delta_{\mu j} = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \tag{3.18}$$

The equations (I.9) for the determination of parameters $\alpha_1^*, \dots, \alpha_k^*$ of the generating solution according to (3.4)-(3.7), (3.12) and (3.17) will be:

$$P_s(\alpha_1^*,\dots,\alpha_k^*) \equiv \frac{\omega}{2\pi}\frac{\sigma_s}{\kappa_s}\int_0^{2\pi/\omega}\left[-\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_s} + \frac{\partial L}{\partial \varphi_s} + Q_s\right]_0 dt = 0$$
$$(s = 1,\dots,k) \tag{3.19}$$

If the characteristical equation of the system of the first variations (3.14) has no other roots equal by modulus to one, besides k roots equal to one, i.e. roots corresponding to periodical solutions (3.15), and all the other roots have moduli less than one, then the necessary and sufficient conditions of asymptotic stability will be negativity conditions of the re parts of the equation (1.10) roots. If together with the roots the moduli of which are less than one a characteristic equation has roots equal by modulus to one (but different from one)

then the negativity real parts of equation (1.10) roots will be only a necessary stability condition.

In this case additional relationships are added to the above-mentioned stability conditions, which however in the vibrators' synchronization problems are usually effected [5,9] and that is why the question of stability is solved by means of real parts equation (1.10) roots investigation.

Let us consider in detail the equations (3.19) and introduce function

$$\Lambda = \Lambda(\alpha_1,\dots,\alpha_k) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}[L]_0\, dt, \tag{3.20}$$

representing the mean for a period value of Lagrange function $L = T - \Pi$ calculated for the generating solution (3.5) and (3.10).

Let us calculate the derivative $\partial\Lambda/\partial\alpha_s$. As a result of simple transformations, including integration by parts of respective equations (3.8) (3.10) we find

$$\frac{\partial\Lambda}{\partial\alpha_j} = \frac{\omega}{2\pi}\sum_{s=1}^{k}\int_0^{2\pi/\omega}\left\{\left[\frac{\partial L}{\partial x_s}\right]_0\frac{\partial x_s^\circ}{\partial\alpha_j} + \left[\frac{\partial L}{\partial \dot x_s}\right]_0\frac{\partial \dot x_s^\circ}{\partial\alpha_j}\right\}dt +$$
$$+\frac{\omega}{2\pi}\sum_{s=1}^{k}\int_0^{2\pi/\omega}\left\{\left[\frac{\partial L}{\partial \varphi_s}\right]_0\frac{\partial \varphi_s^\circ}{\partial\alpha_j} + \left[\frac{\partial L}{\partial \dot\varphi_s}\right]_0\frac{\partial \dot\varphi_s^\circ}{\partial\alpha_j}\right\}dt =$$
$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sum_{s=1}^{\nu}\left[\frac{\partial L}{\partial x_s} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot x_s}\right)\right]_0\frac{\partial x_s^\circ}{\partial\alpha_j}\,dt +$$
$$+\frac{\omega\delta_j}{2\pi}\int_0^{2\pi/\omega}\left[\frac{\partial L}{\partial\varphi_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi_j}\right)\right]_0 dt =$$

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$$= -\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sum_{s=1}^{k}[Q_s^{(0)}]\frac{\partial x_s^0}{\partial \alpha_j}\,dt + \\ + \frac{\omega\sigma_j}{2\pi}\int_0^{2\pi/\omega}\left[\frac{\partial L}{\partial \varphi_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_j}\right)\right]_0 dt \qquad (3.21)$$

Using the last relationship the equations (3.19) may be represented as follows:

$$P_s(\alpha_1,\dots,\alpha_k) = \frac{1}{K_s}\left\{\frac{\partial \Lambda}{\partial \alpha_s} + \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sum_{r=1}^{k}[Q_r^{(0)}\frac{\partial x_r^0}{\partial \alpha_s}\,dt + \right. \\ \left. + \frac{\omega\sigma_s}{2\pi}\int_0^{2\pi/\omega}[Q_s]_0\,dt\right\} = 0, \quad (s=1,\dots,k) \qquad (3.22)$$

Let us notice now that equations (3.19) or, what the same equations (3.22), represent essentially the average result for the period of vibrators motion differential equations (3.3). In the process of this averaging in the expressions of functions φ_r and x_s their values in corresponding solution (3.9) and (3.10) have been taken.

The value

$$\frac{\omega\sigma_s}{2\pi}\int_0^{2\pi/\omega}[Q_s]_0\,dt = M_s^{(m)} - R_s^{(m)}, \quad (s=1,\dots,k) \qquad (3.23)$$

in conformance with the above-mentioned in its physical sense represents the mean for the period value of the surplus torque on the shaft of vibrator s, i.e. the sum of the average torques $M_s^{(m)}$ and $-R_s^{(m)}$. The expression

$$W_s = W_s(\alpha_1,\dots,\alpha_k) = W_s^{(\Lambda)} + W_s^{(R)} \qquad (3.24)$$

where

$$W_s^{(\Lambda)} = -\frac{\partial \Lambda}{\partial \alpha_s}, \quad W_s^{(R)} = -\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sum_{r=1}^{k}[Q_r^{(0)}]_0\frac{\partial x_r^0}{\partial \alpha_s}\,dt \qquad (3.25)$$

are called vibration torques.

In the designations (3.23) and (3.24) the equations (3.22) take the form

$$P_s(\alpha_1,\dots,\alpha_k) = \frac{1}{K_s}\left(M_s^{(m)} - R_s^{(m)} - W_s\right) = 0, \qquad (3.26)$$

i.e they express the equilibrium conditions of the mean torques introduced above.

In case of resonance vibration of bodies on which vibrators are mounted for an steady state regime of rotor's motion the relationship $M_s^{(m)} = R_s^{(m)}$ expressing the equality (average) conditions of the torque and the torque resistance forces, should be true. Hence the vibration torques W_s may be interpreted as additional torques, acting upon the unbalanced rotors of vibrators due to the vibrations of the foundations where they are mounted.

Let us notice that in conformance with (3.24) and (3.25) the vibration torques W_s may be presented as consisting of two parts: for the first part the mean for π the period value of Lagrange function plays the role of "potential energy" and the second part reflects the influence of nonconservative generalized forces $Q_r^{(0)}$ upon

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the vibration torque.

The relationships (3.20), (3.24) and (3.25) allow to find vibration torques if the generating solution (3.9) and (3.10) is known. For vibrators mounted on a solid body, which can effect a flat-parallel motion and is bound by arbitrary system of linear elastic elements with the immobile foundation in work [1] the following expression were found for the vibration torque (in conditions of the work mentioned $W_s^{(R)} = 0$):

$$W_s = W_s^{(\alpha)} = W_s(\alpha_1, \dots, \alpha_k) = \sum_{r=1}^{k} \frac{m_r \varepsilon_r m_s \varepsilon_s \omega^2}{2M} \left[P_{sr} \sin(\alpha_s - \alpha_r) + Q_{sr} \cos(\alpha_s - \alpha_r) \right] \tag{3.27}$$

where $P_{sr} = P_{rs}$ and $Q_{sr} = -Q_{rs}$ are values depending on inertia and elastic system parameters as well as on the disposition and direction of vibrators rotation in the synchronous motion considered. The formulas for P_{sr} and Q_{sr} are given in work [1].

Vibration torques reflect the presence in systems of sometimes very strong connections between vibrators conditioned by oscillations vibrating bodies upon which they are mounted. This connection explains many peculiar phenomena in these systems, for instance the phenomena of vibrators self-synchronization.

Let exist function $U^* = U^*(\alpha_1, \dots, \alpha_k)$ such

that

$$\frac{\partial U^*}{\partial \alpha_s} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sum_{j=1}^{\nu} [Q_j^{(*)}]_0 \frac{\partial x_j^0}{\partial \alpha_s} dt + \frac{\omega \varepsilon_s}{2\pi} \int_0^{2\pi/\omega} [Q_s]_0 dt =$$

$$= -W_s^{(R)} + \mathcal{M}_s^{(m)} - R_s^{(m)} \tag{3.28}$$

and let besides all resistance coefficients $k_s > 0$ be identical and equal to k_0. This last condition is fulfilled for identical or nearly identical vibrators.

Then, if there exists a force function

$$\mathcal{D} = -(\Lambda + U^*), \tag{3.29}$$

the equations (3.22) for "generating phases" will be represented in form

$$P_s(\alpha_1, \dots, \alpha_k) = -\frac{\partial \mathcal{D}}{\partial \alpha_s}, \quad (s = 1, \dots, k) \tag{3.30}$$

and therefore the integral criterion of stability adduced in § 2, will be true.

Particularly simple in the formulation of investigation results in the case of a problem on selfsynchronization of identical or nearly identical vibrators when it can be considered that the torques $W_s^{(R)}$ are equal to zero or are negligibly small while the forces resistance average torques $R_s^{(m)}$ and the average rotating torques $\mathcal{M}_s^{(m)}$ for all vibrators are identical [9]:

$$W_s^{(R)} = 0, \quad R_s^{(m)} = R^{(m)}, \quad \mathcal{M}_s^{(m)} = \mathcal{M}^{(m)} \tag{3.31}$$

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As far as, further, due to the autonomity of the initial equations system in case of the selfsynchronization problem the torques $W_s^{(x)} = W_s = -\frac{\partial \Lambda}{\partial \alpha_s}$ may depend only on the difference of phases $\alpha_s - \alpha_j$, but not on the α_s itself (see, for instance, the expression 3.27) then (function Λ and the)

$$\Lambda(\alpha_1+\alpha_0,\ldots,\alpha_\kappa+\alpha_0) \equiv \Lambda(\alpha_1,\ldots,\alpha_\kappa)$$

where α_0 is an arbitrary constant.

Differentiating the last identity by α_0 and assuming then that $\alpha_0 = 0$ we obtain

$$\sum_{s=1}^{\kappa} \frac{\partial \Lambda}{\partial \alpha_s} = -\sum_{s=1}^{\kappa} W_s \equiv 0 \qquad (3.32)$$

In reckoning (3.31) and (3.32) from equations (3.26) and relationship (3.28) it follows

$$R^{(m)} = \mathcal{U}^{(m)}, \quad W_s = W_s^{(x)} = 0, \quad \mathcal{U}^* = const. \qquad (3.33)$$

therefore it may be written

$$\mathcal{D} = -\Lambda = -\frac{\omega}{2\pi}\int_0^{2\pi/\omega} [T-\Pi]_0\, dt \qquad (3.34)$$

In other words, function $\mathcal{D}$ in the case considered is a mean for a period of oscillations [x] of Lagrange function value taken with an opposite sign (and) calculated for the

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x) Let us notice that the angle velocity of the synchronous rotation ω in the particular case of the self-synchronization considered, is determined from the first equation (3.33).

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In these cases, obviously, it may be written $\Lambda = -\Lambda_0$.

§ 4. Example of stability integral criterion use.

As a concrete example of the stability integral criterion use let us consider the problem of selfsynchronization of two identical vibrators mounted upon an elastically supported solid body and having parallel axes, symmetrically situated in relation to the center of gravity (Fig. 1). While the elastic supports in this case are considered so soft, that the greatest of the natural frequencies of the body on shock absorbers is negligibly small in comparison with the frequency of forced oscillations.

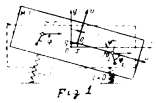

Fig. 1

The motion equations of a vibrating body at the rotation of vibrators, according to the law, determined by equations (3.9) in the given case looks like

$$\begin{aligned} M\ddot{x} &= F[\cos(\omega t+\alpha_1)+\cos(\omega t+\alpha_2)], \\ M\ddot{y} &= -F[\sigma_1 \sin(\omega t+\alpha_1)+\sigma_2 \sin(\omega t+\alpha_2)], \\ I\ddot{\varphi} &= Fr[\sigma_1 \sin(\omega t+\alpha_1)-\sigma_2 \sin(\omega t+\alpha_2)] \end{aligned} \tag{4.1}$$

Here x and y are the coordinates of the gravity center of a body in relation to immobile axes xOy system, φ is the rotation angle of the vibrating member in relation to these axes counted off clockwise, M - the system mass, I - moment of inertia of the solid body and vibrators with respect to the axis, passing through the gravity center, r - the distance from the vibrators' axes of rotation to the body center of gravity, F - the amplitude of the disturbing force developed by each vibrator.

In connection with the assumption concerning the smallness of elastic supports the system potential energy may be considered equal to zero. Therefore after the determination of x, y and φ in steady state motion by integration of equations (4.1), in accordance with the above mentioned, we shall obtain in the case considered:

$$\mathcal{D}(\alpha) = -\Lambda_0 = \Lambda_0 = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} T_e \, dt = \frac{\omega}{4\pi} \int_0^{2\pi/\omega} [M(\dot{x}^2+\dot{y}^2)+I\dot{\varphi}^2]\,dt =$$

$$= \frac{F^2}{2M\omega^2}\left(1+\sigma_1\sigma_2+\sigma_1\sigma_2\frac{Mr^2}{I}\right)\cos\alpha + C_1 \tag{4.2}$$

Here $\alpha = \alpha_2 - \alpha_1$ is [redacted] a relative shift of vibrator rotation phases and C_1 - a value independent from angle α.

Equating the derivative $d\mathcal{D}/d\alpha$ to zero we come to equation $\sin\alpha = 0$, having two substantially different roots $(\alpha)_1 = 0$ and $(\alpha)_2 = \pi$.

It is obvious that in the case of vibrators' rotation in identical directions $(\sigma_1\sigma_2 = 1)$ the minimum function $\mathcal{D}(\alpha)$ responds to the first root provided that the

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inequality

$$\frac{Mv^2}{I} > 2$$

is true.

While the maximum function $\mathcal{D}(\alpha)$ [illegible] second root. When the condition (4.3) is not sent an opposite conclusion is true.

In case of vibrator's rotation in opposite direction $(\sigma_1 \sigma_2 = -1)$ to value $\alpha = (\alpha)_1 = 0$ always respond maximum, while to value $\alpha = (\alpha)_2 = \pi$ the minimum function $\mathcal{D}(\alpha)$.

In conformance with the integral stability criterion at $\sigma_1 \sigma_2 = 1$ and the realization of condition (4.3) only a co-phasal vibrator rotation is stable but when condition (4.3) is not realized only the counterphasal rotation is stable; in the case $\sigma_1 \sigma_2 = -1$ only the [illegible] phasal rotation must be stable.

These conclusions are fully confirmed experimentally and are used in vibrating machines for different purposes.

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НЕЛИНЕЙНЫЕ ЗАДАЧИ ТЕОРИИ ВИБРОТРАНСПОРТА И ВИБРОСЕПАРАЦИИ

NON-LINEAR PROBLEMS OF VIBROTRANSPORTATION AND VIBROSEPARATION

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Introduction.

The study of vibration transportation and vibration separation is a cardinal problem of the theory of working processes of a wide range of vibration machinery such as vibration conveyers, screens, feeders, unloaders, dryers, concentrating tables, vibrosinkers etc.

In turn, the central problem theory of vibration transportation and vibration separation is the problem concerning the motion of a material particle or a loose material media layer upon a rough vibrating surface.

So far, only the simplest particular cases of this problem have been investigated more or less in detail. This is conditioned by an essential non-linearity of equations describing the vibrotransportation processes.

The most difficult question concerning the motion of a loose material layer upon a vibrating surface which comes to the problem of dynamics of loose media complicated by the need to reckon upon the air resistance [1], [2].

However, the problem of a solid single particle motion too involves a lot of difficulties. The material partic-

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les have in fact various and irregular shapes and that is the reason why it is desirable to build up a theory embracing a wide enough class of particles without a complete precision concerning their shape. This can be achieved for "flat" particles which only slip during the movement upon the surface but do not roll from face to face [3], [5]. The movement of these particles is described accurately enough, as a movement of material points differing only in friction coefficient value. In this case the shape is of no importance and the separation at vibration occurs only according to friction coefficients.

The movement of "rounded" particles (not "flat") is to be considered as the movement of solid bodies rolling upon a vibrating plane. In this case the shape of particles is very important.

Presently the that had been problem studied most of all is the problem concerning the movement of a material point upon a rough surface carrying out rectilinear translated oscillations in conformance with the simple harmonic law. The works by G.Lindner [8], L.S.Lebenson [3,4], A.O.Spivakovsky [10], L.G.Lojtzlansky [11], S.A.Berg [12], N.P.Malkin [13], G.D.Terskov [14,15], V.A.Bauman [16], V.A.Olevsky [17,18], I.I.Blechman [5-8], A.Young [20], B.Clockhouse [21], S.Böttcher [22], S.A.Osmakov [23], J.Zaidel [24], D.D.Malkin [25], V.A.Brusin [26] and others deal with this problem.

However, and now this problem cannot be considered as exhaustively solved, particularly for motions without the contact with a plane.

The problem of a point motion upon a vibrating surface inspite of its specific character has a great practical importance. The results of its solution can be used in the problem of a ball and a cylinder motion upon a vibrating plane [5]. Tests have shown [2], [8], [22] that in many cases the mean velocity of a material layer motion is accurately enough determinated by formula for a particle. At last in the works [5-7, 28, 29] it has been shown that many problems, and among them those not concerning the considered problem of mechanics, come to the same differential equations system. The list of these problems will be expanded below.

The problems concerning the motion of a material point upon a vibrating plane carrying out rectilinear non-harmonic oscillations, harmonic oscillations in two mutually perpendicular directions (in particular-circular translated oscillations) as well as longitudinally. - transverse oscillations have been little investigated. The particular cases of the first problem have been considered in the works by G.Lindner [9] L.S.Levenson [4] N.G.Kopilov [30, 31] and others. The case of circular oscillations of a plane has been analysed by L.S.Levenson [4] and V.A.Olevsky [18]. Some problems concerning the

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motion of points and round bodies during transverse and longitudinally transverse oscillations of a plane were studied by Tstsy Shu-Khua [3].

This paper deals first with the problem of a material point motion upon a rough vibrating plane carrying out harmonic translated oscillations in two mutually perpendicular directions with a difference in phases. Rectilinear harmonic and circular oscillations are specific cases of this law of motion.

Further the paper deals with a detailed analysis of the case of the motion of a particle in absence of bouncing and reports some new research results concerning the motion with bouncing.

In conclusion it reports a communication concerning concrete applications of theory to the problems of calculation of vibrating conveyers, vibroseparators, vibrosinkers etc.

§ 1. Equations of a material point movement upon a rough plane carrying out harmonic translated oscillations in two mutually perpendicular directions

Let us consider the material point (flat particle) relative motion upon a plane with a slope to the horizont under some angle α carrying out translated oscillations

upon elliptical trajectories resulting from the summation of oscillation in two arbitrary mutually perpendicular directions, lying in a vertical plane, perpendicular to the vibrating plane considered (Fig. 1).

At an arbitrary choice of an immobile system of coordinates axes $\xi O_1 \eta$ displacement $\tilde{\xi}$ and $\tilde{\eta}$ points of the plane along these axes in the problem examined may be presented in form:

$$\tilde{\xi} = a \sin(\omega t + \varepsilon), \quad \tilde{\eta} = b \sin \omega t \tag{1.1}$$

Here a and b - are amplitudes of constituent oscillations, ω - frequency, and ε - difference in phase between constituents.

The equation of the elliptical trajectory of an arbitrary plane point A in the indicated motion in local immobile axes $\tilde{\xi} O_2 \tilde{\eta}$ is obtained by means of exclusion of time t from equations (1.1):

$$\left(\frac{\tilde{\xi}}{a}\right)^2 + \left(\frac{\tilde{\eta}}{b}\right)^2 - 2\frac{\tilde{\xi}}{a}\frac{\tilde{\eta}}{b}\cos\varepsilon = \sin^2\varepsilon \tag{1.2}$$

The equation (1.2) does not change at substitution of ε for $-\varepsilon$. Practically speaking the ellipse shape at the substitution indicated as it was previously remains. However such substitution leads to a change of the motion direction of the plane points upon the ellipse.

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The case of rectilinear harmonic oscillations of a plane under angle β and with an applitude A is obtained from (1.1) where

$$a = A\cos\beta,\ b = A\sin\beta \text{ and } \varepsilon = 0$$

To the circular oscillations of a plane correspond $a = b = A_0$, $\varepsilon = \pm\frac{1}{2}$ (A_0-radius of circular trajectory oscillations).

Let us introduce the mobile axes of coordinates xOy invariably constrained with the oscillating plane and parallel to the immobile axes $\xi O_1\eta$ and $\tilde{\xi} O_2\tilde{\eta}$ Then the equations of the relative motion of particle of the mass m in projections on axes xOy will be

$$m\ddot{x} = ma\omega^2\sin(\omega t+\varepsilon) - mg\sin\alpha + F \tag{1.3}$$

$$m\ddot{y} = mb\omega^2\sin\omega t - mg\cos\alpha + N \tag{1.4}$$

Here F - projection of the force of Coulomb's friction on axes Ox, N - projection of the normal reaction upon axis Oy, $-m\ddot{\xi} = ma\omega^2\sin(\omega t+\varepsilon)$ and $-m\ddot{\eta} = mb\omega^2\sin\omega t$ - projection of inertia $\bar{J}$ on axis Ox and Oy.

At the motion of the particle upon the vibrating plane ($y=0$, $\dot{x}\neq 0$) according to equation (1.4) the projection of a normal reaction upon axis Oy is given by formula

$$N = N(t) = mg\cos\alpha - mb\omega^2\sin\omega t \tag{1.5}$$

and in conformance with Coulomb's law the projections of the friction force upon axis Ox is

$$F = F(\dot{x}) = \mp fN(t) \tag{1.6}$$

Substituting the expression (1.5) in (1.6) and then (1.6) in (1.3) we obtain after simple transposing the following equation of the motion of a particle upon a plane

$$\ddot{x} = -g\frac{\sin(\alpha\pm\rho)}{\cos\rho} + \frac{M_\pm\omega^2\sin(\omega t+\alpha_\pm)}{\cos\rho} \tag{1.7}$$

Where $\rho = \operatorname{arctg} f$ - angle of kinetic friction and where besides the following symbols are brought in

$$M_\pm = \sqrt{a^2\cos^2\rho \pm 2ab\cos\varepsilon\sin 2\rho + b^2\sin^2\rho}$$

$$M_\pm\cos\alpha_\pm = a\cos\rho\cos\varepsilon \pm b\sin\rho$$

$$M_\pm\sin\alpha_\pm = a\cos\rho\sin\varepsilon \tag{1.8}$$

Whereupon here and further while dual symbols are present in formulae the upper symbols are corresponding to the

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slide of the particle forward ($\dot{x}>0$) and lower - to the slide backwards ($\dot{x}<0$).

The particle in reality remainson the vibrating plane only on condition of the positiveness of the reaction $N(t)$.

In conformance with (1.6) at the execution of [illegible]ity

$$Z_0 \geqslant 1, \quad Z_0 = \frac{g\cos\alpha}{e\omega^2} \tag{1.9}$$

[illegible] projection of reaction $N(t)$ is always positive and the particle getting upon a vibrating plane with a zero transv[illegible] component velocity remains further on the plane.

If, however, the inequality (1.9) is not true then in moment t_0 determined from equation

$$\sin\omega t_0 = \frac{g}{e\omega^2}\cos\alpha = Z_0 \tag{1.10}$$

the function $N(t)$ is converted to zero and the particle whi[illegible] initially remained on the plane looses contact wi[illegible]

The equations of the particle flight ($y>0$) are obtained from (1.3) and (1.4) if $F=N=0$

$$\ddot{x} = e\omega^2 \sin(\omega t+\varepsilon) - g\sin\alpha, \tag{1.11}$$

$$\ddot{y} = e\omega^2 \sin\omega t - g\cos\alpha \tag{1.12}$$

In a state of relative rest ($x=0, y=0$) the particle may stay on the vibrating plane only on condition that the projection of the friction force upon axis Ox

$$F = F^0(t) = mg\sin\alpha - ma\omega^2\sin(\omega t+\varepsilon) \tag{1.13}$$

obtained from equation (1.3) at $x=0$ does not exceed in its absolute value the modul of the maximum value of the static friction force $F_{max} = f_1 N$ or

$$-f_1 N(t) < F^0(t) < f_1 N(t) \tag{1.14}$$

Here f_1 - is the coefficient of the static friction

To obtain a complete system of relationships determining the particle motion it is necessary to indicate the law of the change of velocity of a particle at collision with a vibrating plane. As a rule it is considered that the collision practically occurs instantly while it is suggested that in course of the impact only the component velocities of the particle are subject to change and the effect of the impact on the motion of the plane is to be disregarded.

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Further the majority of research works it is assumed that as a result of the impact the transverse projection of the velocity $\dot{y}$ is equal zero i.e. that the impact is considered as absolutely non-elastic. Only some works attempt to take into consideration the elasticy of bodies in collision. For this purpose the coefficient of restitution is introduced

$$0 < R = -\frac{\dot{y}_n'}{\dot{y}_n} \leqslant 1 \qquad (1.15)$$

Here $\dot{y}_n$ and $\dot{y}_n'$ projections of the particle velocity upon the axis Oy before and after the impact.

The hypothesis accepted by various authors concerning the law of change of the longitudinal component velocity resulting from impact are more varied (see for instance [15], [24] and [25]).

Unlike these works, let us assume that the longitudinal component of velocity of the particle before the impact $\dot{x}_n$ and after the impact $\dot{x}_n'$ are connected by the relationship

$$\frac{\dot{x}_n'}{\dot{x}_n} = 1 - \lambda \qquad (1.16)$$

where $0 \leqslant \lambda \leqslant 1$ the so called coefficient [illegible] stant friction at impact. Let us consider the [illegible] of this coefficient as known. It is to be noted that it can easily be determined experimentally upon the results of the measurement mean velocity of a particle motion (see below).

From (1.16) at $\lambda = 0$ and $\lambda = 1$ we obtain frequently used relationships.

The non-linearity of the above written out equations of the particle motion is generated first by the presence of the Coulomb's the presence of the friction force and second, by the unilateral constraint.

The basic difficulty of the problem solution is conditioned by the fact that the differential equations on separate stages (for instance at slipping or at flight are written in a different analytical from. At the same time at every equation stage it is easy integrated.

It is the search for-transition moments from one stage to the other that is a very difficult part of the problem. The basic investigation problem is to find solutions for the mentioned above equations system to which correspond steady state regimes of the particle motion upon a vibrating plane i.e. motions established after the passing of a great enough time interval. Transition regimes will further be considered only in so far as it is necessary for the solution of stability questions.

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§ 2. General analysis of the problem. Integrals of motion equations at different stages.

Some purely qualitative representation concerning the nature of possible steady state motion of a particle as well about the location of transition moments from one stage to another in a steady state motion is possible to obtain without integration of equations of motion. To illustrate it let us notice that all the t - axis may be divided into two kinds of intervals just as it has been done in the particular case of the problem in work [5]

Let us call one kind of intervals, [termed intervals I.] They are characterised by a positive projection of the normal reaction $N(t)$. The projection is defined by (1.6).

Intervals of the second kind (termed interwals II) let us define as intervals during which the inequality of an opposite meaning is fullfilled i.e. $N(t)<0$. Let us consider the physical meaning of the division into intervals. If a particle having a zero transversal projection of the velocity $\dot{y}$, gets onto a vibrating plane at a moment t of the interval I, it will remain on the plane for some finite period of time till the loss of contact moment of t_0 defined by equations (1.11). If a particle having a zero transversal projection of the velocity $\dot{y}$ gets onto a vibrating plane at a moment belonging to the interval II, the particle immediately leaves the vibrating plane.

Intervals I exist at all possible values of the system parameters. As to the intervals II they take place only unequality is accomplished

$$Z_0 = \frac{g\cos\alpha}{\varepsilon\omega^2} < 1 \qquad (2.1)$$

In fact if $Z_0 > 1$ inequality $N(t)<0$ cannot be completed at any real values of t, in other words, the intervals II do not exist.

Naturally this conditions of the absence of intervals II coincides with the condition (1.9) of the absence of bouncing of a particle got on a vibrating plane with a zero transversal velocity $\dot{y}$. But it should be noted that inequality $Z_0 \geqslant 1$ does not suggest impossibility of a steady state regimes of motion with bouncing. As it will be shown further (see § 6) in case of a restitution coefficient R different from zero, and in presence of a transversal vibration constituent (viz. $\varepsilon \neq 0$) generally speaking there may exist conditions under which a regime with a continuous particle bouncing is maintained even at $Z_0 \geqslant 1$. Only in the case of an absolutely unelastic impact ($R=0$), or at purely transversal vibrations of the plane ($\varepsilon=0$) it is possible to affirm in advance the impossibility of steady state regimes of motions with bouncing if only the particle got on a plane with the velocity $\dot{y}=0$ is not bouncing.

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Generally speaking, each of the intervals I can be divided into three subintervals according to the following relations.

During the subinterval I_{1+} the negative force of friction F° necessary to retain the particle on the plane in a state of relative rest is less than the minimum possible value of the static friction force $-f_1 N(t)$ i.e.

$$F^{\circ}(t) < -f_1 N(t) \tag{2.2}$$

During the subinterval I_{1-} the force $F^{\circ}(t)$ is greater than the maximum value of the static friction force equal to $f_1 N(t)$:

$$F^{\circ}(t) > f_1 N(t) \tag{2.3}$$

And, at last during the subinterval I_{10} the force of friction F° in its absolute value does not exceed the static friction force $f_1 N(t)$ that is:

$$-f_1 N(t) < F^{\circ}(t) < f_1 N(t) \tag{2.4}$$

If a particle having a zero relative velocity ($\dot{x} = \dot{y} = 0$) has got onto a vibrating plane at a moment of the subinterval I_{1+}, this particle will immediately slip forward on this plane ($\dot{x} > 0$). If the moment of getting I_{1-}, the particle will immediately slip back

($\dot{x} < 0$). And at last if the moment is somewhere in the subinterval I_{10}, the particle will stay in a state of relative rest till the beginning of one of the subintervals I_{1+} or I_{1-} or till the beginning of interval II

Inequalities defining intervals and subintervals, may be expressed plainly provided the exspressions for forces according to (1.5) and (1.13) $N(t)$ and $F^{\circ}(t)$ would be substituted.

Resulting relationships are given in table 1.

Table 1

Intervals			Intervals
$Z_0 - \sin\omega t > 0$			
Subintervals I_{1+}	Subintervals I_{10}	Subintervals I_{1-}	$Z_0 - \sin\omega t < 0$
$Z'_{1+} - \sin(\omega t + \chi_{1+}) < 0$	$Z'_{1+} - \sin(\omega t + \chi_{1+}) > 0$	$Z'_{1-} - \sin(\omega t + \chi_{1-}) > 0$	
	$Z'_{1-} - \sin(\omega t + \chi_{1-}) < 0$		

In the table the following symbols besides (1.9) are presented:

$$Z'_{1\pm} = \frac{g}{M_{1\pm}\omega^2} \sin(\alpha \pm \rho_1)$$

$$M_{1\pm} = \sqrt{a^2 \cos^2\rho_1 \pm ab\cos\varepsilon \sin 2\rho_1 + b^2 \sin^2\rho_1}$$

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$$M_{1\pm}\cos\varkappa_{1\pm} = a\cos\rho_1\cos\varepsilon \pm B\sin\gamma$$

$$M_{1\pm}\sin\varkappa_{1\pm} = a\cos\rho_1\sin\varepsilon \qquad (2.5)$$

Here the angle of friction $\rho_1 = \operatorname{arctg} f_1$. The limits for the intervals and subintervals introduced are derived from equations

$$\sin\omega t_0 = z_0 \qquad (2.6)$$

$$\sin(\omega t_{1\pm} + \varkappa_{1\pm}) = z'_{1\pm} \qquad (2.7)$$

These equations have infinite number of roots. We shall mean by $\delta_0 = \omega t$ and $\delta_{1\pm} = \omega t_{1\pm}$ only roots defined by the expressions:

$$\delta_0 = \arcsin z_0$$

$$\delta_{1+} = -\varkappa_{1+} + \arcsin z'_{1+}, \quad \delta_{1-} = \pi - \varkappa_{1-} - \arcsin z'_{1-} \qquad (2.8)$$

Here $\arcsin z$ denotes the principal value of the functions ranging in the limits $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$. Other roots of the equations (2.6), (2.7) are easily expressed by means of the quantities δ_0 and $\delta_{1\pm}$. Besides the phase angle δ_{1-} we shall introduce γ_{1-} which is connected with δ_{1-} by a relationship

$$\gamma_{1-} = \delta_{1-} - \pi + \varkappa_{1-}$$

[illegible] β, $\varepsilon = 0$ [illegible] rectilinear harmonic oscillations with an [illegible] taking place under angle β to the direction the plane the values z_0, δ_0, $\delta_{1\pm}$ and [illegible] well as all other relations turn into corresponding values and into the formulas of [5].

This is not difficult, to show, finding [illegible] above mentioned particular case according to [illegible] (1.9) and (2.5) the following formulas

$$M_{\pm} = A\left|\cos(\beta \mp \rho)\right|, \quad M_{1\pm} = A\left|\cos(\beta \mp \rho_1)\right|$$

$$\varkappa_{0-} = \begin{cases} 0 & \text{при} \quad \sigma = 1 \\ \pi & \text{при} \quad \sigma = -1 \end{cases}$$

$$\varkappa_{1-} = \begin{cases} 0 & \text{при} \quad \sigma_1 = 1 \\ \pi & \text{при} \quad \sigma_1 = -1 \end{cases}$$

$$\left(\sigma = \operatorname{sign}\left[\tfrac{1}{2}\pi - (\beta+\rho)\right], \quad \sigma_1 = \operatorname{sign}\left[\tfrac{\pi}{2} - (\beta+\rho_1)\right]\right) \qquad (2.10)$$

are true

To divide t - axis graphically into intervals and subintervals it is quite sufficient to plot the chart of the function $\sin\omega t$ and $\sin(\omega t + \varkappa_{1\pm})$. The abscissa of the intersection points of these curves with the corresponding horisontal straight line $y = z_0$ and $y = z'_{1\pm}$ will limit the intervals and subintervals.

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As in the case of rectilinear harmonic oscillations [5] some subintervals may not exist. This happens if above mentioned subintervals are "overlapped" by the intervals II. We shall not give the detailed analysis here as this case is properly discussed in article [5].

Let's write both the integrals of the motion of the particle at different stages and the equations defining the moments of transition from one stage of motion to another.

Velocity projection and the displacement of the particle at the stage of slipping on the plane at initial conditions $\dot{x}(t^*)=\dot{x}^*$, $x(t^*)=0$ are given by the following expressions which are obtained by integration of equation (1.7).

$$\dot{x}(t)=-g\frac{\sin(\alpha\pm\rho)}{\cos\rho}(t-t^*)-\frac{M_\pm\omega}{\cos\rho}\left[\cos(\omega t+\varkappa_\pm)-\cos(\omega t^*+\varkappa_\pm)\right]+\dot{x}^*, \tag{2.11}$$

$$x(t)=-g\frac{\sin(\alpha\pm\rho)}{2\cos\rho}(t-t^*)^2+\frac{M_\pm\omega\cos(\omega t^*+\varkappa_\pm)}{\cos\rho}(t-t^*)-\frac{M_\pm}{\cos\rho}\left[\sin(\omega t+\varkappa_\pm)-\sin(\omega t^*+\varkappa_\pm)\right]+\dot{x}^*(t-t^*) \tag{2.12}$$

Formulas (2.11) and (2.12) describe the motion of the particle until it stops or leaves the plane. In the last case the moment of transition from the stage of slipping [illegible] with the speed of the

terval II beginning. If the stage of slipping is followed by the period of rest or by the period of slipping in the opposite direction, the moment of transition $t=t^{**}$ is defined from equation which is obtained by putting expression (2.11) equal to zero:

$$\cos(\varphi_\pm+\varkappa_\pm)=\cos(\delta_\pm^*+\varkappa_\pm)-z_\pm'(\varphi_\pm-\delta_\pm^*)+a_\pm \tag{2.13}$$

The following symbols are introduced:

$$z_\pm'=\frac{g}{M_\pm\omega^2}\sin(\alpha\pm\rho),\quad a_\pm=\frac{\dot{x}^*\cos\rho}{M_\pm\omega} \tag{2.14}$$

$$\varphi_\pm=\omega t^{**},\quad \delta_\pm^*=\omega t^* \tag{2.15}$$

Though there are some roots in equation (2.13) only the least root which is exceeding $\delta_\pm^*$ has. In case when there are no roots greater than $\delta_\pm^*$, the particle would slip continuously in one direction. This is obviously possible as in the case of rectilinear oscillations only if $|\alpha|>\rho$. Otherwise the particle is certain to stop after a certain lapse of time.

As the result of the introduction of auxiliary variables:

$$\varphi_\pm'=\varphi_\pm+\varkappa_\pm,\quad \delta_\pm^{*\prime}=\delta_\pm^*+\varkappa_\pm \tag{2.16}$$

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[illegible] reduced to [illegible] [illegible]. Equation [illegible] describes rectilinear harmonic oscillations. The methods discussed in [5] are completely applicable for the solution of the equation (2.13).

It is possible to use special graphs of this article to solve equation for the most important particular case, when the initial velocity equals zero and $a_{\pm}=0$. For this it is sufficient to introduce phase angles $\delta_{\pm}$ and γ defined by the relationships (2.8,2.9) (it is assumed that $|Z'_{\pm}|<1$) :

$$\delta_{+} = -\varkappa_{+} + \arcsin Z'_{+}, \quad \delta_{-} = \pi - \varkappa_{-} - \arcsin Z'_{-}$$

$$\gamma_{-} = \delta_{-} - \pi + \varkappa_{-} \tag{2.17}$$

If the coefficient of friction of rest equals the coefficient of friction at slipping $f = f_1$ then $Z'_{1\pm} = Z'_{\pm}$, $\varkappa_{1\pm} = \varkappa_{\pm}$, $\delta_{1\pm} = \delta_{\pm}$ and $\gamma_{1-} = \gamma_{-}$.

Using (2.17), equation (2.13) for the given particular case ($\dot{x}^{*} = 0$) can be reduced to

$$\cos(\varphi_{\pm} + \varkappa_{\pm}) = \cos(\delta^{*}_{\pm} + \varkappa_{\pm}) - \sin(\delta_{\pm} + \varkappa_{\pm})(\varphi_{\pm} - \delta^{*}_{\pm}) \tag{2.18}$$

Hence it is clear that equation (2.13) comes to of the article (2.19) [5] by substitution (2.16) and

$$\delta'_{\pm} = \delta_{\pm} + \varkappa_{\pm} \tag{2.19}$$

The above mentioned graphs are just plotted for equation (2.19) of the article [5]. That is why these graphs are applicable to the solution of equation (2.18). too.

Let us consider the motion of the particle at the stage of flying. Let us assume that the initial conditions are:

$$x(t_0^{*}) = 0, \quad \dot{x}(t_0^{*}) = \dot{x}_0^{*};$$

$$y(t_0^{*}) = 0, \quad \dot{y}(t_0^{*}) = \dot{y}_0^{*} \tag{2.20}$$

By integrating equation (1.12) one obtains for the velocity projection and the displacement of the particle in flight the following expression

$$\dot{x}(t) = -g\sin\alpha(t - t_0^{*}) - a\omega[\cos(\omega t + \varepsilon) - \cos(\omega t_0^{*} + \varepsilon)] + \dot{x}_0^{*},$$

$$x(t) = -g\frac{\sin\alpha}{2}(t - t_0^{*})^2 + a\omega(t - t_0^{*})\cos(\omega t_0^{*} + \varepsilon) - a[\sin(\omega t + \varepsilon) - \sin(\omega t_0^{*} + \varepsilon)] + \dot{x}_0^{*}(t - t_0^{*}) \tag{2.21}$$

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$$\dot{y}(t) = -g\cos\alpha\,(t - t_o^*) - \varepsilon\omega(\cos\omega t - \cos\omega t_o^*) + \dot{y}_o^*,$$

$$y(t) = -\frac{g\cos\alpha}{2}(t - t_o^*)^2 + \varepsilon\omega(t - t_o^*)\cos\omega t_o^* -$$

$$- \varepsilon(\sin\omega t - \sin\omega t_o^*) + \dot{y}_o^*(t - t_o^*)$$

(2.22)

Formulas (2.21), (2.22) described the motion of the particle only untill it gets onto the plane. The moment of falling $t_n > t_o$ is defined as the root of equation closest to t_o :

$$\frac{1}{2} z_o(\varphi_n - \delta_o^*)^2 + (\sin\varphi_n - \sin\delta_o^*) -$$

$$-(\varphi_n - \delta_o^*)(\varepsilon^* + \cos\delta_o^*) = 0.$$

(2.23)

This equation is obtained by putting expression $y(t)$ (2.22) equals zero at the moment $t = t_n$

In the formula (2.23) new symbols are introduced

$$\varphi_n = \omega t_n, \quad \delta_o^* = \omega t_o^*, \quad \varepsilon^* = \frac{\dot{y}_o^*}{\varepsilon\omega}$$

(2.24)

A besides old symbols (1.10) are used. Equation (2.23) for $\dot{y}_o^* = 0$, $\varepsilon^* = 0$ coincides with the transcendental

equation connects phase angles of falling and the beginning of flying in the problem of the motion of a particle on rectilinear vibrating plane [5].

In this important particular case it is possible to use graphs supplied in [5].

It should be noted that general equation (2.23) coincides with the corresponding equation for rectilinear oscillations.

Let present formulas describing displacement of a particle and the increase of its velocity during one stage of motion. These formulas are very important for defining the mean velocity of the motion of a particle on the plane with steady state regime.

The displacement of the particle $S_\pm$ during the stage of slipping on the plane which ends in stop is calculated with the help if equetion (2.12). Using symbols (2.14) and (2.15) gives the following equation

$$S_\pm = x(t^{**}) = \frac{M_\pm}{\cos\rho}\left\{-\mathcal{Z}_\pm\frac{(\varphi_\pm - \delta_\pm^*)^2}{2} - \left[\sin(\varphi_\pm + \chi_\pm) - \right.\right.$$

$$\left.\left. - \sin(\delta_\pm^* + \chi_\pm)\right] + (\varphi_\pm - \delta_\pm^*)\cos(\delta_\pm^* + \chi_\pm)\right] +$$

$$+ \frac{\dot{x}^*(\varphi_\pm - \delta_\pm^*)}{\omega}$$

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In substituting (2.16) and (2.19) equation (2.25), we reduce equation (2.25) to the same form as the corresponding expression for rectilinear oscillations of the plane formula (3.1) [5]. Hence it is possible to use the results derived for the above mentioned simplest case, in particular, graphs from the articles [15] and [5] for function F, and with the help of it is easy to define $S_{\pm}$ at $\dot{x}^{*}=0$ (the case of sliding from the state of rest).

Let us consider the case when the slipping stage is succeeded by the flying stage. In this case the transition mount (from slipping to flight) is the initial moment $t_0+\frac{2\pi\kappa}{\omega}$ of the interval II defined by equation (1.11), κ being integer or zero.

The expression for the particle displacement in a stage in the case given is derived from (2.25) by means of substitution of the phase angle of particle $\varphi_{\pm}$ stop by the phase angle of the flight beginning $\delta_0+2\pi\kappa=\omega t+2\pi\kappa$

$$S_{\pm}=x\left(t_0+\frac{2\pi\kappa}{\omega}\right)=\frac{M_{\pm}}{\cos\rho}\left\{-\chi_{\pm}\frac{(\delta_0-\delta_{\pm}^{*}+2\pi\kappa)^2}{2}-\right.$$

$$-\left[\sin(\delta_0+\chi_{\pm})-\sin(\delta_{\pm}^{*}+\chi_{\pm})\right]+$$

$$\left.+(\delta_0-\delta_{\pm}^{*}+2\pi\kappa)\cos(\delta_{\pm}^{*}+\chi_{\pm})\right\}+$$

$$+\frac{\dot{x}^{*}(\delta_0-\delta_{\pm}^{*}+2\pi\kappa)}{\omega}\qquad(2.26)$$

The increment of the velocity of the particle $\Delta\dot{x}_{\pm}$ during the time of slipping before the beginning of flying according to (2.11) will be

$$\Delta\dot{x}_{\pm}=\dot{x}\left(t_0+\frac{2\pi\kappa}{\omega}\right)-\dot{x}^{*}=\frac{M_{\pm}\omega}{\cos\rho}\left\{-\chi_{\pm}(\delta_0-\delta_{\pm}^{*}+\right.$$

$$\left.+2\pi\kappa)-\left[\cos(\delta_0+\chi_{\pm})-\cos(\delta_{\pm}^{*}+\chi_{\pm})\right]\right\}\qquad(2.27)$$

The displacement of the particle S_n during the time of flight according to (2.21) and (2.24) is given by the formula

$$S_n=-\frac{g\sin\alpha}{2\omega^2}(\varphi_n-\delta_0^{*})^2-a\left[\sin(\varphi_n+\varepsilon)-\sin(\delta_0^{*}+\varepsilon)\right]+$$

$$+a(\varphi_n-\delta_0^{*})\cos(\delta_0^{*}+\varepsilon)+\frac{\dot{x}_0^{*}(\varphi_n-\delta_0)}{\omega}\qquad(2.28)$$

The increment of the particle during the time of its flight is evaluated according to equation (1.21) taking into consideration the decrease of the velocity at the collision of the particle with the plane according to (1.15):

$$\Delta\dot{x}_n=\dot{x}_n'-\dot{x}_0^{*}=\dot{x}(t_n)(1-\lambda)-\dot{x}_0^{*}=$$

$$=-a\omega(1-\lambda)\left\{\frac{g}{a\omega^2}\sin\alpha(\varphi_n-\delta_0^{*})+\right.$$

$$\left.+\left[\cos(\varphi_n+\varepsilon)-\cos(\delta_0+\varepsilon)\right]\right\}-\lambda\dot{x}_0^{*}\qquad(2.29)$$

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The formulae mentioned above are the basis for the investigation of steady state regimes of the particle motion upon a vibrating plane as well as for the calculation of the average velocity (or average acceleration) in these regimes.

§ 3. The steady state regimes of particle motion and their classification

Before proceeding with the investigation of the steady state particle motions upon an oscillating plane, it should be expedient to introduce their classification understressing practically the most interesting cases.

The type of particle motion is essentially characterised by transition moments from one stage to the other. At the same time since the outside force (inertia in a relative motion) is periodical it would be logical to assume that the location of transition moments along the t - axis in steady state motions will as a rule, be periodical i.e. that in case of a steady state particle motion such a minimum number s transition moments from one motion change to another may be indicated

$$t_1^{(0)} < t_2^{(0)} < \ldots < t_s^{(0)}, \tag{3.1}$$

so that all following transition moments are determined by the relationships

$$t_1^{(\kappa)} = t_1^{(0)} + \kappa T, \quad t_2^{(\kappa)} = t_2^{(0)} + \kappa T,$$

$$t_s^{(\kappa)} = t_s + \kappa T$$

$$(\kappa = 1, 2, \ldots)$$

(3.2)

The time moments $t_1^{(0)}, t_2^{(0)} \ldots, t_s^{(0)}$ by use of which all consequents are expressed according formulae (3.2) let us call typical transition moments and the quantity T - period of transition moments - period of steady state regimes switchings.

Steady state motions with a finite period of switching multiple to the period of the plane oscillations T_0 will be called proper, and motions with a period of switching which is not multiple to T_0 or with an infinite great period of switching - non-proper.

The most frequently met and most interesting from the point of view of their application are the proper motions with a period of switching not to big in comparison with the period of disturbing force.

So, for instance, it appears (see further) that in the motion of a particle without bouncing at an angle slope of the plane to the horizon α, lesser by the modulus of the friction angle φ, all the steady state regimes have a period of switching equal to the oscillations period T_0.

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Non-proper steady state motions of a particle upon a vibrating plane are either impossible or of no considerable inte es

It is expedient to begin a systems tic study of steady state motions by a consideration of proper motions in order of growth of the switching periods.

In addition to the classification introduced above it is expedient to divide all the steady state motions into two groups, and according to another criterion.

To the first group we shall refer the motions with a periodic longitudinal velocity component when a particle in course of every switching period is transfered along the plane for equal distances. Such regime will be called regular.

To the second group we shall refer the regimes where the particle velocity is a non-periodic function of time so that the displacement of the particle along the plane for a period of switching is a variable value. Let us agree to call such regimes accelerated.

From the determination of regular regimes it follows that the period of switching in these regimes is a period of a particle velocity change and acceleration change. Thus the solutions of equations for particle motions (1.3), (1.4) corresponding to regular regimes are as

$$x = V_{cp}\, t + \varphi(t), \quad y = \psi(t) \tag{3.3}$$

where V_{cp} is the constant mean velocity of the particle

displacement along a plane, and $\varphi(t)$ and $\psi(t)$ - periodical functions of time with a period equal to the period of switching T .

From the fact, that the right parts of the equations for particle motion (1.3), (1.4) have a period $T_0 = \frac{2\pi}{\omega}$ it follows that in proper steady state motions the particle acceleration is a periodic function of time the period of which equal to the switching period T . Therefore for steady state regular accelerated particle motions the solutions of motion equations (1.3), (1.4) may be written as

$$x = W_{cp}\frac{t^2}{2} + Vt + \varphi_1(t), \quad y = \psi_1(t) \tag{3.4}$$

where W_{cp} - mean longitudinal particle acceleration, V - constant and $\varphi_1(t)$ and $\psi_1(t)$ - periodical functions of zime with a period equal to a period of switching T .

The values of typical transition moments $t_1^{(0)}$, $t_2^{(0)}\ldots$, $t_5^{(0)}$ in steady state motions their character, number and switching period are determined by the basic parameters of the problem (i.e. by angles α and β , the amplitude of plane acceleration $A\omega^2$, friction coefficients f and f_1) and as a rule, do not depend on initial conditions.

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The location of transition moments $t_j^{(0)}$ has a, prior dash importance as their knowledge allows with help of the formula 1.2 to calculate the mean velocity of the particle motions upon a plane for regular regimes, and the mean accelerations for the accelerated regimes.

Different methods for finding out typical transition moments are possible. Further we shall prefer the so-called reverse method [5] which allows to obtain conditions for the existence of many important types of steady state regimes in an analytical form, that is to avoid the application of a integration by stage or of any other method requiring cumbersome calculations.

By means of the reverse method first on ground of an analysis of motion equations all conceivable types of steady state regimes are established. Then general integrals of the motion equations for separate stages are found and after this the regions of existence and steadiness is determined for every conceivable ty pe of steady state motion.

4. Stability of steady state regimes of particle motions on transition moments.

When investigating the stability of steady state motions in the problem considered it is expedient to make use of the Sp[illegible]ral determination of stability [5].

[illegible] together with a steady state regime (undisturbed motion") we shall consider some "disturbed motion" when the transition from some determined motion stage to another occurs not at the moment $t_0 = t_j^{(k)}$ as in a steady state regime but at another time moment τ_0 :

Let us call the steady state regime stable if at least at a small enough positive η and big enough (but finite) $T^{(0)}$ at the completion of the unequality

$$|\tau_0 - t_0| < \eta \tag{4.1}$$

for $\tau_j^{(k)} > T^{(0)}$ the following conditions are effected:

1. to each transition moment $t_j^{(k)}$ from one stage to the other in an undisturbed motion corresponds one transition moment $\tau_j^{(k)}$ of the same character in a disturbed motion, and, inversely, to each transition moment $\tau_j^{(k)}$ corresponds one transition moment $t_j^{(k)}$;

2. the inequalities are true

$$|\tau_j^{(k)} - t_j^{(k)}| < \varepsilon \tag{4.2}$$

where ε is, as much as desired, small in advance chosen, positive number.

If together with the first condition instead of the second more strong conditions are fullfilled

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$$\lim_{t \to \infty} \tau_j^{(\kappa)} = t_j^{(\kappa)} \quad (j = 1,2, \ldots s) \tag{4.3}$$

we shall say that the steady state regime is asymptotically stable, on transition moments.

At last let us call the: steady state regime in the large if the first condition and the relationships (4.3) are effected independ on the assumption concerning the completion of the condition (4.1).

There may be cases when the stability of the regime is interesting not for all typical transition moments $t_j^{(o)}$ but only for some of them. Then we shall speak about a conditional stability on corresponding transition moments.

The requirement for the stability of a steady state regime on the transition moments is less rigid than the requirement for stability in the sense of A.M.Ljapunov. It is easy to refer to an example of a motion stable in the sense mentioned above, and unstable according to Ljapunov.

The use in the problem considered of the conception concerning the stability, on transition moments allows in many cases to confine oneself to more simple calculations and reasoning than those, that would be constrained with the stability investigation in the sense of A.M.Ljapunov. On the other hand for the problem given where essentially only the particle mean velocity and the mean acceleration represent interest, it is quite enough to prove the stability of the steady state regime on transition moments.

In fact from the continuous-character of the dependence of the particle displacement for a stage from the initial and the finite stage moments (see § 2), it follows that in case of a regular steady state regime stability on transition moments the mean particle velocity for the switching period in a disturbed motion differs so small as desired (at a big enough t) from the mean velocity in an undisturbed motion if only the disturbance is small enough. The same may be said in relation to the mean for a switching period acceleration of a particle in an accelerated steady state motion.

It is to be emphasized, that the determination mentioned above may be useful at the investigation of properties of solutions differential equations with right parts which in different regions of change of variables are set by different analytical expressions.

Let us notice too, that when solving the problem considered, and others close to it by means of the point transformation method [26] virtually the inverse method of looking for steady state motions is used and their stability is investigated on transition moments, so that the

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difference exists only in terminology.

We shall mostly be satisfied with consideration of stability "in law" thus the problem comes to the solution of a linear recurrent equations system and to the studying of conditions for the realization of some relationships.

§ 5. Steady states motion regimes in absence of bouncing. Case of rectilinear plane oscillations

The motion of a particle without loss of contact with a vibrating plane may occur, as it has been mentioned above only on condition of execution of [illegible] restitution coefficient R at the impact the particle against the plane is equal to zero, (1.9) and at the same time an indispensable and sufficient condition for the particle motion without bouncing at $1 \geqslant R > 0$, this condition is only necessary at insufficient [illegible] there are steady state regimes of particle motions with continuous contact, certainly, they are practically unstable as the particle reaching the plane with a transverse component velocity equal or near to zero at the execution of condition (1.9) will not bounce any more.

Using the inverse method let us consider the steady state motion regimes in absence of bouncing.

As it has been established in § 2, the particle which reached the vibrating plane with a zero transverse velocity component $\dot{y}$ and with an arbitrarily longitudinal velocity component $\dot{x} = \dot{x}^*$ either slips downwards upon the plane without stopping (this is possible only on condition $|\alpha| > \rho$) or stops after a certain finite time interval. Thus, in absence of bouncing, two types of steady state particle motions are possible:

a) uninterrupted slipping in one direction (downwards). Such motion is accelerated and has an infinite great switching period;

b) regimes with stops which, naturally, are regular regimes since in accelerated regimes a particle cannot stop.

Let us first study the regimes with stops. We have to begin with a regime having the switching period T equal to the period of plane oscillations $T_0 = \frac{2\pi}{\omega}$. For this purpose let us look at Fig. 2 where at $z_0 \geqslant 1$ the most general case of time axis division in subintervals is represented. At it has been noticed this division is periodical: every other time interval $t = T_0$, time lengths corresponding to determined subintervals are recuring.

Fig. 2 shows moments of transition from one stage of motion to another which in conformance with the principles of division may belong to one or to another subinter-

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val. At this, the moment of transition from one lasting stop x) to slipping forward is denoted by $t_{o+} = \varphi_{o+}/\omega$, by $t_{-+} = \varphi_{-+}/\omega$ - from slipping backwards to slipping forward, by $t_{+-} = \varphi_{o-}/\omega$ from a lasting stop to a slipping back. The moments t_{+o}, t_{+-} and t_{-o} correspond to reverse transitions respectively.

If the transition moments get to the boundary of two subintervals as, for instance, it is the case for moments t_{o+} and t_{o-} it is convenient to relate them to a beginning subinterval. Then each transition moment will belong only to one of the subintervals. For instance moment t_{+-} always belongs to subinterval I_{1-}

Examination of Fig. 2 leads to the conclusion that only the following steady state regimes of particle motion with stops having a switching period $T = T_o = 2\pi/\omega$ are possible. The arrows directed to the right on Fig. 2 mark time intervals corresponding to the stages of particle slipping forward; the arrows directed to the left correspond to the stages of slipping backwards; (the time intervals corresponding to relative rest are not marked).

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x) Under "lasting stop" in contrast to an instantenuous stop is meant the stay of a particle in a state of relative rest during any more, or less lasting, (however finite) time interval.

Regime 1. Slipping of the particle forward backwards with two lasting stops in every period;

Regime 2. Slipping forwards and backwards with instantenuous stops in every period;

Regime 3. Slipping forwards and backwards with one lasting and one instantenuous stop in every period. Two kinds of such a regime are possible, they differ one from that in one case the instantenuous stop follows the slipping forwards (regime 3A) and in the other - it follows the slipping backwards (regime 3B)

Regime 4. Slipping in one direction with one lasting stop in every period. This regime has also two versions corresponding to the slipping forwards (regime 4A) and to the slipping backwards (regime 4B).

A further analysis of steady state regimes of particle motion in absence of [illegible] will be limited by linear harmonic plane oscillations with [illegible] taking place under angle β . As it has been mentioned above even this relatively simplest problem [illegible] the large amount of investigations till recently has been fully enough studied. In particular the case of unequal coefficients of cinetic friction f and the friction f_1 has not been considered definitely.

In the problem case considered in all relationships of §§ 1 and 2 must be assumed

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$$a = A\cos\beta\,,\quad b = A\sin\beta\,,\quad c = 0 \tag{5.1}$$

Then in conformance with (1.10), (2.5), (2.8) - (2.10), (2.14) and (2.17) we shall have:

$$z_0 = \frac{g}{A\omega^2}\,\frac{\cos\alpha}{\sin\beta}\,,\quad z'_{1\pm} = \frac{g}{A\omega^2}\,\frac{\sin(\alpha\pm\beta_1)}{|\cos(\beta\mp\beta_1)|}\,,$$

$$z'_{\pm} = \frac{g}{A\omega^2}\,\frac{\sin(\alpha\pm\beta)}{|\cos(\beta\mp\beta)|}\,,$$

$$\delta_0 = \arcsin z_0\,,\quad \delta_{1+} = \arcsin z'_{1+},$$

$$\gamma_{1-} = -\arcsin z'_{1-},$$

$$\delta_+ = \arcsin z'_+\,,\quad \gamma_- = -\arcsin z'_-\,. \tag{5.2}$$

Table II contains summary of investigation results of steady state regimes of particle motion without of particle motion without bouncing having a switching period equal to the period of plane oscillations or infinitely large. If may be shown that other motions without bouncing are impossible.

The obtaining of necessary and sufficient conditions of existence and stability of regimes is carried out in conformance with the work [5]. In the present communication these conditions are given in a precised and simplified form obtained by the author of the same work.

It is noteworthy, that in the new form all conditions are expressed by means of four parameters z'_{1+}, z'_+, z'_{1-} and z'_- (or respectively δ_{1+}, δ_+, γ_{1-} and γ_-) bound with the initial parameters by relationships (5.2).

In table 2 the following transcendental functions of these parameters have been used

$$\psi(a,b,c) = \cos b + \cos c - (\pi + c - b)\sin a,$$

$$\nu_+ = \pi - \nu_- = \pi\,\frac{\sin\gamma_-}{\sin\delta_+ + \sin\gamma_-}\,,$$

$$\gamma = \frac{\nu_+}{\sin\nu_+}\sin\delta_+ = \frac{\nu_-}{\sin\nu_-}\sin\gamma_- \tag{5.3}$$

as well as function $f(a,b)$ the graphs of which are given in work [5].

It is to notice that the conditions of the existence of every regime given in the table, either desintegrate into two relationships groups which turn into each other at the substitution of δ_{1+} for γ_{1-}, δ_+ for γ_-, γ_{1-} for δ_{1+} and γ_- for δ_+ (or respectively z'_{1+} for $-z'_{1-}$, z'_+ for $-z'_-$ and inversely) or do not change at the substitution mentioned. Hence, the regions of existence of the regimes in the space of parameters

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Steady state regimes of a particle in absence of bouncing

Form of steady state regime motion		Necessary and sufficient conditions of existence and stability of steady state regimes	
1		2	3
Accelerated motion	forwards	$\dot{z}'_+ < 0,\ \dot{z}'_- < 0\ (\alpha < -\varphi)$	$\dot{z}'_+ \dot{z}'_- > 0$ $\lvert\alpha\rvert > \varphi$
	backwards	$\dot{z}'_- > 0,\ \dot{z}'_+ > 0\ (\alpha > \varphi)$	
Regular regime I		$\dot{z}'_{1+} < 1,\ \dot{z}'_{1-} > -1$	$\psi(\delta_+, \delta_{1+}, \gamma_{1-}) < 0$ $\psi(\gamma_-, \gamma_{1-}, \delta_{1+}) < 0$
Regular regime 2		$\dot{z}'_{1+} < 1,\ \dot{z}'_{1-} > -1$	$\nu_- - \arcsin\chi \geqslant \delta_{1+}$ $\nu_+ - \arcsin\chi \geqslant \gamma_{1-}$ $\nu_- + \arcsin\chi > \gamma_{1-}$ $\nu_+ + \arcsin\chi > \delta_{1+}$ $\lvert\chi\rvert < 1$
Regular regime 3	Regime 3 A	$\dot{z}'_{1+} < 1,\ \dot{z}'_{1-} > 1$	$\psi(\delta_+, \delta_{1+}, \gamma_{1-}) \geqslant 0$ $\psi(\delta_+, \delta_{1+}, \pi - \gamma_{1-}) < 0$ $\nu_- - \arcsin\chi < \gamma_{1-}$
	Regime 3 B		$\psi(\gamma_-, \gamma_{1-}, \delta_{1+}) \geqslant 0$ $\psi(\gamma_-, \gamma_{1-}, \pi - \delta_{1+}) < 0$ $\nu_+ - \arcsin\chi < \gamma_{1-}$
Regular regime 4	Regime 4 A	$\dot{z}'_{1+} < 1,\ \dot{z}'_{1-} \leqslant -1$	$\psi(\delta_+, \delta_{1+}, \pi + \delta_+) < 0$
		$\dot{z}'_{1+} < 1,\ \dot{z}'_{1-} > -1$	$\psi(\delta_+, \delta_{1+}, \pi - \gamma_{1-}) \geqslant 0$ $\psi(\delta_+, \delta_{1+}, \pi + \delta_+) < 0$
	Regime 4 B	$\dot{z}'_{1+} \geqslant 1,\ \dot{z}'_{1-} > -1$	$\psi(\gamma_-, \gamma_{1-}, \pi + \delta_-) < 0$
		$\dot{z}'_{1+} < 1,\ \dot{z}'_{1-} > -1$	$\psi(\gamma_-, \gamma_{1-}, \pi - \delta_{1+}) \geqslant 0$ $\psi(\gamma_-, \gamma_{1-}, \pi + \delta_-) < 0$
Relative rest		$\dot{z}'_{1+} \geqslant 1$	$\dot{z}'_{1-} \leqslant -1$

Table 2

Switching period	Phase angles corresponding to transition moments	Formulas for V_{cp} and W_{cp}
4	5	6
∞	—	$W_{cp} = -\frac{g\sin(\alpha+\varphi)}{\cos\varphi}$ $W_{cp} = -\frac{g\sin(\alpha-\varphi)}{\cos\varphi}$
$\frac{2\pi}{\omega}$	$\varphi_1^{(0)} = \varphi_{0+} = \delta_{1+}$ $\varphi_2^{(0)} = \varphi_{+0} = f(\delta_+, \delta_{1+})$ $\varphi_3^{(0)} = \varphi_{0-} = \delta_{1-} = \pi + \gamma_{1-}$ $\varphi_4^{(0)} = \varphi_{-0} = \pi + f(\gamma_-, \gamma_{1-})$	$V_{cp} = \frac{A\omega}{\pi\cos\varphi}\left[\cos(\beta-\varphi)F(\delta_+, \delta_{1+}) - \right.$ $\left. -\cos(\beta+\varphi)F(\gamma_-, \gamma_{1-})\right]$
$\frac{2\pi}{\omega}$	$\varphi_1^{(0)} = \varphi_{-+} = \nu_- - \arcsin\chi$ $\varphi_2^{(0)} = \varphi_{+-} = \nu_+ + \pi - \arcsin\chi$	$V_{cp} = \frac{A\omega}{\pi\cos\varphi}\left[\cos(\beta-\varphi)F(\delta_+, \varphi_{-+}) - \right.$ $\left. -\cos(\beta+\varphi)F(\gamma_-, \varphi_{+-} - \pi)\right] = 2\pi A\omega\sqrt{1-\chi^2} \times$ $\times \cos\beta\cos\nu_+\left[-1 + \frac{2}{\pi}f\,\mathrm{tg}\,\beta\,(\mathrm{tg}\,\nu_+ - \nu_+ + \frac{1}{2}\pi)\right]$
$\frac{2\pi}{\omega}$	$\varphi_1^{(0)} = \varphi_{0+} = \delta_{1+}$ $\varphi_2^{(0)} = \varphi_{+-} = f(\delta_+, \delta_{1+})$ $\varphi_3^{(0)} = \varphi_{-0} = \pi + f(\gamma_-, \varphi_2^{(0)} - \pi)$	$V_{cp} = \frac{A\omega}{\pi\cos\varphi}\left[\cos(\beta-\varphi)F(\delta_+, \delta_{1+}) - \cos(\beta+\varphi) \times \right.$ $\left. \times F(\gamma_-, \varphi_3^{(0)} - \pi)\right]$
	$\varphi_1^{(0)} = \varphi_{0-} = \delta_{1-} = \pi + \gamma_{1-}$ $\varphi_2^{(0)} = \varphi_{-+} = \pi + f(\gamma_-, \gamma_{1-})$ $\varphi_3^{(0)} = 2\pi + f(\delta_+, \varphi_2^{(0)} - 2\pi)$	$V_{cp} = \frac{A\omega}{\pi\cos\varphi}\left[\cos(\beta-\varphi)F(\delta_+, \varphi_3^{(0)} - 2\pi] - \right.$ $\left. -\cos(\beta+\varphi)F(\gamma_-, \gamma_{1-})\right]$
$\frac{2\pi}{\omega}$	$\varphi_1^{(0)} = \varphi_{0+} = \delta_{1+}$ $\varphi_2^{(0)} = \varphi_{+0} = f(\delta_+, \delta_{1+})$	$V_{cp} = \frac{A\omega}{\cos\varphi}\cos(\beta-\varphi)F(\delta_+, \delta_{1+})$
	$\varphi_1^{(0)} = \varphi_{0-} = \delta_{1-} = \pi + \gamma_{1-}$ $\varphi_2^{(0)} = \varphi_{-0} = \pi + f(\gamma_-, \gamma_{1-})$	$V_{cp} = \frac{A\omega}{\cos\varphi}\cos(\beta+\varphi)F(\gamma_-, \gamma_{1-})$
—	—	$V_{cp} = 0$

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$\delta_{1+}, \gamma_{1-}, \delta_{+}$ and γ_{-} are symmetric in relation to the two dimensional multiformity, determined by the equalities: $\delta_{1+} = \gamma_{1-}$ and $\delta_{+} = \gamma_{-}$ (or equalities $z'_{1+} = -z'_{1-}$, $z'_{+} = -z'_{-}$).

Let us study in detail the regions of the regimes existence.

Let us consider first an ideal case when the coefficient of static friction f_1 is equal to the coefficient of cinetic friction f. At the study of a range of applied problems, the assumption of equality of friction coefficients does not lead to serious errors since f_1 does not differ strongly from f.

In the indicated case $z'_{1+} = z'_{+}$, $z'_{1-} = z'_{-}$ ($\delta_{1+} = \delta_{+}$, $\gamma_{1-} = \gamma_{-}$) and therefore the character of the steady state motion regime is determined only by two dimensionless parameters.

The regions of the steady state motion regimes existence for this case in the plane of parameters z'_{+} and $-z'_{-}$ are shown in Fig. 3. Plotting of regions is effected in conformance with the conditions of the existence regimes given in table 2.

The quadrant of plane determined by unequalities

$$z'_{+} < 0, \quad -z'_{-} < 0$$

corresponds to realy impossible values of parameters since, according to (5.2) the unequalities indicated come to the requirement of a simultaneous execution of relationships $\alpha < -\rho$ and $\alpha > \rho$, which are contradictory owing to the positivity of of the friction angle ρ. It is substantial that in the case considered, first, all the region of possible values of parameters z'_{+} and z'_{-} is occupied by regions of existence of the regimes considered and, second that the regions of existence of separate regimes are not (overlapping) one upon the another. Therefore in the case given to each combination of system parameters values corresponds some fully determined regime of system motion.

A similar situation as it may be proved by the analysis of conditions of regime existence given in table 2 occurs also when the friction coefficients f_1 and f are not identical but the plane inclination angle to horizon α is less by the modulus of the angle of kinetic friction $\rho = \operatorname{arctg} f$. In this case there can be no accelerated particle motion. The existence conditions for each regular regime and relative rest in all cases are such, that when the existence conditions of one of these regimes are realised the existence conditions of no other regime can be fullfilled.

In case when $|\alpha| > \rho_1 = \operatorname{arctg} f_1$, i.e. when the angle of plane inclination to the horizon in its absolute

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value is bigger than the static friction angle there is a single-valuedness too. And exactly in this case regardless of other parameters values, there occurs the accelerated motion of the particle down the plane.

A more complicated picture corresponds to the case when the plan inclination angle to the horizon is bigger than the angle of kinetic friction but less than the static friction angle or

$$\rho < |\alpha| < \rho_1 \tag{5.4}$$

Even in the absence of inclined plane vibrations in the last case according to initial conditions the particle either rests, or slips down the plane, without stopping and with a constant acceleration.

In presence of vibrations, in case of the realisation of relationship (5.4) two motion regimes may exist at the same time. One of them is a regular regime (particularly relative rest), and the other is a non-stopping accelerated motion down the plane.

In case, when one of the two possible motion regimes is stable, and the other unstable actually a stable regime is realised. If both regimes are stable ("in few") then the character of the originating steady state regime depends on initial conditions.

At the same time if the plane vibrations are intensive enough, that is at a big enough acceleration amplitude $A\omega^2$, there is always an accelerated slipping of the particle downwards.

Let us pass now to the question of stability of the steady state particle motions without bouncing. It is to be noticed first of all that the stability of motions without bouncing on transition moments in its sense is a conditional stability since only disturbances are considered when the particle does not leave the vibrating plane. However the last circumstance is of no importance in cases when the restitution coefficient R is assumed to be equal to zero. Moreover, as it will be seen further (see § 6) the conditionally stable in the sense indicated above, particle motions without bouncing are practically always absolutely stable too.

In order to establish stability "in great" of all motions without bouncing considered at $|\alpha| < \rho$ it is enough to remark that according to the said above in this case the regions of the regimes existence are overlapping i.e. at determined values of the system parameters values of the system parameters only one steady state regime may take place with a switching period equal to the period of plane oscillations or relative rest. Therefore if it will be assumed that this regime is unstable we should inevitably arrive to a contradictory conclusion that there must exist another steady state motion regime possessing a finite or infinitely great switching period and not

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approaching in course of time to the regime given. Thus, at $|\alpha| < \rho$ all possible steady state motion regimes are asymptotically stable "in great" in all the region of their existence determined by the inequalities of table 2.

An identical situation takes place in case $|\alpha| > \rho_1 = \arctg f_1$, when there cannot occur any regimes with stops and only one steady state regime of particle motion exists - a non-stopping accelerated slipping down the plane. In this case a nonstopping motion is stable on transition moments "in great" since any other motion (in which the slipping of the particle downwards began at an arbitrary moment of time $t = t_0$) turns in in course of time in this motion.

A more complicated picture corresponds to the case $\rho < |\alpha| < \rho_1$, when at the same parameters sometime two types of steady state motion regimes are possible one of then regular and the other - accelerated (non-stopping slipping down).

It can be shown that in this case there exists only a non-stopping accelerated particle slipping and no other steady state motion regimes are possible then the non-stopping motion is stable "in great" on transition moments i.e. it is established regardless from the value of the particle slipping moment beginning.

Really if in the conditions mentioned it would not

be so, this would mean that a certain regime with stops has been established. But then taking into consideration the mentioned above inevitably there should exist one of the regular regimes considered above and this contradicts the initial assumption.

In case when at $\rho < |\alpha| < \rho_1$ not only a non-stopping slipping but any regular motion regime is possible each of the existing steady state regimes generally speaking is stable on transition moments "in few" in all the region of its existence, excluding its boundaries.

In fact at values of parameters not lying on the boundary of the region of the accelerated motion existence (i.e. at $|\alpha| \neq \rho$) there is an continuous diapason of values of slipping beginning moments to which the continuous particle slipping is corresponding. Therefore the change of the particle slipping beginning moment not lying at the boundary of the diapason indicated does not change the character of the motion. At the same time in the case considered, as it is easy to show for regular regimes inside the region of their existence small enough deviations of any transition moment do not lead to the change of the steady state regime.

Thus all possible particle motion steady state regimes in absence of bouncing are stable (in any case "in few") in all the region of their existence except, may be its boundary. A more complicated picture occurs in the case

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with motion with bouncing when the stability regions of steady state regimes do not coincide with the regions of their existence.

In the last column of table 2 are given formulas for the mean velocity in regular regimes and mean acceleration in accelerated motions. As a rule, the calculation of these values is most interesting for applications. It has to be noticed that the velocity calculations according to these formulae is esentially simplified owing to the presence of function F, graphs brough for instance in the work [5].

§ 6. Steady state regimes of the particle motion in presence of bouncing.

The essential feature of the motion with bouncing is the great variety of steady state regimes Kinds. Besides, regular and accelerated regimes with a switching period $T = pT$ where p - any whole number may exist at the motion in presence bouncing. Another particularity of steady state regimes with bouncing is the non-coincidence of some regimes stability regions with the regions of their existence. Owing to the great variety of particle motion with bouncing steady state regimes the individual examination of each of them in a general form is impossible. That is why, we shall examine here only some most important (and at the same time most simple in character) particle motion with bouncing steady state regimes viz the motions which are characterized by the fact that in every switching period there is only one stage of the particle flight.

Two types of this class regimes are possible. To the first, let us refer regimes which are characterised by are the fact that flight stage coincides with the initial moment of interval II i.e. with $t_0 = \delta_0/\omega$ and the end of the stage (i.e. the moment of fall $t_n = \varphi_n/\omega$) belongs to the interval I. After falling upon the plane the particle remains there till the beginning of interval II. Then the particle tears itself away from the plane and the cycle is repeated.

To the second type of steady states regimes of motion with one flight stage in each switching period we shall refer the peculiar regimes when the particle is all the time in a state of flight, and is coming in contact with the vibrating plane only during isolated time moments following each other after the whole number p of plane oscillations.

The essential feature of the problem given (as well as of a range of identical problems) is that the results corresponding to the assumption concerning the absolutely non-elastic impact of the particle at its fall upon a plane ($R=0$) are not obtained as a particular case, from the results obtained in the assumption concerning a partialy or wholy elastic impact ($0<R\leq 1$).

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In table 3 for the case of rectilinear oscillations gives the summary investigation results of the abovementioned particle motion steady state regimes in presence of bouncing. These results partly stated in the work [5] were obtained by means of the method described in §§ 3 and 4. It is to notice that columns 1-6 are true for elliptical oscillations too.

The following symbols are used in table 3.

$$w_0 = 1/z_0 \tag{6.1}$$

$$w_{0p} = \sqrt{\pi^2 p^2 + 1}, \quad w_{0p}^* = \sqrt{\pi^2 p^2 + 4} \tag{6.2}$$

$$w_{0p}' = 1/\sin\delta_{0p}' \tag{6.3}$$

where δ_{0p}' - is the root of the transcedent equation

$$tg\,\delta_{0p}' = \frac{1}{(p-\frac{1}{2})\pi - \delta_{0p}'}, \tag{6.4}$$

which by substitution of the unknown $z = (p-\frac{1}{2})\pi - \delta_{0p}'$ is brough to the equation $tg\,z = z$. The calculations give

$$w_{01}' = 1, \quad w_{02}' = 4,603; \quad w_{03}' = 7,791; \quad w_{04}' = 10,94$$

Further

$$\varphi_p^* = arctg\,[M_p(w_0)\,tg\,\varphi]$$

$$M_p(w_0) = 1 - \frac{(\varphi_n - \delta_0) + (\cos\varphi_n - \cos\delta_0)w_0}{2\pi p} \tag{6.5}$$

$$N_p(w_0) = M_p(w_0) - \frac{1}{2} + \frac{w_0\cos\varphi_n}{2\pi p} \tag{6.6}$$

$$2\pi p' = 2\pi p'(w_0, R) = \begin{cases} \sqrt{w_0^2 - 1} + \sqrt{w_0^2 - 4} & \text{при } R = 0 \\ w_0\frac{1+R}{1-R} + \sqrt{\frac{w_0^2(1+R)^2 - 4(1+R^2)^2}{(1-R^2)(1+R^2)}} & \text{при } R \neq 0 \end{cases} \tag{6.7}$$

where $\varphi_n = \varphi_n(\delta_0, \delta_0)$ - function calculated in the work [5].

The mean velosity formula brought in the second line of the ninth column is true only in the particular case of regimes where there is a slipping stage in one direction and a flight stage. Conditions which are sufficient for the existence of such regimes are the inequalities:

$$\left[\frac{2\pi p}{\lambda}M_p(w_0) + \psi_\pm^{(p)}(w_0, w_\pm', \delta_\pm, \delta^*)\right] tg\,\beta <$$

$$< \mp\left[\frac{2\pi p}{\lambda} + K^{(p)}(w_0, w_0^*) + \psi_\pm^{(p)}(w_0, w_\pm', \delta_\pm, \delta^*)\right] tg\,\beta \tag{6.8}$$

where

$$\psi^{(p)}(w_0, w_\pm', \delta_\pm, \delta^*) = (\delta_0 + 2\pi p - \varphi_n)\frac{(1-\delta_\pm^*)}{2} + w_\pm'(1 - \delta_\pm\cos\delta_0)$$

$$K^{(p)}(w_0, w_0^*) = -(\varphi_n - \delta_0) - w^*(\cos\varphi_n - \cos\delta_0)$$

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$$w_{\pm}^{-1} = \frac{1}{z_{\pm}^{1}} = \frac{A\omega^2}{g}\,\frac{\cos(\beta \mp \rho)}{\sin(\alpha \pm \rho)}\,, \quad w^{*} = \frac{A\omega^2}{g}\,\frac{\cos\beta}{\sin\alpha}$$

$$\sigma_{\pm} = sign\left[\pm\frac{\pi}{2} \mp (\beta \mp \rho)\right], \quad \sigma_{\pm}^{*} = sign(\rho \pm \alpha)$$

(5.9)

The approximate universal formula for V_{cp} mentioned in the last column of table 3 gives a sufficient for a range of applications precision at the fulfillement of the une quality

$$w_0 > w_0^{*} = 2\,\frac{1-R}{1+R}$$

(5.10)

ensuring the presence of a sufficiently intensive particle bouncing.

The regions of existence and stability of steady state regimes considered at $R = 0$ are shown in Fig. 4 where the regions of existence stability and of steady state regimes with a continuous bouncing are hatched in two directions, and the regions of existence and stability (conditional according to the basic transition moments, shown in the sixth column table 3) of other regimes with one stage flight in every switching period - in one direction.

For the case $0 < R \leq 1$ the regions of existence and stability of steady state regimes with a continuous par-

ticle bouncing where the period of flight [illegible] the switching period of switchings $T = p\,T_0$ [illegible] presented on Fig. 5 (these regions are [illegible] drawing). The region of existence and stability of motion without bouncing regimes determined by the inequality $w_0 \leq 1$ is shown in the same figure.

From the juxtaposition of Fig. 4 and Fig. 5 it is seen that for the values of $R \approx 0$, the regions of existence and stability of regimes are very close to corresponding regions especially found for the [illegible] although they do not exactly coincide [illegible]

Such conformity of results obtained for $R = 0$ and for $R \neq 0$ but at $R \to 0$ has a substantial importance. In fact otherwise we should have a striking "leaping" change of existence conditions at the transition from value $R = 0$ to the so small as desired values of R.

The presence of such a jump certainly should witness the unsatisfactory description of the process of collision with the help of the conception of the restitution coefficient.

In separate plane regions of the parameters w_0 , R located near the straight line $R = 0$ regimes with continuous bouncing are impossible. On the contrary, near the straight line $R = 1$ the regions of existence of regimes corresponding to different values of p are overlapping each other and upon the existence and stability regions of motion without bouncing. Otherwise, at

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values of R near to one several regimes correspond to defined values of parameters W_0 and R. Depending on the initial conditions of motion only one regime is actually established.

Besides, in presence of always acting disturbances, characteristic for real systems, as a rule in such systems ultimately one "most stable" regime is established. So, for instance, it has been observed repeatedly that exactly such regimes are established in the region of existence of regimes without bouncing (i.e. at $W_0 \leq 1$) even if regimes with bouncing exist and are stable (in few).

Let us notice that the existence and stability regimes considered in case $0 < R \leq 1$ are wholy identical with the conditions obtained by means of a somewhat different method by R.F. Brunstein and A.E. Kobrnsky [34] in the problem of the motion of a "vibrator jumping on the stair case".

In conclusion let us mention that at a coefficient of instantenuous friction λ, differing from zero, the accelerated particle motion with bouncing (down the plane) becomes impossible even at $|\alpha| \geq \beta$ i.e. when in absence of bouncing the particle without fail is slipping with acceleration down upon the plane. This paradox is explained by the fact that in presence of bouncing in case $\lambda \neq 0$ there is a more intensive energy consumption than in the absence of bouncing. When the consumption is related only with the presence of kinetic friction. Here it may be possible to compare with the well known fact of the unlimitedness of resonance amplitudes of an oscillator in case of Coulon's friction and the limitation of these amplitudes at a small as desired sticky friction.

§ 7. Technical applications of theory

The results of the above given investigations of vibrodelivering are well in correspondence and agreement for all many vibrating machines used chiefly in mining and metallurgy.

Let us enumerate the basic applications of the theory:

a) in determination of those principal parameters of vibrating machines causing separate bodies (or approximately, of a material layer, is determined, and its dependence on basic parameters is cleared. Often the character of the motion regime is of interest too ([2], [8], [16], [20], [21] and others).

b) in calculations of vibration machines intended for separation of material particles according to friction coefficients and shape - the separation ability which determines the machine sensibility [5] and the rate of the process estimated.

Owing to the analogy of differential equations with the equations of a range of other problems the theory results are used in the following cases:

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Steady state regimes of particle motion in presence of bouncing with one flight stage in each switching period

Table 3

Values of restitution coefficient at impact	General necessary and sufficient conditions of existence and stability of regimes with one flight stage in each period of switchings.	Type of regime	Switching period	Necessary and sufficient conditions of existence and stability of a given type	Phase angles corresponding to transition moments t_0^0 in steady state regimes	Character of regime
1	2	3	4	5	6	7
$R=0$	$\omega_{op}' < \omega_0 < \omega_{op}''$	I (with the particle stay upon a vibrating plane)	$\frac{2\pi p}{\omega}$ $p=1,2,\ldots$	$\omega_{op}' < \omega_0 < \omega_{op}$	$\varphi_1^{(0)} = \delta_0 = \arcsin\frac{1}{\omega_0}$; $\varphi_2^{(0)} = \varphi_n(\delta_0, \delta_0)$	regular accelerated
		II (with continuous bouncing)	$\frac{2\pi p}{\omega}$ $p=1,2,\ldots$	$\omega_{op} < \omega_0 < \omega_{op}''$	$\varphi_1^{(0)} = \delta_0^* = \arccos\frac{\pi p}{\omega_0}$	regular accelerated
$0<R\leq 1$		III (with continuous bouncing)	$\frac{2\pi p}{\omega}$ $p=1,2,\ldots$	$\omega_{op(R)} < \omega_0 < \omega_{op(R)}''$	$\varphi_1^{(0)} = \delta_0^* = \arccos\frac{\pi p}{\omega_0}\frac{1-R}{1+R}$	regular accelerated

1) For the particular case of regimes with one stage of flight and one [illegible]

Conditions of presence of a given character regime	Mean velocity or mean accelerazion of particle in regular regimes (in case of rectilinear plane oscillations)	Approximated general formula for calculation of the particle mean velocity in regular motions with bouncing
8	9	10
$\lambda=0$, $\lvert\alpha\rvert < \varphi_p^*$		
$1>\lambda>0$	$V_{cp} = \frac{2\pi pg}{\omega}\left\{\frac{1}{2\pi p}\frac{A\omega^2}{g}\cos\beta\cos\varphi_n \pm f\cos\alpha \cdot \left[N_p(\omega_0) - \frac{dN_p(\omega_0)}{d\omega_0}\frac{2-\lambda}{2\lambda}\sin\alpha\right]\right\}^{1/2}$	
$\lambda=0$, $\lvert\alpha\rvert > \varphi_p^*$	$W_{cp} = -g\cos\alpha(\operatorname{tg}\alpha \pm \operatorname{tg}\varphi_p^*) = -g\frac{\sin(\alpha \pm \varphi_p^*)}{\cos\varphi_p^*}$	
$\lambda=0$, $\alpha=0$	$V_{cp} = \frac{\pi pg}{\omega \operatorname{tg}\beta}$	
$1>\lambda>0$	$V_{cp} = \frac{\pi pg}{\omega}\left(\operatorname{ctg}\beta\cos\alpha - \frac{2-\lambda}{\lambda}\sin\lambda\right)$	
$\lambda=0$, $\alpha\neq 0$	$W_{cp} = -g\sin\alpha$	

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c) at calculation of the vibrosliders of files and grooves in order to obtain the mean velocity of sinking or extraction and for the choice of system parameters [28] , [29] , [30] .

d) at calculations of automotive vibrating ground scraper for the definition of the motion mean velocity and for choice of system parameters [35] .

e) in calculations of vibrocentrifuges [37] .

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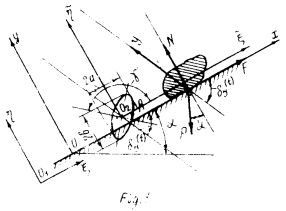

Fig. 1

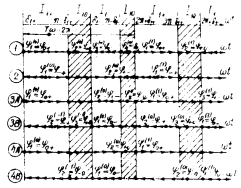

Fig. 2

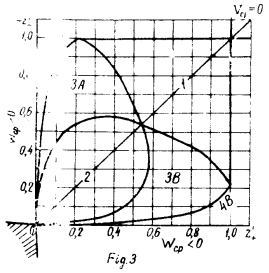

Fig. 3

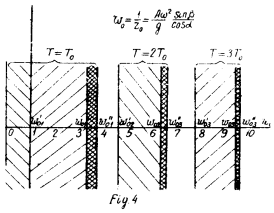

Fig. 4

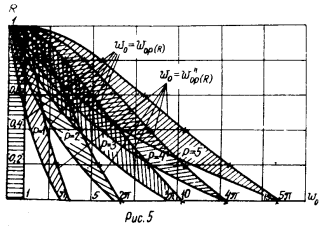

Рис. 5

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ИССЛЕДОВАНИЕ УСТОЙЧИВОСТИ НЕЛИНЕЙНОЙ СИСТЕМЫ АВТОМАТИЧЕСКОГО УПРАВЛЕНИЯ ПРИ ВЫНУЖДЕННЫХ ВИБРАЦИЯХ

In this article we calculate the permissible noise amplitude for an airplane-autopilot control system, and make recommendations on the choice of certain autopilot parameters from the point of view of reducing the influence of noise.

The noise may be caused by elastic vibrations of the airplane fuselage, electric fluctuations in the control system circuits, local vibrations, etc.

We assume that the vibration noise obeys a sinusoidal law

$$f_2(t) = B \sin \Omega t \qquad /1/$$

with an amplitude B and a frequency Ω that is considerably higher than the possible range of frequencies of the useful signal in the system. This means that the vibration component of the signal is in practice blocked by the linear elements located past the nonlinear one, and consequently, the control system is an open-loop one as far as noise goes. In other words, the system is a low-pass filter. The overwhelming majority of airplane control systems have this property.

A more general case, when the signal due to noise may be closed by the feedback loop, and the noise itself may be a random function of time, has been considered in detail in /1 and 2/.

If the control system is subject to noise, the input of the nonlinear element will have two components

$$x = x^{\circ}(t) + A \sin(\Omega t + \varphi), \qquad /2/$$

where $x^{\circ}(t)$ is a slowly variable component /the useful signal/, and $A \sin(\Omega t + \varphi)$ is the noise component.

The amplitude A and the phase φ of this component are determined by the transfer function of the linear part of

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the system W_c' from the noise input point to the input to the nonlinear element by means of the formula

$$A = B|W_c'|, \quad \varphi = \arg W_c'. \qquad /3/$$

A Fourier expansion of the output of the nonlinear element will have the form

$$y = F^{\circ}(x) + \text{harmonic components} \qquad /4/$$

where $F^{\circ}(x)$ is the useful signal, which is defined as the d-c component of the Fourier series. /One can neglect small variations of this component over the vibration period

$$T_n = \frac{2\pi}{\Omega}.$$

The harmonic components may be disregarded, since they are suppressed in the linear elements following the nonlinearity. It follows from Equations /2 and 4/ that F° is a function of two variables, A and x°. Usually $F^{\circ}(x^{\circ})$ /at fixed A/ is a smooth function, even if the initial nonlinearity is jumplike, and can be linearized for small variation of the useful signal. Then the equation of the nonlinear element in terms of the useful signal becomes

$$F^{(o)}(x) = K_n x^{\circ}, \qquad /5/$$

where K_n is the gain of the useful signal in the nonlinear element.

The quantity K_n depends on A and is determined by the formula /I/

$$K_n = \left(\frac{\partial F^{\circ}(x^{\circ}, A)}{\partial x^{\circ}}\right)_{x_0 = 0} \qquad /6/$$

It is found that for most typical nonlinear elements, K_n has the simple analytic expression /see Appendix/.

Thus, if the amplitude of the noise component at the input of the nonlinear element is known, then the nonlinear element can be linearized with regard to the useful signal, with a gain K_n. Then the nonlinear element will be seen by the useful signal as a linear element of gain K_n. The

control system can now be investigated by any method of linear theory.

Assume that we have the following automatic control system /Fig. I/:

The airplane equation /longitudinal motion/ /3, Fig. I/ is

$$(p^3 + a_1 p^2 + a_2 p + a_3)\vartheta = K_0(T_0 p + 1). \qquad /7/$$

/Here and henceforth, $p \equiv \frac{d}{dt}$./

The equation of the sensing element /I, Fig.I/ is

$$x_1 = (K_1 + K_2 p)[\vartheta + f_1(t) + f_2(t)]. \qquad /8/$$

The feedback equation /4, Fig. I/, is

$$x_f = K_f \delta. \qquad /9/$$

The equations of the steering motor /2. Fig. I/ and amplifier are also assumed given.

On the basis of the physical processes that take place in these blocks, they can be considered as the series connection of a nonlinear element and an inertial-type integrating element /Fig. 2/.

Let the noise frequency be known, say $\Omega = 100 \text{sec}^{-1}$, and let $T = 0.08$ sec. Then, after passing through the steering motor and the amplifier, the fundamental harmonic of the noise will be attenuated, and its amplitude will become

$$|W(j\omega)| = \frac{1}{\Omega\sqrt{T^2\Omega^2+1}} \approx \frac{1}{T\Omega^2} = \frac{1}{800} \qquad /10/$$

of the amplitude at the output of the nonlinear element. In practice, therefore, there will be only one noise component at the input of the nonlinear element, and the assumptions made at the beginning of the article are satisfied.

Then the equation of the steering machine and the amplifier, with respect to the useful signal, becomes

$$(Tp + 1)p\delta = K_n x; \quad x = x_1 - x_f. \qquad /11/$$

Considering Equation /7-9/ and /II/, we can represent the system as consisting of two blocks /Fig. 3/, the airplane and

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In Fig. 3 $K_a = \frac{1}{\quad}$ is the autopilot gain, while $T_1 = \frac{T}{K_n K_f}$ and $K_f T_2 = \frac{1}{K_n K_f}$ are the time constants of the autopilot.

Thus, the time constants of the autopilot depend essentially on the nonlinear gain and this means the noise affects the dynamics of the system and may lead under certain conditions to a loss of stability. The characteristic equation of the system becomes

$$A_0 p^5 + A_1 p^4 + A_2 p^3 + A_3 p^2 + A_4 p + A_5 = 0. \quad /12/$$

In accordance with the Hurwitz criterion for the stability of a fifth-order system, it is necessary and sufficient that the following inequalities be satisfied

$$A_0 > 0;\ A_1 > 0;\ A_2 > 0;\ (A_1 A_2 - A_0 A_3) > 0;$$
$$A_3 > 0;\ A_4 > 0;\ A_5 > 0;$$
$$(A_1 A_2 - A_0 A_3)(A_3 A_4 - A_2 A_5) - (A_1 A_4 - A_0 A_5)^2 > 0. \quad /13/$$

After inserting the values of the parameters and making certain simplifications, the last inequality becomes

$$H(K_n) = K_n^3 - 14.2 K_n^2 + 6.51 K_n > 0. \quad /14/$$

A plot of Equation /14/ is shown in Fig. 4.

In the interval $0 \le K_n \le 0.48$ condition /14/ is satisfied, but the second condition of /13/ is not satisfied.

When $K_n > 13.7$, all three Hurwitz conditions are satisfied. Consequently, the system is stable only if

$$K_n > 13.7. \quad /15/$$

If the nonlinear element has a saturation-type characteristic /Fig. 5/, then K_n, as a function of the noise amplitude at the input of the nonlinear element A, is given by the following formula /see Appendix/

$$K_n = \frac{2K}{\pi} \arcsin \frac{b}{A}. \quad /16/$$

From /16/, taking /15/ into account, we can readily

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sible amplitude A_{max} of the maximum noise component at the input of the nonlinear element

$$A_{max} = \frac{b}{\sin \frac{\pi K_{n\,min}}{2K}}, \quad /17/$$

where $K_{nmin} = 13.7$.

If, for example, $K = 80$ and $b = 0.5$, then $A_{max} = 1.87$, and the noise amplitude B_{max} will be

$$B_{max} = \frac{A_{max}}{\sqrt{K_1^2 + K_2^2 \Omega^2}} \cong \frac{A_{max}}{K_2 \Omega}. \quad /18/$$

When

$$A_{max} = 1.87,\ \Omega = 100,\ K_1 = 0.9,\ K_2 = 0.4, \quad /19/$$

we have

$$B_{max} = 0.047.$$

The amplitude of the velocity of the vibrations, corresponding to the resultant value B_{max}, can also be readily determined from /18/

$$B'_{max} \cong \frac{A_{max}}{K_2} = \frac{1.87}{0.4} = 4.7. \quad /20/$$

Thus, it is enough to apply to the input of the autopilot a vibration noise $B'_{max} = 4.7$ deg/sec in order for the control system to become unstable.

In this example it was found that B'_{max} was independent of the vibration frequency. This is explained by the fact that the component of the noise frequency vibration amplitude at the input of the nonlinear element is determined principally by the derivative of the vibration noise.

The effect of vibration in the system can be reduced by a rational choice of the autopilot parameters. For example, from an examination of the curve of Fig. 6, which shows the stability boundary of the system in the plane of the parameters K_n and K_f, it is seen that the most unfavorable feedback coefficients, from the point of view of the effect

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of noise, are K_f = 0.2 to 0.3.

If K_f is chosen on the basis of other conditions /for example, from the condition of insuring specified accuracy or speed of response/, it is possible to vary other parameters of the autopilot, for example, K, β, K_2. If these parameters also fail to give the required stability margin, the structure of the autopilot must be modified.

APPENDIX

Useful signal gains in the presence of vibration noise for certain nonlinear elements are shown in Fig. 7.

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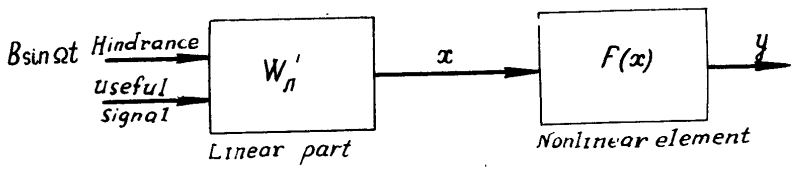

Fig. 1

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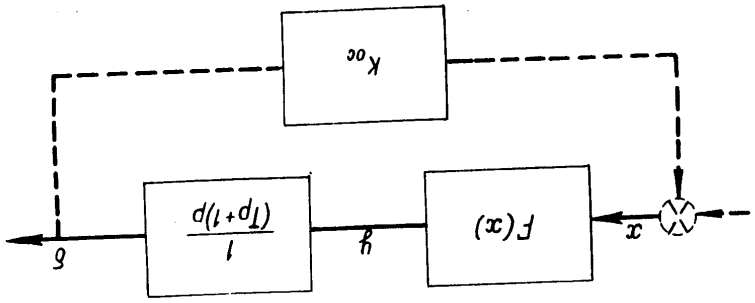
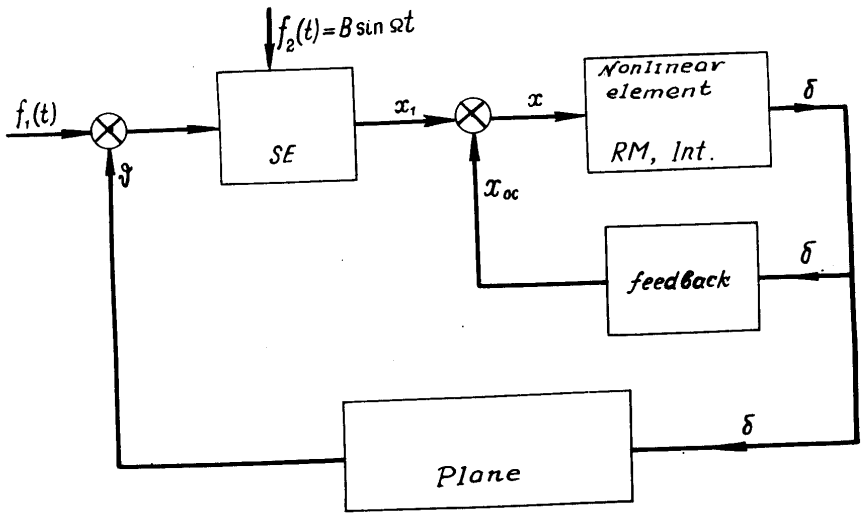


Fig. 3



$f_1(t)$ - operate signal ; $f_2(t)$ - Hindrance

Fig. 2

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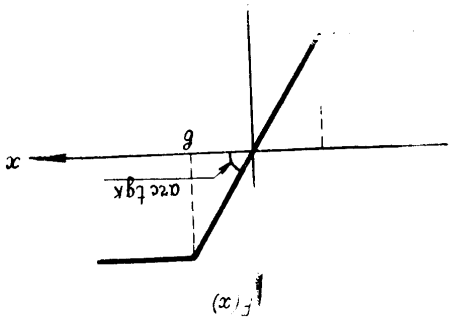

$F(x)$

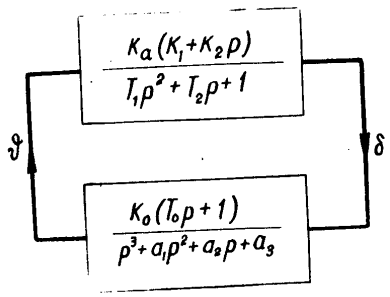

Fig. 4

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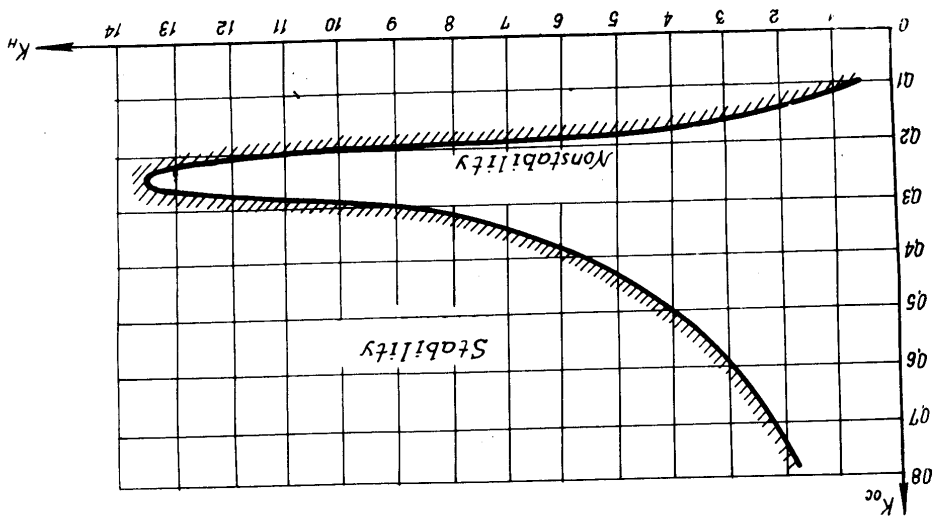

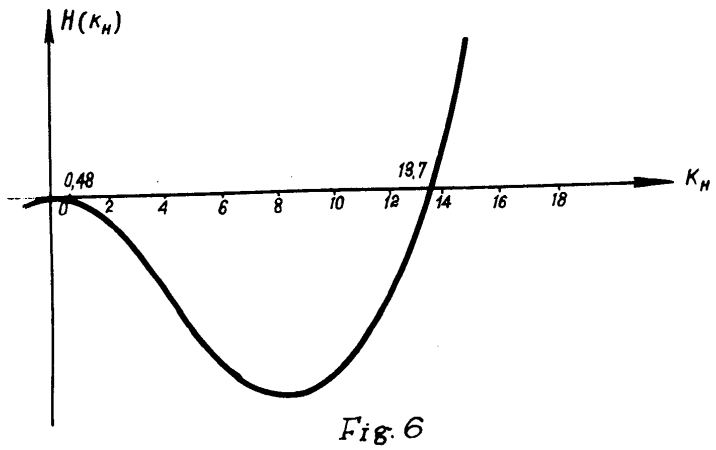

Fig. 6

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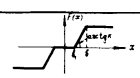
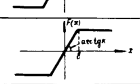
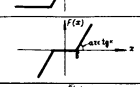
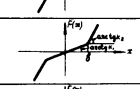
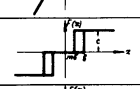
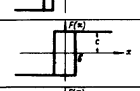
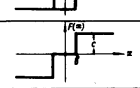
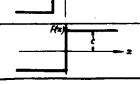
Intensification Coefficients of Useful Signal under the Variation Hindrance for Some Nonlinear Elements

№ п/п	Form of nonlinearity	Expression for $K_n(A)$
1		$K_n = \frac{2K}{\pi}\left(\arcsin\frac{b}{A} - \arcsin\frac{b_1}{A}\right)$
2		$K_n = \frac{2K}{\pi}\arcsin\frac{b}{A}$
3		$K_n = K - \frac{2K}{\pi}\arcsin\frac{b}{A}$
4		$K_n = K_2 - \frac{2(K_2-K_1)}{\pi}\arcsin\frac{b}{A}$
5		$K_n = \frac{c}{\pi A}\left[\frac{1}{\sqrt{1-\left(\frac{b}{A}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{m b}{A}\right)^2}}\right]$
6		$K_n = \frac{2c}{\pi A}\frac{1}{\sqrt{1-\frac{b^2}{A^2}}}$
7		$K_n = \frac{2c}{\pi A}\frac{1}{\sqrt{1-\frac{b^2}{A^2}}}$
8		$K_n = \frac{2c}{\pi A}$

Note: All the formulae are valid for $A > b + |x^0|$

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МЕТОД ИНТЕГРАЛЬНЫХ МНОГООБРАЗИЙ В НЕЛИНЕЙНОЙ МЕХАНИКЕ

§ I. General Idea of the Method of Integral Manifolds

The purpose of the present paper is to outline the basic results obtained recently by further developing and generalizing the method of integral manifolds in nonlinear mechanics, and applying it to new classes of problems. The idea of the method, which belongs to N.N.Bogoliubov, was stated by him in 1945 in a monograph: "On Some Statistical Methods in Mathematicyl Physics". In this monograph, a particular problem concerned with the properties of the solutions of equations in standard form on an infinite time interval is considered. However, the idea and the methods of proof of a number of theorems are very effective as well as versatile, and can be applied to investigate a sufficiently wide range of problems.

At present the ideas of the theory of integral manifolds in nonlinear mechanics have been greatly developed and generalized, and came to be extensively used in the investigation of complex phenomena which take place in various dynamical system described by differential equations containing a small parameter.

Let us outline briefly the general idea of the method, before we proceed to deal with a number of concrete classes of equations, for which the method of integral manifolds allows to obtain interesting theorems revealing some general properties of integral curves, and to consider the basic methods of the theory and analyze the developments and

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results recently obtained.

First of all we shall give the analytical definition of an integral manifold for a system of differential equations

$$\frac{dx}{dt} = X(t, x, \varepsilon), \qquad /I.I/$$

where x, X are n-vectors of an n - dimensional Euclidian space E_n, t -time, ε - small positive parameter, and the righ-hand sides of system /I.I/ satisfy certain sufficiently general conditions for $-\infty < t < \infty$ and x contained in the open set $U_n \in E_n$.

Suppose that to each t in the interval $(-\infty, \infty)$ there corresponds a certain set S_t of points x which can be represented analytically in a parametrical form by means of equations

$$x = f(t, C_1, C_2, \ldots, C_s), \qquad /I.2/$$

where $f(t, C_1, C_2, \ldots C_s)$ satisfy Lipschitz's conditions with respect to C_1, C_2, C_s in the whole range of their variation.

Then S_t will be said to be an s - dimensional $(s \leq n)$ integral manifold for equation /I.I/, if for every solution $x = x(t)$ of this equation the relation

$$x(t) \in S_t \qquad /I.3/$$

which holds for some moment $t = t_0$, will hold for every real value of t.

Geometrically interpreted, the integral manifold S_t represents a hypersurface having the property that if some value of the solution of a system of equations lies on the integral manifold /on the hypersurface/, then the whole of the solution will also lie on the manifold /on the hypersurface/.

Originally /Reference I/, the idea of the theory of integral manifolds was applied to the following particular problem. A particular case of equations /I.I/ is considered, when the right-hand side of the system is proportional to a small parameter, the so-called equations in standard form:

$$\frac{dx}{dt} = \varepsilon X(t, x). \qquad /I.4/$$

Assuming that for every $x \in U_n$ we have

$$\lim_{T \to \infty} \frac{1}{T} \int_t^{t+T} X(t, x)\, dt = X_0(x) \qquad /I.5/$$

uniformly with respect to t in the interval $(-\infty, \infty)$ simultaneously with system /I.4/ the averaged equation

$$\frac{dx}{dt} = \varepsilon X_0(x) \qquad /I.6/$$

is considered. Then a fairly complex problem is solved for equations /I.4/, viz., to establish a series of agreements between such properties of exact solutions /those of system /I.4/ and approximate solutions /those of the averaged system /I.6/ as are dependent on their behaviour in the infinite interval.

Thus, for instance, in many cases of importance /as shown below/ the averaged equations /I.6/ allow invariant manifolds of toroidal type. In this connection, the following question is of interest: will the integral manifolds of the exact equations /I.4/ lie in sufficiently close neighbourhood to the toroidal manifolds above and to what extent will they be stable?

It schould be noted that by using certain changes of variables, to exact equations /I.4/ can be reduced to a special form in which the right-hand sides differ from those of the averaged equations by terms of higher order of smallness. Hence, the problems arising herein possess a certain

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analogy to the problem of the existence of periodic solutions in A. Poincaré's local theory.

However, while in Poincaré's theory the problem is reduced to the investigation, by employing the theorem of implicit functions, of the solvability of a system of ordinary equations with a finite number of unknows, which contains a small parameter, the theory of integral manifolds deals with functional equations determining the functions which characterize the integral manifolds in question.

At this point, it is to be noted specifically that independently of the abovementioned problem of the correspondence between the exact and approximate solutions of system /I.4/ /i.e. between the solutions of system /I.4/ and those of system /I.6/ /, which is a problem of substantiating the averaging method /and the asyptotic methods, in general/, the construction of even a local theory of integral manifolds for system /I.4/ is of great interest in itself owing to the fact that the qualitative investigation of the solutions of system /I.4/ will be greatly simplified, if these lie on a manifold of smaller dimensionality than the initial phase space.

A special feature of the ideas being developed in the method of integral manifolds is the new approach to the qualitative theory of differential equations. We consider here two systems of differential equations - the exact equations /I.4/ and the approximate equations /I.6/, the difference between their right-hand sides being asymptotically small. It is known that individual solutions, as a rule, are very sensitive to small changes of the right-hand sides of the equations.

Our theory deals not with individual solutions /curves/ but with integral manifolds /hypersurfaces/. And it appears that an integral manifold is a more stable formation with respect to small changes of the right-hand sides of equations than individual solutions. In many cases theorems of the following type can be proved: if the approximate equations /I.6/ have a certain integral manifold, then the exact equations /I.4/ will also allow an integral manifold which lies in an asymptotically narrow neighbourhood of the approximate system. At the same time, similar theorems for individual solutions are obtainable only under sufficiently rigid conditions imposed on the right-hand sides of equations /I.4/ and /I.6/.

It seems quite opportune to emphasize here the fact that the very existence of an integral manifold for the exact equations /I.4/ is of great importance also for the study of their individual solutions, for now, instead of considering the whole phase space, it is possible to concentrate upon the consideration of solutions lying on the integral manifold - the hypersurface.

The above concerns especially the case of stable integral manifolds when there is contraction of the whole phase space towards the integral manifold.

After the preliminary remarks above and briefly characterizing the basic features of the method of integral manifolds, as applied to equations of the /I.4/ type, we proceed first of all to the consideration of the typical systems of differential equations containing a small parameter, for which the applying the ideas of the method of integral manifolds proves to be very effective.

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§ 2. Characteristic Types of Differential Equations Encountered in Nonlinear Mechanics

It is known that the methods of nonlinear mechanics are effectively applicable to the investigation of such oscillatory processes for which corresponding differential equations contain a small parameter, such that when it is zero these equations are exactly integrable. The form of these nonlinear differential equations containing a small parameter, as well as the very manner in which the small parameter is introduced, can be most diverse.

However, in many cases, by a change of variables, the differential equations of oscillations can be reduced to special forms in which the small parameter enters in such a manner that it is possible not only to employ asymptotical methods in order to obtain approximate solutions, but also to perform evaluations necessary for qualitative studies by using the method of integral manifolds.

In what follows we turn to a number of characteristic types of differential equations and to some changes of variables by which it is possible to reduce the system to a form convenient for further consideration.

First of all, consider the well known [1] reduction of the differential equations of oscillations to the so-called standard form, which is very convenient in connection with applying the averaging principle.

Let a nonlinear oscillatory system with n degrees of freedom be characterized by the following expressions for the kinetical and potential energies:

$$T = \frac{1}{2}\sum_{\kappa,j=1}^{n} a_{\kappa j}\dot{q}_\kappa \dot{q}_j, \qquad V = \frac{1}{2}\sum_{\kappa,j=1}^{n} c_{\kappa j} q_\kappa q_j, \tag{2.1}$$

where $q_1, q_2, \ldots, q_n$ are generalized coordinates $a_{\kappa j}, c_{\kappa j}$ - constants and, in addition, the quadratic forms T and V are definitely positive. Then, as it is well known, by using the linear transformation

$$q_j = \sum_{\kappa=1}^{n} \varphi_j^{(\kappa)} z_\kappa \tag{2.2}$$

normal coordinates $x_1, x_2, \ldots, x_n$ can be introduced for which

$$T = \frac{1}{2}\sum_{\kappa=1}^{n} \dot{z}_\kappa^2, \qquad V = \frac{1}{2}\sum_{\kappa=1}^{n} \omega_\kappa^2 z_\kappa^2, \tag{2.3}$$

($\varphi_{\kappa j}$, ω_κ being respectively the normal functions and natural frequencies corresponding to the quadratic froms /2.1/ and the Lagrangian equations of unperturbed motion become

$$\frac{d^2 z_\kappa}{dt^2} + \omega_\kappa^2 z_\kappa = 0 \qquad (\kappa = 1, 2, \ldots, n), \tag{2.4}$$

Assume now that the oscillatory system under consideration is acted upon by a small perturbation of the form:

$$\varepsilon Q_\kappa = \varepsilon\left[Q_\kappa^{(0)}(q_i, \dot{q}_i) + \sum_{\alpha}\left[Q_{\kappa 1}^{(\alpha)}(q_i, \dot{q}_i)\cos\Omega_\alpha t + Q_{\kappa 2}^{(\alpha)}(q_i, \dot{q}_i)\sin\Omega_\alpha t\right]\right], \tag{2.5}$$

where Ω_α are the frequencies of perturbation forces, ε - a small positive parameter.

Then, changing /2.5/, too, to normal coordinates, the following systems of nonlinear differential equations is

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obtained:

$$\frac{d^2 z_\kappa}{dt^2} + \omega_\kappa^2 z_\kappa = \varepsilon Z_\kappa(t, z_i, \dot{z}_i) \qquad (i = 1, 2, \ldots, N) \qquad /2.6/$$

in which Z_i are determined from the conditions of equivalent works represented by formulas

$$Z_\kappa = \sum_{j=1}^{N} Q_j \varphi_\kappa^{(j)} \qquad (\kappa = 1, 2, \ldots, N). \qquad /2.7/$$

Equations /2.6/, by a transformation of variables

$$z_\kappa = x_\kappa e^{i\omega_\kappa t} + x_{-\kappa} e^{-i\omega_\kappa t},$$

$$\dot{z}_\kappa = i\omega_\kappa x_\kappa e^{i\omega_\kappa t} - i\omega_\kappa x_{-\kappa} e^{-i\omega_\kappa t} \qquad /2.8/$$

are readily reduced to equations in standard form

$$\frac{dx_\kappa}{dt} = \varepsilon X_\kappa(t, x_i) \qquad (\kappa = \pm 1, \pm 2, \ldots, \pm N), \qquad /2.9/$$

in which

$$X_\kappa = \frac{e^{-i\omega_\kappa t}}{2 i \omega_\kappa} Z_\kappa, \qquad /2.10/$$

and, in addition, $-\omega_{-\kappa} = \omega_\kappa$, $Z_{-\kappa} = Z_\kappa$.

For simplification in the following, system /2.9/ will be represented in the form:

$$\frac{dx}{dt} = \varepsilon X(t, x) \qquad /2.11/$$

where x, X are n-vectors / $n = 2N$ in the case under consideration/ of an n-dimensional Eucledian space E_n, and the right-hand sides of system /2.11/ are periodic or almost periodic functions of t in accordance with the manner in which the functions εQ_κ were dependent on t.

To equations of the /2.11/ type can be also reduced many problems concerned with the action of extremal forces of high frequency on oscillatory systems. Thus, for instance, let the motion of such a system be characterized by equations of the form

$$\frac{d^2 q_s}{dt^2} = F_s\left(\omega t, q_i, \frac{dq_i}{dt}\right) \qquad (s = 1, 2, \ldots, N), \qquad /2.12/$$

where $F_s\left(\omega t, q_i, \frac{dq_i}{dt}\right)$ are periodic /or almost periodic/ functions of t, and ω - a "large" parameter.

Then, introducing new unknowns

$$\tau = \omega t, \quad q_s = x_s, \quad \frac{dq_s}{dt} = y_s, \qquad /2.13/$$

the following equations are obtained instead of system /2.12/:

$$\frac{dx_s}{d\tau} = \varepsilon y_s, \quad \frac{dy_s}{d\tau} = \varepsilon F_s(\tau, x_i, y_i), \qquad /2.14/$$

where $\varepsilon = \frac{1}{\omega}$ is a small parameter.

To equations of type /2.11/ is also reducible the more general case than that considered above, viz., when the conditions of a dynamical system are characterized by an angular variable α and n variables $x_1, x_2, \ldots, x_n$ and described by the following system of equations:

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$$\frac{dx_s}{d\tau} = X_s(\alpha, x_1, \dots, x_n) \quad (s = 1, 2, \dots, n),$$

$$\frac{d\alpha}{d\tau} = \lambda\omega(x_1, \dots, x_n) + A(\alpha, x_1, \dots, x_n), \qquad /2.15/$$

where λ is a large parameter, $\lambda\omega$ corresponds to the frequency of rotation α; $X_s(\alpha, x_1, \dots, x_n)$, $A(\alpha, x_1, \dots, x_n)$ are periodic /or almost periodic/ functions of the angular variable α of period 2π. /Systems of type /2.15/ will be encountered when hygroscopical systems are studied, in the theory of accelerators, etc./.

Introducing a new variable $\tau = \lambda t$, system /2.15/ can be represented as

$$\frac{dx_s}{d\tau} = \varepsilon X_s(\alpha, x_1, \dots, x_n) \quad (s = 1, 2, \dots, n),$$

$$\frac{d\alpha}{d\tau} = \omega(x_1, \dots, x_n) + \varepsilon A(\alpha, x_1, \dots, x_n), \qquad /2.16/$$

where $\varepsilon = \frac{1}{\lambda}$ is a small parameter, or eliminating τ,

$$\frac{dx_s}{d\alpha} = \varepsilon X_s^*(\alpha, x_1, \dots, x_n, \varepsilon) \quad (s = 1, 2, \dots, n), \qquad /2.17/$$

where

$$X_s^*(\alpha, x_1, \dots, x_n, \varepsilon) = \frac{X_s(\alpha, x_1, \dots, x_n)}{\omega(x_1, \dots, x_n) + \varepsilon A(\alpha, x_1, \dots, x_n)}.$$

Equations of a more general type than /2.6/ can also be reduced to equations of type /2.11/.

It is well known /3/ that in studying non-stationary processes in nonlinear oscillatory systems we have to deal often with systems of differential equations, such as

$$\frac{d}{dt}\left\{\sum_{i=1}^{N} a_{ij}(\tau)\dot{q}_i\right\} + \sum_{i=1}^{N} b_{ij}(\tau)q_i = \varepsilon Q_j(\tau, \Theta, q_1, \dots, q_N, \dot{q}_1, \dots, \dot{q}_N, \varepsilon)$$

$$(j = 1, 2, \dots, N), \qquad /2.18/$$

where q_i - generalized coordinates, εQ_j - external perturbing forces, $\tau = \varepsilon t$ - slow time /slow as compared to the natural unit of time - of the magnitude of the period of natural oscillations/, ε - a small positive parameter, $a_{ij}(\tau) = a_{ji}(\tau)$, $b_{ij}(\tau) = b_{ji}(\tau)$ - the inertial and quasi-elastic coefficients varying slowly with time $\frac{d\Theta}{dt} = \nu(\tau)$.

A system of type /2.18/, by a transformation of variables

$$q_i = \sum_{\kappa=1}^{N} \varphi_i^{(\kappa)}(\tau) Z_\kappa \quad (i = 1, 2, \dots, N), \qquad /2.19/$$

$\varphi_i^{(\kappa)}(\tau)$ denoting the normal functions which depend on the parameter τ and satisfy the orthogonality condition, can be represented as

$$\frac{d^2 Z_\kappa}{dt^2} + \omega_\kappa^2(\tau) Z_\kappa = \frac{\varepsilon}{m_\kappa(\tau)} Z_\kappa(\tau, \Theta, Z_1, \dots, Z_N, \dot{Z}_1, \dots, \dot{Z}_N, \varepsilon). \qquad /2.20/$$

Then, introducing another change of variables

$$Z_\kappa = x_\kappa e^{i\int\omega_\kappa(\tau)dt} + x_{-\kappa} e^{-i\int\omega_\kappa(\tau)dt},$$

$$\dot{Z}_\kappa = i\omega_\kappa(\tau) x_\kappa e^{i\int\omega_\kappa(\tau)dt} - i\omega_\kappa(\tau) e^{-i\int\omega_\kappa(\tau)dt}, \qquad /2.21/$$

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the following system of equations is obtained:

$$\frac{dx}{dt} = \varepsilon X(\tau, \theta, x, \varepsilon), \qquad /2.22/$$

where $\frac{d\theta}{dt} = \nu(\tau)$, $\tau = \varepsilon t$. The right-hand sides of these equations are dependent on the slow time τ and are periodic /or almost periodic or quasi-periodic/ in θ in accordance with the manner in which the functions Θ_j depend on θ.

Notice that to the form of /2.22/ can be also reduced a more general system than /2.18/, viz., for instance, that describing the oscillations of a system with many degrees of freedom, which is characterized in the unperturbed state /i.e. for $\varepsilon = 0$, $\tau = const$ / by the Lagrangian function of the form

$$\mathcal{L} = \frac{1}{2}\left\{\sum_{i,j=1}^{n} a_{ij}(\tau)\dot{q}_i\dot{q}_j + 2\sum_{i,j=1}^{n} g_{ij}(\tau)q_{ij}(\tau)q_i\dot{q}_j - \sum_{i,j=1}^{n} b_{ij}(\tau)q_i q_j\right\}, \qquad /2.23/$$

where $g_{ij}(\tau) = -g_{ij}(\tau)$.

In addition to the types of differential equations considered heretofore /which are largely reducible to equations in standard form/, it is well worthwhile to use the idea of integral manifolds in studying somewhat more general types of differential equations describing dynamical processes in complex oscillating systems.

In this connection, we shall consider also systems which are near to autonomous, such as

$$\frac{dx}{dt} = X(x) + \varepsilon X^*(\nu t, x, \varepsilon), \qquad /2.24/$$

where ε is a small positive parameter, and systems which are near to non-autonomous

$$\frac{dx}{dt} = X(t, x) + \varepsilon X^*(\nu t, x, \varepsilon), \qquad /2.25/$$

ε being a small positive parameter, x, X, X^* - n-dimensional vectors of Eucledian space E_n.

With a number of assumptions to be considered later, numerous problems of the theory of oscillations /such as single frequency oscillations of systems with many degrees of freedom and remote from being harmonic, relaxation oscillations, etc./ will be reduced to systems of types /2.24/ and /2.25/.

We shall also consider more general systems of differential equations of the form

$$\frac{dx}{dt} = X(\tau, x) + \varepsilon X^*(\tau, \theta, x, \varepsilon), \qquad /2.26/$$

$$\frac{d\theta}{dt} = \nu(\tau) + \varepsilon X^*(\tau, \theta, x, \varepsilon) \qquad /2.27/$$

/ $\tau = \varepsilon t$ - slow time, $\frac{d\theta}{dt} = \nu(\tau)$ /, in which the possibility of a slow variation /slow in the sense referred to above/ of a number of parameters is taken into account.

For the present we restrict ourselves to the types of equations above, although it should be noticed that it would be possible to mention other systems of differential equations of practical importance, which are encountered in the the ory of nonlinear oscillations and for which the study of integral manifolds would be of great interest.

Before we proceed to transform the above types of equations to a form convenient for further consideration, let it

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be mentioned that, in addition to the advantages to be achieved by applying the method of integral manifolds, as described in the previous section, great importance is also attached to the investigation of stable two-dimensional integral manifolds in connection with the problem of the construction of approximate solutions for systems of the types /2.6/, /2.18/, and /2.24 to 2.27/, which correspond to single-frequency oscillatory processes.

§ 3. Transformation of the Differential Equations to a form Convenient for Further Study

We now turn to the transformation of the above types of equations to new variables. In so doing, we shall attempt to reduce these equations to such a form which would take into consideration their specific features and enable us to evaluate the terms /of the highest order of smallness/ which are being dropped in order to derive approximate equations, or averaged equations in particular, and to readily obtain numerous evaluations required to construct the manifolds and bring out their properties.

First of all we proceed to the transformation of a system of differential equations in standard form. Since transformations of a similar kind have found wide and effective application in the methods employed in the study of integral manifolds and the construction of approximate solutions, we shall develop them here in sufficient detail for the first case only, remembering that exhaustive details concerning operations involved in the derivations are to be found in a monograph by N.N.Bogoliubov and Yu.A.Mitropolski /2/.

Thus, we consider the system of equations /2.II/

$$\frac{dx}{dt} = \varepsilon X(t,x), \qquad /2.II/$$

where x, X are n-dimensional vectors of the n-dimensional Eucledian space E_n.

Simultaneously with system /2.II/ we shall consider the equation of the first approximation

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi). \qquad /3.I/$$

Consider first a more general case when the equations of the first approximation /3.I/ have a periodical solution

$$\xi = \xi(\omega t) \qquad /3.2/$$

of period $\frac{2\pi}{\omega}$ in t : $\xi(\varphi + 2\pi) = \xi(\varphi)$.

Assume that in the ρ neighbourhood of this periodical solution the vector-function $X(t,x)$ has the following properties:

a/ $X(t,x)$ and its partial derivatives of the first order with respect to x are bounded and uniformly continuous with respect to x in the region

$$-\infty < t < \infty\ ,\ x \in \mathcal{D}_\rho, \qquad /3.3/$$

where $\mathcal{D}_\rho$ is the ρ-neighbourhood of the periodical solution /3.2/;

b/ At every point $\mathcal{D}_\rho$:

$$\frac{1}{T}\int_t^{t+T} X(t,x)\,dt \to X_0(x) \quad \text{as} \quad T \to \infty \qquad /3.4/$$

uniformly with respect to t in the interval $(-\infty, \infty)$

Now, for the equations of first approximation write

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the variational equation corresponding to the periodical solution /3.2/ and we obtain a homogeneous system of linear differential equations with periodical coefficients

$$\frac{d\delta\xi}{dt} = \varepsilon X'_{ox}(\xi(\omega t))\delta\xi. \qquad /3.5/$$

By virtue the definition of function /3.2/, we have identically

$$\omega\xi'_{\varphi}(\varphi) = \varepsilon X_o(\xi(\varphi)). \qquad /3.6/$$

Differentiating this identity with respect to φ, it is easily seen that with the arbitrary constant δu_o the relation

$$\delta\xi = \xi'_{\varphi}(\omega t)\delta u_o \qquad /3.7/$$

is a solution of the variational equation /3.5/.

Before we start transforming equations /2.II/, let us make some well known remarks concerned with the theory of differential equations with periodic coefficients.

On the basis of Floquet-Liapunoff's theorem concerning the properties of linear homogeneous differential equations with periodic coefficients, by using transformations

$$\delta\xi = \xi'(\omega t)\delta u_o + A(\omega t)\delta u, \qquad /3.8/$$

where $A(\varphi)$ is a matrix consisting of n rows and $n-1$ columns, whose elements are periodic functions of φ of period 2π and the continuous first derivatives, δu -a vector with components $\delta u_1, \delta u_2, \ldots, \delta u_{n-1}$, equation /3.5/ with periodic coefficients reduces, as it is well known, to a system of differential equations with constant coefficients

$$\frac{d\delta u_o}{dt} = 0, \quad \frac{d\delta u}{dt} = H\delta u \qquad /3.9/$$

such that the roots of the equation

$$Det|J_{n-1}p - H| = 0, \qquad /3.10/$$

where J is a unitary $n-1$ -dimensional quadraticmatrix, H is an $n-1$ -dimensional quadratic matrix, are characteristic exponents for system of equations /3.5/.

It is to be noted that, generally speaking, both matrices $A(\varphi)$ and H, as well as vector δu are complex, though the coefficients of equation /3.5/ are real.

In passing, we establish a relationship to be used later on. Substituting /3.8/ into equation /3.5/, we obtain

$$\frac{d\xi'(\varphi)}{dt}\delta u_o + \xi'(\varphi)\frac{d\delta u_o}{dt} + \frac{dA(\varphi)}{d\varphi}\omega\delta u + A(\varphi)\frac{d\delta u}{dt} =$$

$$= \varepsilon X'_{ox}(\xi(\varphi))\xi'(\varphi)\delta u_o + X'_{ox}(\xi(\varphi))A(\varphi)\delta u. \qquad /3.11/$$

Taking into consideration equations /3.9/, as well as the fact that $\xi'_{\varphi}(\omega t)\delta u_o$ is a solution of equations /3.5/, the following identity is obtained:

$$\frac{dA(\varphi)}{d\varphi}\omega + A(\varphi)H = \varepsilon X'_{ox}(\xi(\varphi))A(\varphi), \qquad /3.12/$$

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and

$$\frac{\partial \bar{A}(\varphi)}{\partial \varphi}\omega + \bar{A}(\varphi)\bar{H} = \varepsilon X'_{0x}(\xi(\varphi))\bar{A}(\varphi). \qquad /3.13/$$

where $\bar{A}(\varphi)$ and $\bar{H}$ are conjugate with respect to $A(\varphi)$ and H.

We now proceed to the transformation of equation /2.11/ to new variables. Re-write it in the form:

$$\frac{dx}{dt} = \varepsilon X_0(x) + \varepsilon Z(t,x), \qquad /3.14/$$

where

$$Z(t,x) = X(t,x) - X_0(x). \qquad /3.15/$$

Introduce new variables φ, $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-1})$ according to formulas

$$x = \xi(\varphi) + \frac{1}{2}\left\{A(\varphi)\mathcal{B} + \bar{A}(\varphi)\bar{\mathcal{B}}\right\}, \qquad /3.16/$$

where $\mathcal{B}$ and $\bar{\mathcal{B}}$ are mutually conjugate.

Substituting /3.16/ into /3.14/ and taking into consideration identities /3.12/, /3.13/, we find:

$$\left[\xi'(\varphi) + \frac{1}{2}(A'_\varphi(\varphi)\mathcal{B} + \bar{A}'_\varphi(\varphi)\bar{\mathcal{B}})\right]\left(\frac{d\varphi}{dt} - \omega\right) + \frac{1}{2}\left[A(\varphi)\left(\frac{d\mathcal{B}}{dt} - H\mathcal{B}\right) + \bar{A}(\varphi)\left(\frac{d\bar{\mathcal{B}}}{dt} - \bar{H}\bar{\mathcal{B}}\right)\right] = \varepsilon\left[X_0\left\{\xi + \frac{1}{2}(A(\varphi)\mathcal{B} + \bar{A}(\varphi)\bar{\mathcal{B}})\right\} - X_0(\xi) - \frac{1}{2}X'_{0x}(\xi)\left[A(\varphi)\mathcal{B} + \bar{A}(\varphi)\bar{\mathcal{B}}\right] + \varepsilon Z\left\{t, \xi(\varphi) + \frac{1}{2}\left[A(\varphi)\mathcal{B} + \bar{A}(\varphi)\bar{\mathcal{B}}\right]\right\}. \qquad /3.17/$$

System /3.17/ can now be represented in the form

$$\left[\xi'(\varphi) + \frac{1}{2}(A'_\varphi(\varphi)\mathcal{B} + \bar{A}'_\varphi(\varphi)\bar{\mathcal{B}})\right]\left(\frac{d\varphi}{dt} - \omega\right) + \frac{1}{2}\left[A(\varphi)\left(\frac{d\mathcal{B}}{dt} - H\mathcal{B}\right) + \bar{A}(\varphi)\left(\frac{d\bar{\mathcal{B}}}{dt} - \bar{H}\bar{\mathcal{B}}\right)\right] = Y, \qquad /3.18/$$

where Y is a real vector function.

From this system we obtain the variables

$$\frac{d\varphi}{dt} - \omega, \qquad \frac{d\mathcal{B}}{dt} - H\mathcal{B} = R \qquad /3.19/$$

such that the condition

$$R = \bar{R} \qquad /3.20/$$

is satisfied, and, consequently,

$$\frac{d\mathcal{B}}{dt} - \bar{H}\mathcal{B} = \bar{R}$$

It schould be noted that insofar as the values of are complex, there is always a certain arbitrariness in the choice of ; besides, choosing the additional condition in the form of /3.20/ does not appear to be necessarily done.

Taking into considerations /3.20/, we obtain from system /3.18/ a system of linear equations with real coefficients in the variables of /3.19/:

$$\left[\xi'(\varphi) + \frac{1}{2}(A'_\varphi(\varphi)\mathcal{B} + \bar{A}'_\varphi(\varphi)\bar{\mathcal{B}})\right]\left(\frac{d\varphi}{dt} - \omega\right) + \frac{1}{2}(A(\varphi) + \bar{A}(\varphi))\left(\frac{d\mathcal{B}}{dt} - H\mathcal{B}\right) = Y. \qquad /3.21/$$

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Assume that the determinant of this system

$$D(\varphi,\mathcal{B})=\left|\xi'(\varphi)+\frac{1}{2}\left(\bar{A}'_{\varphi}(\varphi)\mathcal{B}+\bar{\bar{A}}'_{\varphi}(\varphi)\bar{\mathcal{B}}\right),\frac{1}{2}\left(A(\varphi)+\bar{A}(\varphi)\right)\right| \qquad /3.22/$$

is different from zero for $\mathcal{B}=0$ *)

Then, by virtue of its continuity, it will be also continuous for some δ -neighbourhood of the point $\mathcal{B}=0$.

Let U_δ denote the δ-neighbourhood of the point $\mathcal{B}=0$ in which $D(\varphi,\mathcal{B})\neq 0$ and ΩU_δ the region of variation of $(\varphi,\mathcal{B})$ for which $\mathcal{B}\in U_\delta$.

Notice that it is always possible to find a small positive δ, such that in the region ΩU_δ the following inequality holds:

$$\frac{1}{2}\left|A(\varphi)\mathcal{B}+\bar{A}(\varphi)\bar{\mathcal{B}}\right|<\rho$$

and, consequently, x as determined by /3.16/ lies in the region $\mathcal{D}_\rho$.

Solving system /3.21/ in region U_ρ for the variables /3.19/, we get:

$$\frac{d\varphi}{dt}=\varepsilon\omega+\varepsilon W(t,\varphi,\mathcal{B}),$$
$$\frac{d\mathcal{B}}{dt}=\varepsilon H\mathcal{B}+\varepsilon B(t,\varphi,\mathcal{B}), \qquad /3.23/$$

where the function $W(t,\varphi,\mathcal{B})$ and vector function $B(t,\varphi,\mathcal{B})$ and their partial derivatives of the first order with respect to $\varphi,\mathcal{B}$ are bounded and uniformly continuous with respect to $\varphi,\mathcal{B}$.

Furthemore, for every point of region

$$-\infty<t<\infty,\ (\varphi,\mathcal{B})\in\Omega U_\delta \qquad /3.24/$$

$$\frac{1}{T}\int_t^{t+T}W(t,\varphi,\mathcal{B})dt\to W(\varphi,\mathcal{B}),\ \frac{1}{T}\int_t^{t+T}B(t,\varphi,\mathcal{B})dt\to B(\varphi,\mathcal{B}) \qquad /3.25/$$

uniformly with respect to t for $T\to\infty$.

Simultaneously, the functions $W(\varphi,\mathcal{B}), B(\varphi,\mathcal{B})$ together with their partial derivatives of the first order become zero at $\mathcal{B}=0$ and satisfy Lipschitz's conditions with respect to φ and $\mathcal{B}$ with Lipschitz's constant for which the following condition holds:

$$\rho(\sigma)\to 0 \quad as \quad \sigma\to 0 \quad (\sigma\sim\delta).$$

For further study of equations /3.23/, it is convenient to introduce new variables and transform them into

$$\frac{dg}{dt}=\mathcal{G}(\varepsilon)+P(t,g,h,\varepsilon),$$
$$\frac{dh}{dt}=Hh+R(t,g,h,\varepsilon), \qquad /3.26/$$

such that the functions $P(t,g,h,\varepsilon)$ and $R(t,g,h,\varepsilon)$ are sufficiently small for small h and ε.

We now proceed to give a general outline of this transformation, omitting the details described in [2].

*) For simplification, the dependence of functions on their conjugate quantities will be given neither here nor in the following.

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The main points of the transformation are certain properties of the functions to be considered in the following and we shall state them in the form of a lemma.

Lemma. - Let function possess the following properties:

a/ $f(t,x)$ a continuous function of t and x is defined for all real t and $x \in E$, where E is a compact subset from E_n;

b/ at every point of E

$$\frac{1}{T}\int_{t}^{t+T} f(t,x)\,dt \to 0 \quad \text{as} \quad T \to \infty \qquad /3.27/$$

uniformly with respect to t and x;

c/ there exist constants M and λ for which

$$|f(t,x)| \le M, \quad |f(t,x')-f(t,x'')| \le \lambda |x'-x''|. \qquad /3.28/$$

Then, for every $\eta>0$ there exists a continuous function $\zeta(\eta) \to 0$ as $\eta \to 0$, such that for the function under consideration $f(t,x)$ the following inequality holds:

$$\left|f_\eta(t,x)\right| = \left|\int_{-\infty}^{t} e^{-\eta(t-\tau)} f(\tau,x)\,d\tau\right| \le \frac{\zeta(\eta)}{\eta}. \qquad /3.29/$$

From this transformation an important corollary follows, which is essential for the transformation above.

$$W_1(\xi,\varphi,\beta) = W(t,\varphi,\beta) - W(\varphi,\beta),$$

$$B_1(\xi,\varphi,\beta) = B(t,\varphi,\beta) - B(\varphi,\beta) \qquad /3.30/$$

assuming as set E region ΩU_δ.

Since functions /3.30/ are periodic in φ of period 2π, it is evident that Ω can be represented as a circle, and the $E = \Omega U_\delta$ will be a compact region in the metric space defined by the topological product of Ω and an $n-1$ -dimensional Eucledian space.

In addition, by virtue of /3.25/ at any point of region ΩU_δ

$$\frac{1}{T}\int_{t}^{t+T} W_1(\xi,\varphi,\beta)\,dt \to 0,$$

$$\frac{1}{T}\int_{t}^{t+T} B_1(t,\varphi,\beta)\,dt \to 0, \quad T \to \infty \qquad /3.31/$$

uniformly with respect to t.

Therefore, on the basis of the lemma above, for any positive η a function $\zeta(\eta)$ can be always constructed, such that $\zeta(\eta)$ as $\eta \to 0$, for which the following inequalities hold:

$$|W_{1\eta}(\xi,\varphi,\beta)| \le \frac{\zeta(\eta)}{\eta}, \quad |B_{1\eta}(t,\varphi,\beta)| \le \frac{\zeta(\eta)}{\eta},$$

$$-\infty < t < \infty, \quad (\varphi,\beta) \in \Omega U_\delta, \qquad /3.32/$$

where

$$W_{1\eta}(t,\varphi,\beta) = \int_{-\infty}^{t} e^{-\eta(t-\tau)} W_1(t,\varphi,\beta)\,dt,$$

$$B_{1\eta}(t,\varphi,\beta) = \int_{-\infty}^{t} e^{-\eta(t-\tau)} B_1(t,\varphi,\beta)\,dt. \qquad /3.33/$$

We now construct the expressions

$$u(t,\varphi,\beta) = \iint_{\Omega U_\sigma} \delta_a(\varphi-\varphi')\Delta_a(\beta-\beta')W_{1\eta}(\xi,\varphi',\beta')\,d\varphi'd\beta',$$

$$v(t,\varphi,\beta) = \iint_{\Omega U_\delta} \delta_a(\varphi-\varphi')\Delta_a(\beta-\beta')B_{1\eta}(t,\varphi',\beta')\,d\varphi'd\beta', \qquad /3.34/$$

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where $\Delta_a(\mathcal{B})$ and $\delta_a(\varphi)$ are functions, such that

$$\Delta_a(\mathcal{B}) = \begin{cases} A_a\left\{1-\frac{|\mathcal{B}|^2}{a^2}\right\}^{2q}, & |\mathcal{B}| \le a, \\ 0, & |\mathcal{B}| > a, \end{cases} \qquad /3.35/$$

a being a certain given sufficiently small number, A_a a constant as determined from the condition of normalization:

$$\int_{U_\delta} \Delta_a(\mathcal{B})\, d\mathcal{B} = 1, \qquad /3.36/$$

where

$$d\mathcal{B} = d\mathcal{B}_1 d\mathcal{B}_2 \cdots d\mathcal{B}_{n-1}$$

and $\delta_a(\varphi)$ a function of one real variable φ given for the interval by means of the relation

$$\delta_a(\varphi) = \begin{cases} \Phi_a\left\{1-\frac{\varphi^2}{a^2}\right\}^{2q}, & |\varphi| \le a, \\ 0, & |\varphi| > a, \end{cases}$$

$$\int_{-a}^{a} \delta_a(\varphi)\, d\varphi = 1, \qquad a < \pi \qquad /3.37/$$

and extends over the whole real axis by virtue of the condition of periodicity of period 2π, q - a certain fixed integer whose magnitude can be taken arbitrarily large.

It should be noted that a and q can be chosen in the form of a_ε and q_ε as certain functions of the parameter ε such that

$$a_\varepsilon \to 0, \quad q_\varepsilon \to 0, \quad \varepsilon\, \mathcal{G}(a_a)\, \frac{\xi(q_\varepsilon)}{q_a} \to 0,$$

$$\mathcal{G}(a_\varepsilon)\, \xi(q_\varepsilon) \to 0 \quad as \quad \varepsilon \to 0 \qquad /3.38/$$

and that expressions /3.34/ and their partial derivatives with respect to φ and $\mathcal{B}$ up to $2q$ - order were bounded in the region $-\infty < t < \infty$, $\varphi \in \Omega$, $\mathcal{B} \in U_\delta$ by the value $\mathcal{G}(a)\, \frac{\xi(q)}{q}$.

Now, changing the variables in system /3.23/ according to

$$\varphi = g + \varepsilon u(t, g, h), \quad \mathcal{B} = h + \varepsilon v(t, g, h), \qquad /3.39/$$

where g and $h(h_1, h_2, \ldots, h_{n-1})$ are the new variables, and performing simple transformations, we arrive at system /3.26/ for which positive numbers ε_0, ρ_1 $(\rho_1 < \delta)$ can be pointed out, such that the following requirements will be satisfied:

a/ function $\mathcal{G}(\varepsilon)$ is defined for $0 < \varepsilon < \varepsilon_0$,

b/ functions $P(t, g, h, \varepsilon)$ and $R(t, g, h, \varepsilon)$ are defined in region

$$-\infty < t < \infty, \quad g \in \Omega, \quad h \in U_{\rho_1}, \quad 0 < \varepsilon < \varepsilon_0 \qquad /3.40/$$

and possess a period 2π with respect to the angular variable g.

c/ $\mathcal{G}(\varepsilon)$ and $P(t, g, h, \varepsilon)$ assume only real values;

d/ the following inequalities hold in region /3.40/:

$$|P(t, g, 0, \varepsilon)| \le M(\varepsilon), \quad |R(t, g, 0, \varepsilon)| \le M(\varepsilon), \qquad /3.41/$$

where $M(\varepsilon) \to 0$ as $\varepsilon \to 0$;

e/ for any positive $\sigma < \rho_1$ in region

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$$-\infty < t < \infty, \quad g \in \Omega, \quad h \in U_\sigma, \quad 0 < \varepsilon < \varepsilon_0$$

/3.42/

the functions $P(t, g, h, \varepsilon)$ and $R(t, g, h, \varepsilon)$ satisfy Lipschitz's conditions with constant $\lambda(\varepsilon, \sigma) \to 0$ for $\varepsilon \to 0, \ \sigma \to 0$;

f/ all the real parts of the roots of equation

$$Det\,|p J_{n-1} - H| = 0$$

/3.43/

are nonzero.

We now proceed to transform the system of equations /2.22/.

Assume that for system /2.22/ conditions of types a/ and b/ /p. 17 / are satisfied, viz.,

a/ $X(\tau, \theta, x, \varepsilon)$ and its partial derivatives of the first order with respect to x are bounded and uniformly continuous with respect to x in the region

$$-\infty < t < \infty, \ x \in D_\rho,$$

/3.44/

where D_ρ is the ρ-neighbourhood of the "periodic" solution $\xi = \xi(\tau, \theta)$ of the averaged system:

$$\frac{d\xi}{dt} = \varepsilon X_0(\tau, \xi, 0);$$

/3.45/

b/ then at every point D_ρ:

$$\frac{1}{T}\int_t^{t+T} X(\tau, \theta, x, 0)\,dt \to X_0(\tau, \xi, 0) \quad as \ T \to \infty \qquad (\tau = const)$$

/3.46/

uniformly with respect to t in the interval $(-\infty, \infty)$.

Now, assuming that for every $-\infty < t < \infty$ the $n-1$ characteristic exponent of the variational equation corresponding to system /3.45/ and solution $\xi = \xi(\tau, \theta)$ has nonzero real parts, and successively changing the variables of type /3.16/:

$$x = \xi(\tau, \varphi) + \frac{1}{2}\left\{A(\tau, \varphi)\beta + \bar{A}(\tau, \varphi)\bar{\beta}\right\}$$

/3.47/

and type /3.39/:

$$\varphi = g + \varepsilon u(\tau, t, g, h), \quad \beta = h + \varepsilon v(\tau, t, g, h)$$

/3.48/

system 2.22/ will be finally reduced to the following equations:

$$\frac{dg}{dt} = \varphi(\varepsilon, \tau) + P(\tau, \theta, g, h, \varepsilon),$$

$$\frac{dh}{dt} = H(\tau)h + R(\tau, \theta, g, h, \varepsilon)$$

/3.49/

in which the functions in the right-hand sides satisfy a number of conditions similar to those given on p.23 for the right-hand sides of equations /3.26/.

Let us go on now to the transformation of system /2.24/:

$$\frac{dx}{dt} = X(x) + \varepsilon X^*(\vartheta t, x, \varepsilon).$$

/2.24/

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We shall describe this transformation in sufficient detail, for it is simpler and somewhat different from that given above.

We first assume that system /2.24/ describes a weak periodic, of period $\frac{2\pi}{\nu}$, perturbation of relaxation oscillations.

To begin with, let for system of equations

$$\frac{dx}{dt} = X(x) \qquad /3.50/$$

which is obtained from system /2.24/ by putting $\varepsilon = 0$ and which we shall call non-perturbed equations in what follows, a stable periodic solution

$$x = x^0(\omega t + \varphi) \qquad /3.51/$$

in $\psi = \omega t + \varphi$ of period 2π be known

$$x^0(\psi + 2\pi) = x^0(\psi) \qquad /3.52/$$

depending on one arbitrary constant φ.

We assume that this periodic solution /3.52/ corresponds to the herein considered stable relaxation oscillation of the non-perturbed system.

Then, $n-1$ characteristic exponents

$$\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \qquad /3.53/$$

for the system of equations with periodic coefficients

$$\frac{d\delta x}{dt} = X'_x(x^0)\delta x \qquad /3.54/$$

/ δx being an n-dimensional vector $X'_x(x_0)$ - an

n-dimensional square matrix with periodic coefficients/ which is a variational system of equations corresponding to solution /3.52/, will have negative real parts, while the n-th characteristic exponent α_0 is obviously zero due to the fact that $\frac{\partial x^0}{\partial \varphi}$ is a solution of system /3.54/.

Suppose that the functions

$$X(x) + \varepsilon X^*(\nu t, x, \varepsilon) \qquad /3.55/$$

are unboundedly differentiable with respect to x and ε in the region

$$-\infty < t < \infty, \quad x \in D_\rho, \quad 0 < \varepsilon < \varepsilon_0, \qquad /3.56/$$

where D_ρ is a ρ-neighbourhood of the curve

$$x = x(\omega t + \varphi) \qquad /3.57/$$

in an n-dimensional Eucledian space E_n.

We first assume that, in addition to the negativity requirement of the real parts of the characteristic exponents /3.53/, the following conditions will also hold:

$$m_1\gamma_1 + m_2\gamma_2 + \cdots + m_{n-1}\gamma_{n-1} \neq \gamma_q,$$
$$m_1 + m_2 + \cdots + m_{n-1} \geq 2, \qquad /3.58/$$

where γ_q $(q = 1, 2, \ldots, n-1)$ are the real parts of the characteristic exponents: $m_1, m_2, \ldots, m_{n-1}$ non-negative integers.

Now, if we consider that the vector functions $X(x)$ are analytic in x, then the general solution of system /3.50/ in the neighbourhood of the periodic solution /3.51/

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is known to be

$$x = f(\omega t + \varphi, Ce^{\alpha t}), \qquad /3.59/$$

where the scalar φ and $n-1$- dimensional vector $C(C_1, C_2, \ldots, C_{n-1})$ are the constants of integration $\alpha(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ are the characteristic exponents of system /3.54/; simultaneously

$$f(\Psi + 2\pi, h) \equiv f(\Psi, h),$$
$$f(\Psi, 0) = x^0(\Psi), \qquad /3.60/$$

where the $n-1$ -dimensional vector function $f(\Psi, h)$ in analytic and regular for sufficiently small values of h.

Let us introduce new variables $\Psi, h(h_1, h_2, \ldots, h_{n-1})$ for x into equations /2.24/ according to the formula

$$x = f(\Psi, h). \qquad /3.61/$$

Then, having performed a series of elementary operations, we obtain the following system of differential equations in the new variables Ψ, h instead of system /2.24/:

$$\frac{d\Psi}{dt} = \omega + \varepsilon P(\vartheta t, \Psi, h, \varepsilon),$$
$$\frac{dh}{dt} = \alpha h + \varepsilon R(\vartheta t, \Psi, h, \varepsilon), \qquad /3.62/$$

where the function $P(\vartheta t, \Psi, h, \varepsilon)$ and the $n-1$ -dimensional vector function $R(\vartheta t, \Psi, h, \varepsilon)$ possess the same properties in the region

$$-\infty < t < \infty, \quad 0 < \varepsilon < \varepsilon_0, \quad \Psi \in \Omega, \quad h \in U_\delta \qquad /3.63/$$

as those possessed by the vector function $X^*(\vartheta t, x, \varepsilon)$

Consider now the general case when the $n-1$ -th characteristic exponent /3.53/ of solution /3.51/ has negative real parts, however condition /3.58/ may not hold.

In analogy to one of the cases considered above, we introduce new variables $\Psi, h(h_1, h_2, \ldots, h_{n-1})$ by using the transformation

$$x = x^0(\Psi) + \frac{1}{2}\{A(\Psi)h + \bar{A}(\Psi)\bar{h}\}, \qquad /3.64/$$

where $A(\Psi)$ is a matrix consisting of n rows and $n-1$ columns, whose elements are periodic functions in Ψ of period 2π, have continuous derivatives with respect to Ψ, and satisfy the identity

$$\frac{\partial A(\Psi)}{\partial \Psi}\omega + A(\Psi)H = X'_x(x^0)A(\Psi), \qquad /3.65/$$

$\bar{h}, \bar{A}(\Psi)$ are complex conjugate with respect to $h, A(\Psi)$, H is a square matrix of the $n-1$ -th order for which the roots of the equation

$$Det|pJ - H| = 0 \qquad /3.66/$$

/ J -unitary matrix of the $n-1$ -th order/ are the characteristic exponents $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ for the system of variational equations /3.54/.

Substituting /3.64/ into /2.24/, we find:

$$\frac{\partial x^0}{\partial \Psi}\frac{d\Psi}{dt} + \frac{1}{2}\left\{\frac{\partial A(\Psi)}{\partial \Psi}h + \frac{\partial \bar{A}(\Psi)}{\partial \Psi}\bar{h}\right\}\frac{d\Psi}{dt} + \frac{1}{2}\left\{A(\Psi)\frac{dh}{dt} + \bar{A}(\Psi)\frac{d\bar{h}}{dt}\right\} =$$
$$= X\left\{x^0 + \frac{1}{2}[Ah + \bar{A}\bar{h}]\right\} + \varepsilon X^*\left\{\vartheta t, x^0 + \frac{1}{2}[Ah + \bar{A}\bar{h}], \varepsilon\right\}. \qquad /3.67/$$

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The right-hand side of the system obtained can be represented in the form:

$$X\left\{x^{\circ}+\frac{1}{2}[Ah+\bar{A}\bar{h}]\right\}+\varepsilon X^{*}\left\{\vartheta t, x^{\circ}+\frac{1}{2}[Ah+\bar{A}\bar{h}], \varepsilon\right\}=$$

$$=X(x_{\circ})-X(x_{\circ})+\frac{1}{2}X'_{x}(x^{\circ})[A(\psi)h+\bar{A}(\psi)\bar{h}]-$$

$$-\frac{1}{2}X'_{x}(x_{\circ})\bar{A}(\psi)h+\bar{A}(\psi)\bar{h}]+X\left\{x_{\circ}+\frac{1}{2}[Ah+\bar{A}\bar{h}]\right\}+ \qquad /3.68/$$

$$+\varepsilon X^{*}\left\{\vartheta t, x^{\circ}+\frac{1}{2}[Ah+\bar{A}\bar{h}], \varepsilon\right\}=X(x^{\circ})+$$

$$+\frac{1}{2}X'_{x}(x^{\circ})[A(\psi)h+\bar{A}(\psi)\bar{h}]+Y(\vartheta t, \psi, h, \varepsilon),$$

where

$$Y(\vartheta t, \psi, h, \varepsilon)^{*)}=-X(x^{\circ})-\frac{1}{2}X'(x_{\circ})[A(\psi)h+$$

$$+\bar{A}(\psi)\bar{h}]+X\left\{x^{\circ}+\frac{1}{2}[Ah+\bar{A}\bar{h}]\right\}+$$

$$+\varepsilon X^{*}\left\{\vartheta t, x^{\circ}+\frac{1}{2}[Ah+\bar{A}\bar{h}], \varepsilon\right\}. \qquad /3.69/$$

It is obvious that for $h=0$,

$$Y(\vartheta t, \psi, h, \varepsilon)\big|_{h=0}=\varepsilon X^{*}(\vartheta t, x^{\circ}, \varepsilon) \quad \text{as } \varepsilon \to 0 \qquad /3.70/$$

On the basis of /3.51/ we have the identity

$$\frac{\partial x^{\circ}}{\partial \psi}\omega=X(x^{\circ}). \qquad /3.71/$$

Then, by subtracting /3.71/ from /3.67/ and taking into account identity /3.65/, as well as its conjugate, the following system of equations can be written down:

$$\left\{\frac{\partial x^{\circ}}{\partial \psi}+\frac{1}{2}\left(\frac{\partial A}{\partial \psi}h+\frac{\partial \bar{A}}{\partial \psi}\bar{h}\right)\right\}\left(\frac{d\psi}{dt}-\omega\right)+\frac{1}{2}A(\psi)\left[\frac{dh}{dt}-Ha\right]+$$

$$+\frac{1}{2}\bar{A}(\psi)\left[\frac{d\bar{h}}{dt}-\bar{H}\bar{h}\right]=Y(\vartheta t, \psi, h, \varepsilon), \qquad /3.72/$$

where the real functions $Y(\vartheta t, \psi, h, \varepsilon)$ are unbounded, differentiable with respect to h and ε in the region /3.56/, periodic in ϑt and ψ of period 2π, and become values of the order ε for $h=0$.

Let us determine the variables

$$\frac{d\psi}{dt}-\omega, \quad \frac{dh}{dt}-Hh=R \qquad /3.73/$$

from system /3.72/ and we specify as before that the condition

$$R=\bar{R}$$

holds. Then, system /3.72/ assumes the form:

$$\left\{\frac{\partial x^{\circ}}{\partial \psi}+\frac{1}{2}\left[\frac{\partial A}{\partial \psi}h+\frac{\partial \bar{A}}{\partial \psi}\bar{h}\right]\right\}\left(\frac{d\psi}{dt}-\omega\right)+\frac{1}{2}[A(\psi)+\bar{A}(\psi)]\left(\frac{dh}{dt}-Hh\right)=$$

$$=Y(\vartheta t, \psi, h, \varepsilon). \qquad /3.74/$$

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*) For the sake of simplicity, the dependence of the functions on $\bar{h}$ is not shown in what follows.

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Suppose that the determinant of this system

$$D(\psi,h)=\left|\frac{\partial x^{\circ}}{\partial \psi}+\frac{1}{2}\left[\frac{\partial \bar{A}}{\partial \psi}h+\frac{\partial \bar{\bar{A}}}{\partial \psi}\bar{h}\right],\ \ \frac{1}{2}\left[\bar{A}(\psi)+\bar{\bar{A}}(\psi)\right]\right| \qquad /3.75/$$

is nonzero for $h=0$.

Then, by virtue of its continuity, it will differ from zero also in some δ -neighbourhood of the point $h=0$ and we can always find a small positive δ , such that the following inequality holds:

$$\frac{1}{2}\left|\bar{A}(\psi)h+\bar{\bar{A}}(\psi)\bar{h}\right|<\rho$$

for $|h|<\delta$ and, consequently, x belongs to the region /3.56/.

Solving system /3.74/ for the variables /3.73/ in the region

$$|h|<\delta,\ \ 0<\varepsilon<\varepsilon_0,\ -\infty<t<\infty,\ \psi\in\Omega \qquad /3.76/$$

we obtain the following system of differential equations in the new variables ψ and h :

$$\frac{d\psi}{dt}=\omega+P(\vartheta t,\psi,h,\varepsilon),$$
$$\frac{dh}{dt}=Hh+R(\vartheta t,\psi,h,\varepsilon). \qquad /3.77/$$

Now positive numbers can be pointed out such that the following requirements will be satisfied:

a/ The functions $P(\vartheta t,\psi,h,\varepsilon)$ and $R(\vartheta t,\psi,h,\varepsilon)$ are defined in the region /3.76/, they are of period 2π in ψ and ϑt and unboundedly differentiable with respect to h and ε .

b/ For $h=0$ the functions $P(\vartheta t,\psi,h,\varepsilon)$ and $R(\vartheta t,\psi,h,\varepsilon)$ are magnitudes of the order ε .

c/ The real parts of the roots of equation /3.66/ are negative.

Thus, equations /2.24/ with the assumptions above and by introducing new variables according to /3.61/ or /3.64/ will immediately reduce to form /3.77/ or /3.49/ which is convenient for further consideration due to the fact that for $h=0$ the functions P and R are of the order ε . In this manner series of estimates can be obtained and an algorithm constructed for the determination of approximate solutions.

Consider now equation /2.24/ for somewhat different conditions which hold for the right-hand sides of these equations.

a/ Assume that for the system of unperturbed equations

$$\frac{dx}{dt}=X(x) \qquad /3.50/$$

a family of periodic solutions is known

$$x=x^{\circ}(\omega t+\varphi,\alpha)\ \ (\omega t+\varphi=\psi) \qquad /3.78/$$

of period 2π in ψ , depending on two arbitrary constants α and φ, ω being a function of α in the general case

$$\omega=\omega(\alpha) \qquad /3.79/$$

b/ For the system of variational equations

$$\frac{d\delta x}{dt}=X'_x(x^{\circ})\delta x \qquad /3.80/$$

corresponding to the periodic solution /3.78/, $n-2$ characteristic exponents

$$\alpha_2,\ \alpha_3,\ldots,\alpha_{n-1} \qquad /3.81/$$

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have negative real parts, while the two characteristic exponents γ_{n-1}, γ_n are equal to zero by virtue of the existence of a two parametric family of periodic solutions /3.78/.

$$x = x^0(\omega t + \varphi, a) \qquad /3.78/$$

$$\Delta(a) \neq 0 \quad \text{for} \quad a \in \overline{\alpha} \quad / \overline{\alpha} \text{ being}$$

the interval of variation $a\ (a_1, a_2)$ on which a does not become zero/, where

$$\Delta(a) = \left| \frac{\partial x^0}{\partial \varphi}, \frac{\partial x^0}{\partial a}, \frac{1}{2}(A + A) \right|_{t=0} \qquad /3.82/$$

in which $A(\varphi, a)$ is a matrix of n rows and $n-2$ columns, whose elements depend on a and are periodic in φ of period 2π.

1/ There exists a certain neighbourhood $U_\rho^{\varepsilon_0}$ of the curve $x = x^0(\omega t + \varphi, a)$, $\varepsilon = 0$ in an $n+1$ -dimensional space for which the functions

$$X(x) + \varepsilon X^*(\vartheta t, x, \varepsilon)$$

and their partial derivatives of any order with respect to x and ε are bounded and uniformly continuous for

$$-\infty < t < \infty.$$

2/ The functions $X^*(\vartheta t, x, \varepsilon)$ are periodic in t of period $\frac{2\pi}{\vartheta}$ uniformly with respect to x and ε in the region $U_\rho^{\varepsilon_0}$.

The above conditions are equivalent to saying that in an unperturbed system described by equations /3.50/ there exists a family of periodic motions, which depends on two arbitrary constants /"amplitude" a and phase φ / and is representable analytically in the form of a fymily of closed curves /3.78/ in an n -dimensional Eucledian space. Equations /2.24/ describe a motion due to the perturbation of the above family of stable periodic motions, brought about by small perturbing forces $\varepsilon X^*(\vartheta t, x, \varepsilon)$ periodic in t of period 2π.

We now proceed to the transformation of equations /2.24/ under the assumptions made above, and assume first /4/ that in addition to the negativity condition of the real parts of the characteristic /3.81/, there also holds the condition of linear independence of the real parts of these exponents γ_φ, i.e.

$$m_2\gamma_2 + m_3\gamma_3 + \cdots + m_{n-1}\gamma_{n-1} \neq \gamma_\varphi,$$

$$m_2 + m_3 + \cdots + m_{n-1} \geq 2, \qquad /3.83/$$

where $m_2, m_3, \ldots, m_{n-1}$ are non-negative integers, and the vector functions $X(x)$ are analytic in x.

Now, as in the previous case, let us introduce into equations /2.24/ new variables $\psi, a, h\ (h_2, h_3, \ldots, h_{n-1})$ for x, according to the equality

$$x = f(\psi, a, h), \qquad /3.84/$$

where $f(\omega(a)t + \varphi, a, C e^{\alpha t})$ is an n -dimensional vector function periodic in t of period $\frac{2\pi}{\omega(a)}$ representing the general solution of equation /3.50/ in the neighbourhood of the periodic solution /3.78/; $C(C_2, C_3, \ldots, C_{n-1}), \psi, a$ are integration constants $\alpha(\alpha_2, \alpha_3, \ldots, \alpha_{n-1})$ and $f(\psi, a, 0) = x^0(\psi, a)$.

Then, performing the necessary calculations, the following system of differential equations in the new variables $\psi, a, h\ (h_2, h_3, \ldots, h_{n-1})$ is obtained:

$$\frac{d\psi}{dt} = \omega(a) + \varepsilon \bar{P}(\vartheta t, \psi, a, h, \varepsilon),$$

$$\frac{da}{dt} = \varepsilon Q(\vartheta t, \psi, a, h, \varepsilon),$$

$$\frac{dh}{dt} = \alpha h + \varepsilon R(\vartheta t, \psi, a, h, \varepsilon) \qquad /3.85/$$

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in which the functions $P(\vartheta t, \psi, a, h, \varepsilon)$, $Q(\vartheta t, \psi, a, h, \varepsilon)$ and the $n-2$ -dimensional vector function $R(\vartheta t, \psi, a, h, \varepsilon)$ possess in region

$$-\infty < t < \infty, \quad \psi \in \Omega, \quad 0 < \varepsilon < \varepsilon_0, \quad h \in U_\delta \qquad /3.86/$$

the same properties as those possessed by the vector function $X^*(\vartheta t, x, \varepsilon)$.

In the general case, when $n-2$ characteristic exponents /3.81/ have negative real parts, but condition /3.83/ does not hold, we introduce into system /2.24/ new variables ψ, a, $h(h_2, h_3, \ldots, h_{n-1})$ for x by means of the transformation [5]

$$x = x^*(\psi, a) + \frac{1}{2}\{A(\psi, a)h + \bar{A}(\psi, a)\bar{h}\}, \qquad /3.87/$$

where $A(\psi, a)$ is a matrix of n rows and $n-2$ columns, whose elements are periodic functions of ψ of period 2π having bounded and uniformly continuous derivatives of any order with respect to ψ, a.

Transforming as above /s.p.32/, we obtain instead of system /2.24/ an equivalent system in the new variables ψ, a, h:

$$\frac{d\psi}{dt} = \omega(a) + P(\vartheta t, \psi, a, h, \varepsilon),$$
$$\frac{da}{dt} = Q(\vartheta t, \psi, a, h, \varepsilon),$$
$$\frac{dh}{dt} = Hh + R(\vartheta t, \psi, a, h, \varepsilon), \qquad /3.88/$$

where H is an $n-2$ -dimensional matrix for which the roots of the equation

$$Det|J_{n-2}\rho - H| = 0 \qquad /3.89/$$

/ J_{n-2} being an $n-2$ -dimensional unitary matrix/ are the characteristic exponents of system /3.80/.

Simultaneously, positive numbers ε_0, δ can be pointed out, such that the functions $P(\vartheta t, \psi, a, h, \varepsilon)$, $Q(\vartheta t, \psi, a, h, \varepsilon)$ and the vector function $R(\vartheta t, \psi, a, h, \varepsilon)$ have the following properties:

1/ $P(\vartheta t, \psi, a, h, \varepsilon), Q(\vartheta t, \psi, a, h, \varepsilon), R(\vartheta t, \psi, a, h, \varepsilon)$ defined in the region

$$-\infty < t < \infty, \ \psi \in \Omega, \ a \in \mathcal{A}, \ h \in U_\delta, \ 0 < \varepsilon < \varepsilon_0 \qquad /3.90/$$

are periodic functions of ϑt and ψ of period 2π and possess bounded and uniformly continuous derivatives of any order with respect to ψ, a, h, ε.

2/ For $h = 0$ the functions $P(\vartheta t, \psi, a, h, \varepsilon), Q(\vartheta t, \psi, h, a, \varepsilon)$, $R(\vartheta t, \psi, h, a, \varepsilon)$ are magnitudes of order ε.

3/ Since the functions $P(\vartheta t, \psi, a, h, \varepsilon), Q(\vartheta t, \psi, h, a, \varepsilon)$, $R(\vartheta t, \psi, h, a, \varepsilon)$ have bounded and uniformly continuous derivatives of any order with respect to ψ, a, h, then for any positive $\sigma < \delta$ in the region

$$-\infty < t < \infty, \ \psi \in \Omega, \ a \in \mathcal{A}, \ h \in U_\sigma, \ 0 < \varepsilon < \varepsilon_0 \qquad /3.91/$$

they will satisfy Lipschitz's inequalities with the Lipschitz constant $\lambda = \lambda(\varepsilon, \sigma)$ for which the following condition holds:

$$\lambda(\varepsilon, \sigma) \to 0 \quad as \quad \sigma \to 0, \ \sigma \to 0. \qquad /3.92/$$

Thus, in this case, too, system of equations /2.24/ reduces to system /3.85/ or /3.88/ belonging to systems of type /3.26/ for which different estimates and properties are conveniently obtained.

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Consider now equation /2.24/ under somewhat different assumptions concerning the right-hand sides [f].

Let the system of unperturbed equations

$$\frac{dx}{dt} = X(x) \qquad /3.50/$$

have an isolated static solution corresponding to the equilibrium point

$$x = 0, \quad X(0) = 0 \quad [X'_x(0) \neq 0] \qquad /3.93/$$

while the vector functions $X(x) + \varepsilon X^*(\vartheta t, x, \varepsilon)$ and their partial derivatives of any order with respect to x, ε in the region

$$-\infty < t < \infty, \quad 0 < \varepsilon < \varepsilon_0, \quad x \in U_\rho \qquad /3.94/$$

where U_ρ is a ρ-neighbourhood of point $x=0$ are bounded and uniformly continuous with respect to x, ε and periodic in t of period $\frac{2\pi}{\nu}$ uniformly with respect to x, ε.

In addition, assume that for the variational equations corresponding to the static solution /3.93/,

$$\frac{d\delta x}{dt} = X'_x(0)\,\delta x \qquad /3.95/$$

the characteristic equation

$$|J_n \rho - A| = 0 \quad [A = X'_x(0)] \qquad /3.96/$$

/ J_n being an n-dimensional unitary square matrix, A an n-dimensional square matrix/ has two purely imaginary roots

$$P_1 = i\omega, \quad P_2 = -i\omega$$

while the rest of the $n-2$ roots have negative real parts.

A problem thus formulated arises in investigating the effect of small perturbing forces $\varepsilon X^*(\vartheta t, x, \varepsilon)$ on the equilibrium position, whose stability cannot be determined on the basis of the variational equations /3.95/ due to the presence of the so-called critical case /two purely imaginary roots/.

For the further study of the behaviour of the solutions of equations /2.24/ in the neighbourhood of the static solution $x=0$, it is well transform system /2.24/ by introducing new variables; however, in order to avoid singularities near point $x=0$, it would be more convenient to employ rectangular coordinates instead of cylindrical.

First note that under the assumptions made above the general solution of system /3.95/ has the form

$$\delta x = C_1 A e^{i\omega t} + C_2 B e^{-i\omega t} + Dh, \qquad /3.97/$$

where C_1, C_2 are arbitrary constants /complex conjugate/, A and K - constant n-dimensional vectors, D - a constant matrix consisting of n rows and $n-2$ columns, h - an $n-2$-dimensional vector, which is a linear combination of the products of polynomials and the damped exponents and contains $n-2$ arbitrary constants.

System /2.24/ can be represented in the form:

$$\frac{dx}{dt} = X'_x(0)\,x + \left\{\bar{X}(x) + \varepsilon X^*(\vartheta t, x, \varepsilon)\right\}, \qquad /3.98/$$

where $\bar{X}(x)$ begins with terms of the second order with respect to x and, consequently, there exist such C and δ_0 $(\delta_0 < \rho)$ that for $|x| < \delta_0$ the following inequality will take

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place:

$$|\overline{X}(x)| \leq Cx^2. \qquad /3.99/$$

Let us now introduce new variables into system /3.98/ according to

$$x = A\xi + B\xi^* + Dh, \qquad /3.100/$$

where $\xi, \xi^*, h\ (h_2, h_3, \ldots, h_{n-1})$ are the new variables, A, B and D -the vectors and matrix given above.

Note that there is always a δ_1 $\left(\delta_1 < \frac{\delta_0}{|A|+|B|+|D|}\right)$ so small that for every $|\xi| < \delta_1, |\xi^*| < \delta_1, |h| < \delta_1$ we have $|x| < \delta_0$.

As a result of the change to new variables, the following system of equations is obtained instead of /3.98/:

$$A\frac{d\xi}{dt} + B\frac{d\xi^*}{dt} + D\frac{dh}{dt} = X'_x(0)(A\xi + B\xi^* + Dh) + \{X(A\xi + B\xi^* + Dh) + \varepsilon X^*(\nu t, A\xi + B\xi^* + Dh, \varepsilon)\} \qquad /3.101/$$

On the other hand, by virtue of /3.95/, /3.97/ and /3.100/ we have:

$$i\omega A\xi - i\omega B\xi^* + DHh = X'_x(0)(A\xi + B\xi^* + Dh), \qquad /3.102/$$

where H is an $n-2$ -dimensional matrix for which the equation

$$|J_{n-2}\rho - H| = 0 \qquad /3.103/$$

has all the roots with negative real parts.

Now, subtracting /3.102/ from /3.101/,

$$A\left(\frac{d\xi}{dt} - i\omega\xi\right) + B\left(\frac{d\xi^*}{dt} + i\omega\xi^*\right) + D\left(\frac{dh}{dt} - Hh\right) = \{X(A\xi + B\xi^* + Dh) + \varepsilon X^*(\nu t, A\xi + B\xi^* + Dh, \varepsilon)\}. \qquad /3.104/$$

It is obvious that the determinant of system /3.104/ is different from zero and, consequently, system /3.104/ is solvable.

Then, solving the system in the region U_{δ_1} for the unknowns

$$\frac{d\xi}{dt} - i\omega\xi, \quad \frac{d\xi^*}{dt} + i\omega\xi^*, \quad \frac{dh}{dt} - Hh, \qquad /3.105/$$

the following system of differential equations is obtained:

$$\frac{d\xi}{dt} = i\omega\xi + P(\nu t, \xi, \xi^*, h, \varepsilon),$$

$$\frac{d\xi^*}{dt} = -i\omega\xi^* + Q(\nu t, \xi, \xi^*, h, \varepsilon),$$

$$\frac{dh}{dt} = Hh + R(\nu t, \xi, \xi^*, h, \varepsilon), \qquad /3.106/$$

where the functions $P(\nu t, \xi, \xi^*, h, \varepsilon), Q(\nu t, \xi, \xi^*, h, \varepsilon)$ and the vector function $R(\nu t, \xi, \xi^*, h, \varepsilon)$ are defined in the region

$$-\infty < t < \infty, \ |\xi| < \delta_1, \ |\xi^*| < \delta_1, \ h \in U_{\delta_1}, \ 0 < \varepsilon < \varepsilon_0, \qquad /3.107/$$

they have bounded and uniformly continuous derivatives of

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any order with respect to $\xi, \xi^*, h, \varepsilon$ and are periodic in t of period $\frac{2\pi}{\rho}$.

For $h=0$ the above functions satisfy the inequalities

$$|P(t,\xi,\xi^*,0,\varepsilon)| \le M(\varepsilon,\rho^2), \quad |Q(t,\xi,\xi^*,0,\varepsilon)| \le M(\varepsilon,\rho^2),$$

$$|R(t,\xi,\xi^*,0,\varepsilon)| \le M(\varepsilon,\rho^2), \qquad /3.108/$$

where $\rho < \delta_1$, simultaneously $M(\varepsilon,\rho^2) \to 0$ as $\varepsilon \to 0$, $\rho^2 \to 0$. Moreover, in region /3.106/ these functions satisfy Lipschitz's conditions with respect to the variables ξ, ξ^* and h with Lipschitz constant $\lambda = \lambda(\varepsilon,\rho^2)$, for which, as well as for $M(\varepsilon,\rho^2)$, the following condition is fulfilled:

$$\lambda(\varepsilon,\rho^2) \to 0 \quad \text{as} \quad \varepsilon \to 0, \ \rho^2 \to 0.$$

Later, it will be shown that in considering system /3.106/ difficulties are apt to arise due to the fact that the behaviour of solutions can be studied only in the vicinity of the origin for small values of ρ /for which ρ^2 and ε are of the same order of smallness/, all such studies to be carried out locally near the origin.

However, these difficulties do not arise in the special case which is often encountered in investigating oscillatory systems and stability problems, when

$$X(x) = Px, \qquad /3.109/$$

where P is a constant n-dimensional square matrix. As a result, we arrive at a system of equations /6/

$$\frac{dx}{dt} = Px + \varepsilon X^*(\nu t, x, \varepsilon), \qquad /3.110/$$

in which the functions $X^*(\nu t, x, \varepsilon)$ are periodic in t of period $\frac{2\pi}{\nu}$ and possess bounded and uniformly continuous derivatives with respect to x, ε for any finite $0 < \varepsilon < \varepsilon_0$.

Assume that the system of unperturbed equations

$$\frac{dx}{dt} = Px \qquad /3.111/$$

has an isolated static solution

$$x = 0,$$

and, also, that the characteristic equation

$$Det\|J_{ij} - P'_{ij}\| = 0, \qquad /3.112/$$

of the variational system of equations corresponding to the solution $x=0$ has two purely imaginary roots $p_1 = i\omega, p_2 = -i\omega$, while the rest of the roots have negative real parts.

Under these assumptions and by using elementary transformations of the variables, system /3.110/ reduces to the form

$$\frac{d\xi}{dt} = i\omega\xi + \varepsilon P(\nu t, \xi, \xi^*, h, \varepsilon),$$
$$\frac{d\xi^*}{dt} = -i\omega\xi^* + \varepsilon Q(\nu t, \xi, \xi^*, h, \varepsilon),$$
$$\frac{dh}{dt} = Hh + \varepsilon R(\nu t, \xi, \xi^*, h, \varepsilon)$$

in which the functions $P(\nu t, \xi, \xi^*, h, \varepsilon)$, $Q(\nu t, \xi, \xi^*, h, \varepsilon)$, $R(\nu t, \xi, \xi^*, h, \varepsilon)$ have the abovementioned /p.44/ properties. However, now for any values of ξ, ξ^*, h and, consequently, in the inequalities /3.108/ and in Lipschitz's conditions the constants μ and λ will depend only on ε and will not depend on ρ/extent of region/.

We shall now briefly consider the transformation of equations /2.25/-/2.27/ to a form convenient for investigation. Under certain assumptions concerning their right-hand sides and and the solution of the unperturbed system /$\varepsilon = 0$/, these systems can be written as

$$\frac{d\psi}{dt} = \omega(c,a) + P(c,\theta,\psi,a,h,\varepsilon), \quad \frac{da}{dt} = Q(c,\theta,\psi,a,h,\varepsilon),$$
$$\frac{dh}{dt} = H(c)h + R(c,\theta,\psi,a,h,\varepsilon) \qquad /3.113/$$

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where the right parts possess the number of properties which depend upon the properties of the right parts of the equations /2.25/ - /2.27/ or to more general shape:

$$\frac{d\psi}{dt} = \omega(\tau, a) + P(\tau, \theta, \psi, a, h, \varepsilon),$$
$$\frac{da}{dt} = Q(\tau, \theta, \psi, a, h, \varepsilon),$$
$$\frac{dh}{dt} = H(\tau, \psi, a)h + R(\tau, \theta, \psi, a, h, \varepsilon) \qquad /3.II5/$$

where $\frac{d\psi}{dt} = \omega(\tau)$, $H(\tau, \psi, a)$ is a function periodic in ψ of period 2π.

In conclusion, let us not that we might not have confined the above transformations to the condition of the existence of a one-parameteric or two-parametric solution of an unperturbed system. These same transformations could have been carried out for a much more general case. Thus, for instance, one might assume that for the system of differential equations /this case will be treated later in more detail in connection with the consideration of some results due to J.K.Hale/

$$\frac{dx}{dt} = X(x) + X^*(t, x), \qquad /3.II6/$$

in which x, X, X^* are n-dimensional vectors, the autonomous system:

$$\frac{dx}{dt} = X(x), \qquad /3.II7/$$

obtained by dropping in system /3.II6/ the terms of higher order in x, has a k-parametric /$k < n$/ family of periodic solutions:

$$x = x^0[\omega(a)t + \varphi, a],$$
$$x^0(\psi + \bar{\alpha}, a) = x^0(\psi, a), \qquad /3.II8/$$

where $a(a_1, a_2, \ldots, a_{k-1})$, φ are constants, $\omega(a) > 0$.

Then, it might be proved that for sufficiently general conditions imposed on the right-hand sides of equations /3.II6/ there exists a matrix $A(\psi, a)$, such that by introducing the variables

$$x = x^0(\psi, a) + A(\psi, a)h$$
$$(h = h_1, h_2, \ldots, h_{n-k+1}) \qquad /3.II9/$$

system /3.II6/ will be transformed into an equivalent system:

$$\frac{d\psi}{dt} = \omega(a) + P_1(\psi, a, h) + P_2(\psi, t, a, h),$$
$$\frac{da}{dt} = Q_1(\psi, a, h) + Q_2(\psi, t, a, h),$$
$$\frac{dh}{dt} = H(a)h + R_1(\psi, a, h) + R_2(\psi, t, a, h). \qquad /3.I20/$$

The functions in the right-hand sides of these equations possess properties which are similar to those given above and enable us to obtain a number of estimates necessary in proving the existence and bringing out the properties of integral manifolds.

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§ 4. Some Fundamental Theorems of the Existence and Properties of Integral Manifolds for the Types of Differential Equations Already Considered

In this paragraph we consider some properies of the solutions of equations of type /3.26/, then we state a number of typical theorems of the existence and the properties of integral manifolds of the systems of differential equations /2.11/, /2.22/, /2.24/-/2.27/, and give a brief outline of the basic steps of proof.

The properties of the solutions of equations of type /3.26/ are largely determined by the lemmas stated below, it being assumed that everywhere in what follows the functions $P(t,g,h,\varepsilon)$, $R(t,g,h,\varepsilon)$, $G(\varepsilon)$ fulfil the conditions pointed out in the previous paragraph /see p.27/ with the functions $P(t,g,h,\varepsilon)$ and $G(\varepsilon)$ always real for any complex h in the region under consideration. Detailed proof of these lemmas is to be found in [2].

Lemma I. - A positive $\bar{\varepsilon}$ $(\bar{\varepsilon} \leq \varepsilon_0)$ can be given such that for every positive $\varepsilon \leq \bar{\varepsilon}$ the system of equations

$$\frac{dg}{dt} = G(\varepsilon) + P(t,g,h,\varepsilon),$$
$$\frac{dh}{dt} = Hh + R(t,g,h,\varepsilon) \qquad /3.26/$$

has an integral manifold representable by a relationship of the form

$$h = f(t,g,\varepsilon) \qquad /4.1/$$

in which $f(t,g,\varepsilon)$ as a function of t, g is defined for $R\Omega$ and fulfils the inequalities

$$|f(t,g,\varepsilon)| \leq D(\varepsilon) \leq \rho, \qquad /4.2/$$

$$|f(t,g',\varepsilon) - f(t,g'',\varepsilon)| \leq \Delta(\varepsilon)|g' - g''|, \qquad /4.3/$$

with $D(\varepsilon) \to 0$, $\Delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

With reference to the angular variable g the function $f(t,g,\varepsilon)$ has a period 2π.

If the functions $P(t,g,h,\varepsilon), R(t,g,h,\varepsilon)$ have in region $t \in R^*$, $g \in \Omega$, $h \in U_\rho$, $0 \leq \varepsilon \leq \varepsilon_0$ bounded and uniformly continuous derivatives with respect to g and h up to m-th order, then the function $f(t,g,\varepsilon)$ will also have bounded and uniformly continuous derivatives with respect to g up to the m-th order.

From this lemma immediately results the following lemma: For the solutions of system /3.26/, which belong to the integral manifold /4.1/

$$h = f(t,g,\varepsilon)$$

the angular variable g satisfies the equation

$$\frac{dg}{dt} = G(\varepsilon) + F(t,g,\varepsilon), \qquad /4.4/$$

where

$$F(t,g,\varepsilon) = P(t,g,f(t,g,\varepsilon),\varepsilon). \qquad /4.5/$$

Simultaneously, on the basis of lemma I it is readily shown that the function $F(t,g,\varepsilon)$ is defined for all real t, g it is a periodic function in g of period 2π and fulfils the following inequalities:

$$|F(t,g,\varepsilon)| \leq M(\varepsilon) \underset{\varepsilon \to 0}{\to} 0, \qquad /4.6/$$

Moreover, $F(t,g,\varepsilon)$ will have continuous derivatives with respect to g up to the m-th order, if $P(t,g,h,\varepsilon)$ possesses continuous derivatives with respect to g and h up to the m-th order

$$|F(t,g',\varepsilon) - F(t,g'',\varepsilon)| \leq N(\varepsilon) \underset{\varepsilon \to 0}{\to} 0. \qquad /4.7/$$

---

* By R we denote the $(-\infty, \infty)$ interval.

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Moreover, $F(t, g, \varepsilon)$ will have continuous derivatives with respect to g up to the m-th order, if $P(t,g,h,\varepsilon)$ possesses continuous derivatives with respect to g and h up to the m-th order.

Lemma II. If there exists a sequence of real numbers $\{\tau_m\}$ such that for some positive $\varepsilon < \bar{\varepsilon}$ the right-hand sides of equations /3.26/ will satisfy uniformly on $R\,\Omega\,U_\rho$ the relationships

$$|P(t+\tau_m, g, h, \varepsilon) - P(t, g, h, \varepsilon)| \underset{m\to\infty}{\to} 0,$$

$$|R(t+\tau_m, g, h, \varepsilon) - R(t, g, h, \varepsilon)| \underset{m\to\infty}{\to} 0,$$

/1.8/

then we have for this ε uniformly on $R\,\Omega$:

$$|f(t+\tau_m, g, \varepsilon) - f(t, g, \varepsilon)| \to 0 \quad \text{as } m\to\infty, \quad /4.9/$$

where $f(t, g, \varepsilon)$ is an integral manifold for the system of differential equations /3.26/.

As a corollary of this lemma we have the following important result:

I/ If the functions $P(t,g,h,\varepsilon)$, $R(t,g,h,\varepsilon)$ are almost periodic functions* in t uniformly with respect to g, h.

---

* A function $f(t, x)$ is said to be an almost periodic function in t uniformly with respect to x, if for every $\eta > 0$ a corresponding positive $\ell(\eta)$ can be found such that in any interval on R of length $\ell(\eta)$ there lies at least one τ /an almost period for η/ for which everywhere on $R\,E_n$ the following inequality holds:

$$|f(t+\tau, x) - f(t, x)| \leq \eta.$$

For these almost periodic functions there exists a denumerable set frequencies $\{\lambda_i\}$ independent of x such that for every λ not belonging to this set the following

/turn to next page/

in the region ΩU_ρ, then $F(t, g, \varepsilon)$ is also an almost periodic function uniformly with respect to $g \in \Omega$ with frequency basis of the functions $P(t,g,h,\varepsilon), R(t,g,h,\varepsilon)$;

2/ If $f(t, x)$ is a function almost periodic uniformly with respect to x, and the set $\{\omega_a\}$ is its frequency basis, then $F(t, x)$ will also be a function almost periodic uniformly with respect to x with the same frequency basis.

Lemma III. Positive constants $\varepsilon', \gamma, C, \sigma_0, \sigma_1$ /with $\sigma_0 < \sigma_1 < \rho$, $\varepsilon' < \bar{\varepsilon}$ / can be pointed out, such that if S characteristic numbers of matrix H have negative real parts, while the other $n - S$ numbers are positive, then for every $\varepsilon < \varepsilon'$ of any real t_0 and every g_0 from Ω in some neighbourhood U_{σ_0} there exists an S-dimensional manifold $M(t_0, g_0, \varepsilon)$ of the points $\{h\}$ having the following properties:

I/ If for $t = t_0$

$$h_t \in U_{\sigma_0}, \qquad h_t \in M(t_0, g_0, \varepsilon)$$

then for some $\bar{t} > t_0$

$$h_{\bar{t}} \in U_{\sigma_1};$$

relationship holds:

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} f(t, x) e^{-i\lambda t}\, dt = 0.$$

Let $\{\omega_a\}$ represent a denumerable set of real numbers having the following properties:

a/ among the ω_a there exist no non-trivial linear relationships

with integral coefficients:

b/ Every λ_i can be represented by a linear combination of ω_a with integral coefficients.

We shall call such a set $\{\omega_a\}$ the frequency basis of a given almost periodic function.

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2/ If for $t = t_o$

$$h_t \in \mathcal{M}(t_o, g_o, \varepsilon),$$

then for all $t > t_o$

$$|h_t - f(\ldots, g_t, \varepsilon)| \leq C e^{-\delta|t-t_o|} |h_o - f(t_o, g_o, \varepsilon)|,$$

with $(\ldots, g_t)$ representing (g_t, h_t) for $t = t_o$ and $\ldots$ a positive constant.

3/ If all the characteristic numbers of the matrix have positive real parts ($S = 0$), then the manifold $\mathcal{M}(t_o, g_o, \varepsilon)$ degenerates into the point $h = f(t_o, g_o, \varepsilon)$.

4/ On the contrary, if the real parts of all the characteristic numbers of matrix H are negative, then the manifold $\mathcal{M}(t_o, g_o, \varepsilon)$ coincides with the whole neighbourhood U_{σ_o}.

Thus, according to this important lemma, if the initial values of some solution of system /3.26/ do not belong to the point S-dimensional manifold $\mathcal{M}(t_o, g_o, \varepsilon)$ lying in an $n-1$-dimensional neighbourhood U_{σ_o}, then in course of time this solution will withdraw from $\mathcal{M}(t_o, g_o, \varepsilon)$ and, in particular, will pass out of some region U_{σ_1}, where $\sigma_1 > \sigma$. And, conversely, the integral curves whose initial values belong to $\mathcal{M}(t_o, g_o, \varepsilon)$, will in course of time tend to an S-dimensional manifold of integral curves which have parametrically represented as $f(t, g_t, \varepsilon)$.

From this lemma follows the important lemma, viz., if the real part of at least one of the roots of the equation

$$Det|\rho J_{n-1} - H| = 0 \qquad /4.10/$$

is positive, then the integral manifold $h = f(t, g, \varepsilon)$ under consideration possesses the property of repulsing all solutions close thereto, except those whose initial values lie on a particular point manifold $\mathcal{M}(t_o, g_o, \varepsilon)$, the dimensionality of the latter being less than that of the whole phase space.

Now if the real parts of all the roots of equation /4.10/ are negative, then, conversely, the given manifold possesses the property of attracting close solutions:

$$|h_t - f(t, g_t, \varepsilon)| \leq C |h_{t_o} - f(t_o, g_{t_o}, \varepsilon)| e^{-\delta|t-t_o|} \qquad /4.11/$$

and also

$$\left| \frac{dg_t}{dt} - G(\varepsilon) - F(t, g_t, \varepsilon) \right| \leq C \lambda(t, \varepsilon) e^{\delta|t-t_o|} |h_{t_o} - f(t_o, g_{t_o}, \varepsilon)| \qquad /4.12/$$

where $\lambda(\varepsilon, \delta) \to 0$ as $\varepsilon \to 0$, $\delta \to 0$.

By using these lemmas, it is no longer difficult to formulate and prove a number of theorems directly for the systems of equations as given in Section 2.

Thus, for instance, for equation /2.11/ the following theorems will hold /details of proof to be found in [2]:

Theorem I. - Let the functions $X(t, x)$ in the right-hand sides of equation

$$\frac{dx}{dt} = \varepsilon X(t, x), \qquad /2.11/$$

where x, X are n-dimensional vectors, fulfil the following conditions for x belonging to some open set $U_n \in E_n$:

a/ $X(t, x)$ is an almost periodic function in t uniformly with respect to x;

b/ $X(t, x)$ and its partial derivatives of the first order with respect to x are bounded and uniformly continuous for $-\infty < t < \infty$, $x \in U_n$.

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Also the equations of the first approximation

$$\frac{dx}{dt} = \varepsilon X_0(x), \qquad /4.13/$$

where

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t,x)\,dt \qquad /4.14/$$

has a constant solution x^0 and the real parts of the characteristic roots of the variational linear equation for x^0 are different from zero.

Under these assumptions, positive numbers ε_0, σ_0 can be found such that for every ε, $0 < \varepsilon < \varepsilon_0$ the following statements are valid: Equation /2.II/ has a unique solution $x = x^*(t)$ defined in the interval $(-\infty, \infty)$ for which $\|x^*(t) - x^0\| \le \sigma_0$; it is almost periodic with frequency basis of the function $X(t,x)$ the stability property of this solution is the same as that of the stable solution x^0.

The following theorem [2] is of greater interest as well as more general.

Theorem II. - Let equation /4.13/ have a periodical solution $x = f(\omega t)$, $f(s) = f(s + 2\pi)$ and the real parts of $n-1$ characteristic exponents of the variational equations are different from zero for the solution $f(\omega t)$ of equation /3.5/. Assume also that the functions $X(t,x)$ satisfy the condition a/ of theorem I in the ρ -neighbourhood of the periodic solution $f(\omega t)$ and have partial derivatives with respect to x up to the second order, which are bounded and uniformly continuous with respect to x for $-\infty < t < \infty$, $x \in U_\rho$.

Under these assumptions it is possible to find positive numbers $\varepsilon_0, \sigma_0, \sigma_0 < \rho$ such that for every ε, $0 < \varepsilon < \varepsilon_0$ the following statements will hold:

I/ Equation /3.26/ has a unique integral manifold S in U_{σ_0} for all t parametrically represented by

$$x = f(t, \Theta, \varepsilon), \qquad /4.15/$$

where f is defined for all real t and Θ, periodic in Θ of period 2π and almost periodic in t uniformly with respect to Θ with frequency basis of function $X(t,x)$ In addition the following inequality is satisfied:

$$\|f(t,\Theta,\varepsilon) - f(\Theta)\| \le \delta(\varepsilon), \qquad /4.16/$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, $f(t,\Theta,\varepsilon)$ has uniformly continuous derivatives with respect to Θ up to the second order.

2/ There exists a function $F(t,\Theta,\varepsilon)$ defined for all real t,Θ, periodic in Θ of period 2π and almost periodic in t uniformly with respect to Θ, having continuous derivatives up to the second order, such that equation /3.26/ on manifold S is equivalent to the equation

$$\frac{d\Theta}{dt} = \varepsilon F(t,\Theta,\varepsilon), \qquad /4.17/$$

3/ If the cylinder of periodic solutions $f(\omega t + \varphi)$ where φ is an arbitrary constant of equations /4.13/, is stable, unstable or conditionally stable with respect to an S -dimensional manifold, then the integral manifold S of equation /3.26/ is stable, unstable or conditionally stable with respect to the S -dimensional manifold.

While the definition of the stability concept of a manifold follows from lemma III, we proceed, however, as in [10]: let $\frac{dx}{dt} = \varepsilon X(t,x)$ be a system of differen-

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tial equations of n -th order and let S be a manifold of solutions of this equation of dimensionality m in an $n+1$-dimensional space (x, t). The manifold S will be referred to as stable if there exist neighbourhoods U_1, U_2 of dimensionality $n+1$ of the manifold S, $U_1 \subset U_2$ such that for every solution $x(t)$ of the above equation $x(t) \in U_2$ will follow from $x(t_0) \in U_1$ for all $t \geq t_0$ and $x(t) \to S$ as $t \to \infty$. If for every $x(t)$ from $x(t_0) \in U_1 - S$ we have $x(t) \not\to S$ as $t \to \infty$, then S will be referred to as unstable. If U_1 is of dimensionality $S < n+1$ and if for every $x(t)$ from $x(t_0) \in U_1$ there follows $x(t) \in U_2$ for $t \geq t_0$ and $x(t) \to S$ as $t \to \infty$, and if $x(t_0) \in U_2 - U_1$ with $x(t) \not\to S$ for $t \to \infty$, then S will be said to be conditionally stable with respect to an S-dimensional manifold.

Now let briefly characterize the main results which are obtained for systems of differential equations of more general form.

Assume that equations

$$\frac{dx}{dt} = X(x) + \varepsilon X^*(\nu t, x, \varepsilon) \qquad /2.24/$$

satisfy the conditions pointed out on pp 29-31 or the more general conditions of p. 32. Then the system in question, as shown above, will reduce to /3.77/ by a transformation of variables, fulfilling the conditions which are required for the validity of the lemmas stated at the beginning of this section. From these lemmas, generalizing theorem II [7] follows the proof of the following theorem:

equations /2.24/ the following conditions the following conditions are satisfied:

a/ The equation of "unperturbed" motion

$$\frac{dx}{dt} = X(x) \qquad /4.18/$$

has a periodic solution

$$x = x^\circ(\psi), \quad x^\circ(\psi + 2\pi) = x^\circ(\psi) \qquad /4.19/$$

depending on one arbitrary constant ψ $(\psi = \omega t + \varphi)$;

b/ The real parts of all $n-1$ characteristic exponents for the variational equation

$$\frac{d\delta x}{dt} = X'_x(x^\circ)\,\delta x, \qquad /4.20$$

depending to the periodic solution /4.19/ are different from zero;

c/ It is possible to find a ρ-neighbourhood D_ρ of orbits of the periodic solution /4.19/ and an ε_0, such that in region $-\infty < t < \infty$, $x \in D_\rho$, $0 < \varepsilon < \varepsilon_0$ the function

$$X(x) + \varepsilon X^*(\nu t, x, \varepsilon) \qquad /4.21$$

are unboundedly differentiable with respect to x and ε and possess bounded and uniformly continuous derivatives in the same region;

d/ The functions $X^*(\nu t, x, \varepsilon)$ are periodic in t of period 2π, or there exists for them an average for t.

Then positive numbers ε and σ $(\sigma < \rho, \varepsilon < \varepsilon_0)$ such that for every positive $\varepsilon < \varepsilon'$ the following statements hold

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1/ Equation /2.24/ has only one manifold lying in the region

$$-\infty < t < \infty, \quad x \in \mathcal{D}_{\sigma}, \quad 0 < \varepsilon < \varepsilon' \qquad /4.22/$$

and parametrically representable by

$$x = x^{0}(\psi) + \frac{1}{2}\left\{A(\psi)h(\vartheta t, \psi, \varepsilon) + \bar{A}(\psi)\bar{h}(\vartheta t, \psi, \varepsilon)\right\} = \Phi(\vartheta t, \psi, \varepsilon), \qquad /4.23/$$

where $\Phi(\vartheta t, \psi, \varepsilon)$ is defined in region $-\infty < t < \infty$, $\psi \in \Omega$, $0 < \varepsilon < \varepsilon'$ and has a period 2π in ψ.

2/ For the solutions lying on manifold /4.23/ system /2.24/ is equivalent to the equation

$$\frac{d\psi}{dt} = \omega + \varepsilon F(\vartheta t, \psi, \varepsilon) \qquad /4.24/$$

in which function $F(\vartheta t, \psi, \varepsilon)$ is defined in the region $-\infty < t < \infty$, $\psi \in \Omega$, $0 < \varepsilon < \varepsilon_0$, unboundedly differentiable with respect to ψ and periodic in ψ of period 2π.

3/ Manifold /4.23/ possesses the same properties of stability, instability, or conditional stability as the cylinder of periodic solutions /4.19/.

The investigation of integral manifolds appears to be more complicated for system /2.24/, if the equation of "unperturbed" motion /4.18/ has a static solution or a family of periodic solutions depending on more than one arbitrary constant /two, for instance/.

In such cases we meet with difficulties which prevent us from studying the properties of integral manifolds in the whole phase space and confine us to the consideration of only local manifolds.

In what follows two theorems are given /for proof turn to [5] and [6] / which generalize and develop theorem I and theorem II, respectively. Note that in proving these theorems the above lemmas can no longer be made use of. For this purpose, other lemmas are necessary generalizing lemmas I-III, and those interested are referred to the above references.

Thus, we first consider the theorem which characterizes the properties of the integral manifold for system /2.24/ in the case equation /4.18/ has an isolated static solution [6].

Theorem IV. - Assume that system /2.24/ satisfies the conditions:

a/ The system of "unperturbed" equations /4.18/ has an isolated static solution

$$x = 0, \quad X(0) = 0 \quad (X'_{x}(0) \neq 0); \qquad /4.25/$$

b/ The functions $X(x) + \varepsilon X(\vartheta t, x, \varepsilon)$ in the region

$$-\infty < t < \infty, \quad x \in \mathcal{D}_{\sigma_0}, \quad 0 < \varepsilon < \varepsilon_0, \qquad /4.26/$$

where $\mathcal{D}_{\sigma_0}$ is a σ_0-neighbourhood of the point $x = 0$ are periodic in ϑt of period 2π and possess bounded and uniformly continuous derivatives with respect to x, ε of any order.

c/ For the variational equations

$$\frac{d\delta x}{dt} = X'_{x}(0)\,\delta x \qquad /4.27/$$

corresponding to the static solution /4.25/, the characte-

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ristic equation

$$\left|\lambda_i \varepsilon - A\right| = 0, \qquad (A = X_x(\sigma)) \qquad /4.28/$$

two purely imaginary roots $\pm i\omega$, while the other roots

respective real parts.

[illegible] numbers $\delta_1, \ldots, \delta_{n-2}$ can be pointed out

[illegible] such that [illegible] the follo-

[illegible] with sets:

[illegible] /2.24/ has only one two-dimensional local* [illegible]

[illegible] parametrically representable by

$$\ldots = \varphi(\xi, \xi^*, \varepsilon) + D\varepsilon = F(t, \xi, \xi^*, \varepsilon) \qquad /4.29/$$

[illegible] are constant n-dimensional vectors, D – a con-

[illegible] of n rows and $n-2$ columns, and lying in the re-

[illegible]. The function $F(t, \xi, \xi^*, \varepsilon)$ is defined in the

[illegible] $\ldots \infty$, $\xi' < \rho_1$, $0 < \varepsilon < \varepsilon'$,

/4.30/

bounded and uniformly continuous derivatives with res-

pect to t, ξ, ξ^*, ε of any order, and possesses a period 2π in t.

2/ On the local integral manifold, equation /2.24/ is equivalent to the following system:

$$\frac{d\xi_t}{dt} = i\omega\xi_t + P_1(t, \xi_t, \xi_t^*, \varepsilon),$$
$$\frac{d\xi_t^*}{dt} = -i\omega\xi_t^* + Q(t, \xi_t, \xi_t^*, \varepsilon), \qquad /4.31/$$

where the right-hand sides satisfy the conditions on p.44.

3/ The local integral manifold $\mathfrak{M}_\varepsilon$ attacts /so long as $|\xi_t| < \rho_1$, $|\xi_t^*| < \rho_1$/ every solution of system /2.24/, whose initial values belong to the region U_{σ_1}.

The above theorem is of interest in connection with the study of the behaviour of integral curves in the vicinity of the origin, and a number of conclusions can be drawn therefrom with regard to the stability problem in the critical case for permanently acting perturbations, etc.

The limitation associated with locality will not arise, as it has been already noted in section 4, in the particular case, when

$$\ldots = \ldots \qquad /4.32/$$

where C is a constant n-dimensional square matrix. In this case a theorem is stated, analogous to that above, however in distinction to theorem IV which has a local character, the attractive property will occur for any finite values of $x(t)$.

Consider also a theorem concerning the properties of a manifold of system /2.24/ in the neighbourhood of a family of periodic solutions depending on two arbitrary constants [5].

Theorem V. – Assume that for system of equations /2.24/ conditions /a-e/, p. 36-37, Section 3, are fulfilled. Then positive numbers ε', σ' ($\sigma' < \sigma_0$, $\varepsilon' < \varepsilon_0$) can be given such that for any positive $\varepsilon < \varepsilon'$ the following statements will be true:

I/ Equation /2.24/ has only one local integral manifold $\mathfrak{M}_\varepsilon$ on region U_{σ_1} and parametrically representable by

---

* $\mathfrak{M}_\varepsilon$ will be said to be a local integral manifold, if from the relations $x(t_0, \xi_0, \xi_0^*, \varepsilon) \in \mathfrak{M}_\varepsilon$, $|\xi_0| < \rho_1$, $|\xi_0^*| < \rho_1$ it follows $x(t, \xi_0, \xi_0^*, \varepsilon) \in \mathfrak{M}_\varepsilon$ for any t so long as $|\xi| < \rho_1$, $|\xi^*| < \rho_1$.

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$$x = x^*(\psi, a) + \frac{i}{2}\left\{A(\psi, a) f(t, \psi, a, \varepsilon) + \bar{A}(\psi, a) \bar{f}(t, \psi, a, \varepsilon)\right\} = F(t, \psi, a, \varepsilon), \qquad /4.33/$$

where $F(t, \psi, a, \varepsilon)$ is defined in region

$$-\infty < t < \infty, \quad \psi \in \Omega, \quad a \in \mathcal{A}, \quad 0 < \varepsilon < \varepsilon' \qquad /4.34/$$

/region $\mathcal{A}$ being characterized by the locality of manifold $\mathcal{M}_\varepsilon$ has a period 2π in t and ψ and possesses bounded and uniformly continuous derivatives with respect to ψ, a, ε of any order.

2/ On the local integral manifold equations /2.24/ are equivalent to the system

$$\frac{d\psi}{dt} = \omega(a) + P_f(t, \psi, a, \varepsilon), \qquad \frac{da}{dt} = Q_f(t, \psi, a, \varepsilon), \qquad /4.35/$$

where the right-hand sides P_f and Q_f are defined in region /4.34/, they are periodic functions of t and ψ of period 2π and have bounded and uniformly continuous derivatives with respect to ψ, a, ε of any order.

3/ The local integral manifold $\mathcal{M}_\varepsilon$ is stable, i.e. it attracts any solutions of syster /2.24/, whose initial values belong to region U_{σ_0} /so long as $a \in \mathcal{A}$/.

The proof of the last two theorems, as already pointed out above, cannot be based directly on lemmas I-III, but requires a number of preliminary lemmas for systems of equations of types /3.88/ and /3.106/ to be stated and proved. In so doing, it is to be noted that the extension of the results stated in lemmas I-III is not a trivial problem, requiring certain essential difficulties to be overcome, which are associated with the probable unboundedness of some quantities under the integral sign, etc. To get over these difficulties, such a limitation, for instance, has to be introduced as consideration of only a local integral manifold.

We now briefly consider the general idea of the method of proving the above theorems of the existence and properties of integral manifolds.

One of the important steps of proof is the reduction of the equations to the form /3.26/, /3.88/ or /3.106/ convenient for further consideration and proof of auxiliary lemmas.

An inherent difficulty in this reduction /especially in reducing the equations to the standard form /2.11/ is the transformation of variables such that in the new equations the time t is contained in the terms of higher order of smallness. Thereafter, a family of solutions for the system in question is sought so as to belong to a specially selected class of functions. For this purpose, the first equation /or the first two equations/ is considered, in which h is replaced by an arbitrary function belonging to the given class, and then the solution of some functional equation is found by using the fixed point law. Then, for the manifold obtained the periodicity or almost-periodicity with respect to t and the property of stability are established from the fact that this manifold satisfies the equation.

At this point, it seems to be particularly useful and interesting to note that the integral manifold can be computed by successive approximations /since the transformation as obtained from the differential, system in the process of proving the existence and oneness of the integral

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manifold is a reverse one, transferring each term $C(D,\Delta)$ of the class of functions $F'(t,\theta,\varepsilon)$ into $C(D,\Delta)$.

A very important point in the method of proof of all the previous theorems, is the fact that by this method the existence of an integral manifold can be proven without making any special assumptions as to the dependence of the right-hand sides of the equations on time. The above allows the method to be applied in considering the properties of integral manifolds of systems of differential equations of type /2.26/.

In this connection, we first give some results underlying the investigation of equations of type /2.26/

Consider the system of differential equations

$$\frac{dg}{dt}=\omega(t)+P(t,g,h,\varepsilon),$$
$$\frac{dh}{dt}=H(t)h+Q(t,g,h,\varepsilon), \qquad /4.36/$$

where h is an $n-1$- dimensional vector, $H(t)$- a bounded square matrix of $n-1$ -th order, $\omega(t)$ - a bounded function for all real t.

Assume that $P(t,g,h,\varepsilon)$ and $Q(t,g,h,\varepsilon)$ are arbitrary bounded functions of t and satisfy the conditions similar to those given on p. 27. In addition, matrix $H(t)$ fulfils the condition

$$|H(t)|\leq N,$$
$$|H(t)-H(t')|\leq\delta|t-t'| \qquad /4.37/$$

the roots $P_i(t)$ $(i=1,2,\dots,n-1)$ of the corresponding characteristic equation

$$Det\|\rho I_{n-1}-H(t)\|=0 \qquad /4.38/$$

$$Re\{P_i(t)\}<-\gamma_i,\ \gamma_i>0, \qquad /4.39/$$

where N,δ,γ_i are properly chosen constants and, consequently, for matrix $U(t,\tau)$ satisfying the equation

$$\frac{dU(t,\tau)}{dt}=H(t)U(t,\tau),\ U(t,\tau)|_{t=\tau}=E \qquad /4.40/$$

the following estimate will hold:

$$|U(t,\tau)|\leq Ke^{-\gamma(t-\tau)} \qquad t\geq\tau, \qquad /4.41/$$

where K and $\gamma=\frac{\gamma_i}{2}$ are positive constants, E is a unitary $n-1$ -dimensional matrix.

With these assumptions, the following theorem holds /8/:

Theorem VI. - Positive constants ρ^*, ε^* can be pointed out such that for every positive $\varepsilon\leq\varepsilon^*$ system of equations /4.36/ has only one one-parametrical integral manifold explicitly representable by relations

$$h=f(t,g,\varepsilon) \qquad /4.42/$$

in which $f(t,g,\varepsilon)$ as functions of t and g are defined for $-\infty<t<\infty$, $-\infty<g<\infty$ and satisfy the inequalities

$$|f(t,g,\varepsilon)|<D(\varepsilon)<\rho^*,$$
$$|f(t,g',\varepsilon)-f(t,g'',\varepsilon)|\leq\Delta(\varepsilon)|g'-g''|, \qquad /4.43/$$

where $\Delta(\varepsilon)\to 0$, $D(\varepsilon)\to 0$ as $\varepsilon\to 0$.

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This integral manifold possesses the property of stability, which that every solution of system /4.36/ $h = h_t$ whose initial values belong to region U_ρ, is in the course of time attracted to the manifold /4.42/ following the relationship

$$|h_t - f(t, g, \varepsilon)| \leq K(\varepsilon, D) e^{-\gamma(t-t_0)}, \qquad /4.44/$$

where $K(\varepsilon, D)$ and γ are positive constants.

On the basis of this theorem systems of type /2.22/, /2.26/ or /2.27/ can be analyzed for the most diverse assumptions superposed on the right-hand sides of the equations. Thus, for instance, for system /2.26/ the following theorem [9] can be proved:

Theorem VII. - Let for system of equations

$$\frac{dx}{dt} = X(\tau, x) + \varepsilon X^*(\tau, \Theta, x, \varepsilon), \qquad /2.26/$$

where $\tau = \varepsilon t$ is the "slow" time, ε - a small positive parameter, $\frac{d\Theta}{dt} = \nu(\tau) > 0$, the following conditions are fulfilled:

a/ For the unperturbed equations

$$\frac{dx}{dt} = X(\tau, x), \qquad /4.45/$$

in which τ is assumed to be a constant parameter, a family of stable periodic solutions

$$x = x^0(\tau, \omega t + \varphi) \qquad /4.46/$$

of period 2π in ψ $(\psi = \omega t + \varphi)$ is known for any τ $(-\infty < \tau < \infty)$ depending on one arbitrary constant and, consequently, $n-1$ characteristic exponents $\alpha_1(\tau), \alpha_2(\tau), \ldots, \alpha_{n-1}(\tau)$ for the variational system of equations written down for equations /4.45/ and solution /4.46/ have negative real parts.

b/ There exist ρ and ε_0 such that in region

$$-\infty < t < \infty, \; x \in D_\rho, \; 0 < \varepsilon < \varepsilon_0, \qquad /4.48/$$

where D_ρ is a ρ-neighbourhood of orbit /4.46/ in an n-dimensional space E_n, the functions

$$X(\tau, x) + \varepsilon X^*(\tau, \Theta, x, \varepsilon) \qquad /4.49/$$

$/\tau = \varepsilon t, \frac{d\Theta}{dt} = \nu(t)/$ are periodic in Θ of period 2π, bounded and have a sufficient number of bounded derivatives with respect to x, t, ε.

Under these conditions positive constants ρ^*, ε^* $(\rho^* < \rho, \; \varepsilon^* < \varepsilon_0)$ can be pointed out such that any positive $\varepsilon < \varepsilon^*$ the system of equations /2.26/ has only one one-parametric manifold representable by relations of the form

$$x(\varepsilon t, \Theta, \Psi) = f(\varepsilon t, \Psi, h(\varepsilon t, \Theta, \Psi, \varepsilon)). \qquad /4.50/$$

For solutions lying on this integral manifold, system /2.26/ is equivalent to one differential equation of the form

$$\frac{d\Psi}{dt} = \omega(\tau) + \varepsilon P(\tau, \Theta, \Psi, \varepsilon), \qquad /4.51/$$

where $\tau = \varepsilon t$; the manifold possesses the property of stability, meaning that every solution of system /2.26/ whose initial values lie in region D_ρ^*, is in the course of time attracted to the manifold following the relationship

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$$|x(t,\Theta,\Psi)-x(t)|\leq Ce^{-\gamma(t-t_0)},$$

/4.52/

where $x^{(t)}$ denotes any solution of system /2.26/, ε and C,γ are positive constants.

It might be well to make some remarks before we proceed to review some generalizations of the above-described results recently obtained. For simplicity, it has been assumed everywhere in the above theorems that the variables φ, Ψ, Θ are scalars. This limitation is, however, in no way essential; we might have as well assumed, as mentioned in /3/, that these variables were vectors $\varphi(\varphi_1,\varphi_2,\dots,\varphi_s)$; $\Theta(\Theta_1,\Theta_2,\dots,\Theta_s)$ and correspondingly $\omega(\omega_1,\omega_2,\dots,\omega_s)$, $P(P_1,P_2,\dots,P_s)$ etc.

In the above theorems it has been assumed that all the characteristic components of matrix H had nonzero real parts. This condition has been made use of in establishing the correspondence in the stability properties of approximate and exact results.

However, the absence of the above condition would not introduce any very great difficulties, with the exception of algebraic operations only which become much more complicated. This important and very interesting question, as pointed out in /1/, can be studied in many cases by a transition to equations of higher approximations /instead of the equation of the first approximation obtained, for instance, by applying the method of averaging/.

In forming equations of higher, say, m-th approximation, we might in some cases resort to the transformation of variables

$$x=\xi+\varepsilon U_1(t,\xi)+\dots+\varepsilon^m U_m(t,\xi)$$

/4.53/

by means of which the equation

$$\frac{dx}{dt}=\varepsilon X(t,x)$$

/4.54/

can be reduced to the form:

$$\frac{d\xi}{dt}=\varepsilon P_1(\xi)+\dots+\varepsilon^m P_m(\xi)+\dots,$$

/4.55/

where

$$P_1(\xi)=X_0(\xi),\quad P_2(\xi)=M_0\left\{\left(\tilde{X}\frac{\partial}{\partial\xi}\right)X\right\},\dots.$$

In this case, for instance, in considering the quasi-statical solution of /4.55/

$$\xi=\xi_0^{(m)}$$

/4.56/

the matrix $H=X'_{0\xi}(\xi_0)=P'_{1\xi}(\xi_0)+\sum_{n=1}^{m}\varepsilon^n P'_{n\xi}(\xi_0^{(m)})$.

/4.57/

will play the roll of matrix $H=X'_{0\xi}(\xi_0)=P'_{1\xi}(\xi_0)$.

The characteristic values of matrix /4.57/, for sufficiently small ε, may be found to have nonzero real parts even in those cases when the real parts of all the characteristic values of matrix H are zero.

It should be also noted that above theorems can be greatly extended and proved for more general and less rigorous conditions superposed on the right-hand sides of the equations and the structure of solutions of the "unperturbed" equations.

Consider now some generalizations of a number of theorems given in this Section, as made recently by J.K.Hale /10/, /11/.

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y using basically the procedure of proof as described
nerally above, some theorems have been proved in /10/,
12/ for somewhat more general conditions.

We now some of these theorems, omitting the proofs.

Consider the system of equations

$$S' = d_1(\varepsilon) + S_1(t,\varphi,s,h,y,\varepsilon) + \varepsilon S_2(t,\varphi,s,h,y,\varepsilon),$$

$$\varphi' = d_2(\varepsilon) + \Phi_1(t,\varphi,s,h,y,\varepsilon) + \varepsilon \Phi_2(t,\varphi,s,h,y,\varepsilon),$$

$$h' = \varepsilon C(t,\varphi,s)h + \varepsilon H_1(t,\varphi,s,h,y,\varepsilon) + \varepsilon_2 H_2(t,\varphi,s,h,y,\varepsilon),$$

$$y' = A(t,\varphi,s)y + Y_1(t,\varphi,s,h,y,\varepsilon),$$

/1.58/

where ε is a real parameter $0 \le \varepsilon \le \varepsilon_0$; s, φ, h, y are vectors of respectively κ, l, m, n dimension $d_1(\varepsilon)$, $d_2(\varepsilon)$ are vectors: all the functions in the right-hand sides of system /1.58/ are periodic in S of period π; H, A, S_2, Φ_2, H_2 are continuous with respect to t, φ, s, h, y and have continuous partial derivatives of the first order with respect to t, φ, s, h, y in some region; the functions S_1, Φ_1 fulfil Lipschitz's condition with respect to the variables φ, s, h, y with Lipschitz's constant $\gamma(\varepsilon,\sigma,\rho)$, where $\gamma(\varepsilon,\sigma,\rho) \to 0$ as $\varepsilon, \sigma, \rho \to 0$ and $\gamma(\varepsilon,0,0) = 0(\varepsilon)$; the functions H_1, Y_1 fulfil Lipschitz's conditions with respect to φ, s, h, y with the constant $\lambda(\varepsilon,\sigma,\rho)$ where $\lambda(\varepsilon,\sigma,\rho) \to \sigma$ as $\varepsilon, \sigma, \rho \to 0$. Moreover, the average of functions S_2, Φ_2, H_2 with respect to t and φ with constant s, h, y, ε is equal to zero.

It is shown that under these conditions there exist functions u, v, w depending on $t, \varphi, s, h, y, \varepsilon$ periodic in S of period π such that system /1.58/ by introducing new variables according to the relations

$$s = \Theta + \varepsilon u(t,\varphi,\Theta,z,y,\varepsilon),$$
$$\varphi = \Psi + \varepsilon v(t,\Psi,\Theta,z,y,\varepsilon),$$
$$h = z + \varepsilon w(t,\Psi,\Theta,z,y,\varepsilon),$$
$$y = y$$

/1.59/

for sufficiently small ε, z, y, can be transformed to an equivalent system

$$\Theta' = d_1(\varepsilon) + H(t,\Psi,\Theta,z,y,\varepsilon),$$
$$\Psi' = d_2(\varepsilon) + \Psi(t,\Psi,\Theta,z,y,\varepsilon),$$
$$z' = \varepsilon C(t,\Psi,\Theta)z + \varepsilon Z(t,\Psi,\Theta,z,y,\varepsilon),$$
$$y' = A(t,\Psi,\Theta)y + Y(t,\Psi,\Theta,z,y,\varepsilon)$$

/1.60/

the functions H, Ψ, Z and Y possessing, respectively, the same properties as the functions S_1, Φ_1, H_1, Y.

Hence, conclusions will follow for system /1.58/, which are known from the general theory of asymptotical methods.

Denoting equations /1.58/ for $S_2 = \Phi_2 = H_2 = 0$ as "averaged equations", it is evident that by means of a small variation of the variables s and h according to relations /1.59/ the complete equation /1.58/ can be transformed into the equivalent system /1.60/ of the same type as the "averaged equations". Thus, it is quite obvious that if the qualitative behaviour of the solutions of the averaged equations is determined by the linear terms, the complete equation will be the same.

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For special forms of system /4.58/, theorems of type [10] are proved.

Theorem VIII. - Consider the system of equations

$$\theta' = \omega(\varepsilon) + \varepsilon B(t,\theta) + H(t,\theta,y,z,\varepsilon),$$
$$z' = \varepsilon Q(t,\theta) + \varepsilon C(t,\theta) z + \varepsilon Z(t,\theta,y,z,\varepsilon),$$
$$y' = Ay + Y(t,\theta,y,z,\varepsilon), \qquad /4.61/$$

where θ, y, z are vectors $\omega(\varepsilon)$ - vector; H, Y, Z fulfil the above conditions, A - constant matrix $Q(t,\theta)$, $B(t,\theta)$, $C(t,\theta)$ are continuous and have continuous first partial derivatives with respect to t and θ.

If the average values of the functions $Q(t,\theta), B(t,\theta)$ with respect to t and θ are equal to zero $M_{t,\theta}\{C(t,\theta)\} = C$, and the characteristic numbers of the matrices A, C have nonzero real parts, then there exists an $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ system /4.61/ has in the neighbourhood $y = 0$, $z = 0$ a unique integral manifold

$$y = f(t,\theta,\varepsilon), \quad z = g(t,\theta,\varepsilon) \qquad /4.62/$$

which possesses the definite property of stability.

In studying the question of the perturbation of limiting cycles, a system of vector equations

$$\frac{dx^i}{dt} = X^i(x^i) \qquad /4.63/$$

is considered in [11], where x^i, X^i are vectors of dimension n_i, $X^i \in C^2(U^{n_i})$ with U^{n_i} denoting open sets in E_{n_i}. It is assumed that for system /4.63/ a family of periodic solutions is known

$$x^i = x^{i0}(\omega_i t + \varphi),$$
$$x^{i0}(s + \pi) = x^{i0}(s) \qquad /4.64/$$

in which $i = 1,2,\dots,\rho$.

Simultaneously with system /4.63/, the following equations are considered:

$$\frac{dx^i}{dt} = X^i(x^1,\dots,x^\rho,t,\varepsilon) \quad (i = 1,2,\dots,\rho) \qquad /4.65/$$

in which ε is a small positive parameter, the functions $X^i(x^1,\dots,x^\rho,t,\varepsilon)$ with their first and second derivatives with respect to $x^1,\dots,x^\rho$ are bounded and uniformly continuous in region $-\infty < t < \infty$, $x^i \in U^{n_i}$, $i = 1,2,\dots,\rho$ and also

$$X^i(x^1,\dots,x^\rho,t,0) = X^i(x^i) \quad (i = 1,2,\dots,\rho). \qquad /4.66/$$

It is obvious that the functions /4.64/ represent a ρ-dimensional torus of solutions of system /4.65/ for $\varepsilon = 0$.

Under these assumptions and using the results obtained for a system of type /4.58/ /to which system /4.65/ can be evidently reduced by a change of variables/, the following theorem is proved [11]:

Theorem IX. - Assume that for each $i = 1,2,\dots,\rho$ the real parts of the $n_i - 1$ -th characteristic exponent of the variational system of equations

$$\frac{d\delta x^i}{dt} = X^{i\prime}_x(x^{i0})\delta x^i \qquad /4.67/$$

are different from zero. Then there exist positive constant ε_0, σ_0 such that for every $0 < \varepsilon < \varepsilon_0$ we have:

I/ Equation /4.65/ has only one integral manifold S lying for all real t in region $U^{n_1}_{\sigma_0} \times \cdots \times U^{n_\rho}_{\sigma_0}$, where $U^{n_i}_{\sigma_0}$ is a σ_0-neighbourhood of x^{i0}. This manifold is parametrically representable by

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$$x^i = f^i(t, \theta_1, \dots, \theta_\rho, \varepsilon) \quad (i = 1, 2, \dots, \rho), \qquad /4.68/$$

where the functions $f^i(t, \theta_1, \dots, \theta_\rho, \varepsilon)$ are defined for all real $t, \theta_1, \dots, \theta_\rho$, periodic in θ_i $(i = 1, 2, \dots, \rho)$ of period π and satisfy the inequalities:

$$\| f^i(t, \theta_1, \dots, \theta_\rho, \varepsilon) - x^{i_0}(\theta_i) \| \leq \delta(\varepsilon), \qquad /4.69/$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and have uniformly continuous derivatives with respect to $\theta_1, \dots, \theta_\rho$ up to the second order.

2/ On manifold S system /4.65/ is equivalent to the equations

$$\frac{d\theta_i}{dt} = \omega_i + \varepsilon F^i(t, \theta_1, \dots, \theta_\rho, \varepsilon), \qquad /4.70/$$

where the functions $F^i(t, \theta_1, \dots, \theta_\rho, \varepsilon)$ are defined for all real $t, \theta_1, \dots, \theta_\rho$ periodic in θ_i $(i = 1, 2, \dots, \rho)$ of period π and have bounded and uniformly continuous derivatives with respect to $\theta_1, \theta_2, \dots, \theta_\rho$ up to the second order.

3/ If the ρ-dimensional torus of solutions /4.64/ of system /4.63/ is stable, unstable or conditionally stable with reference to the manifold of dimension σ then the integral manifold S of /4.68/ is stable, unstable or conditionally stable with reference to the manifold of dimension σ.

In what follows we are going to deal with the application of the above theorems to problems of the theory of nonlinear oscillations, however we shall first consider the characteristic results obtained in studying the behaviour of concrete integral curves lying on the manifold.

§ 5. Examples of Investigation of Integral Curves Lying on a Manifold

Consider some special cases of importance, when the functions /4.15/ $f(t, \theta, \varepsilon)$ and the right-hand sides of equations /4.17/ $F(t, \theta, \varepsilon)$ are periodic functions of time t with some period T /independent of θ/. Such a "case of periodicity" will take place, for instance, when the functions $X(t, x)$ possess this same period T with respect to t.

For these cases of practical interest, based on A.Poincare's classical results on the behaviour of the integral curves of equation of type /4.17/ on the torus, supplemented by A. Danjoys [12], as well as using a number of N.N.Bogoliubov's lemmas [3], it is possible to analyze both the behaviour of the solutions of system /2.II/, lying on the manifold /in the case under consideration this manifold is represented by the torus/, and the behaviour of any solution of system /2.II/.

The results of such an analysis can be summed up in the form of the following theorem [2]:

Theorem X. - Assume that for the equation

$$\frac{dx}{dt} = \varepsilon X(t, x) \qquad /5.1/$$

the conditions of theorem II are fulfilled /p. 57/.

Moreover, assume that one of the following conditions is assumed:

a/ $X(t, x)$ is a periodic function in t of period T uniformly with respect to $x \in U_\rho$ or

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b/ in the equations

$$\frac{d\varphi}{dt} = \varepsilon\omega + \varepsilon W(t,\varphi,\theta),$$
$$\frac{d\theta}{dt} = \varepsilon\mu\theta + \varepsilon B(t,\varphi,\theta) \qquad /5.2/$$

the functions $W(t,\varphi,\theta), B(t,\varphi,\theta)$ have the form

$$W(t,\varphi,\theta) = \overline{W}(t,\varphi+\nu t,\theta),$$
$$B(t,\varphi,\theta) = \overline{B}(t,\varphi+\nu t,\theta), \qquad /5.3/$$

where $\overline{W}(t,\varphi,\theta), \overline{B}(t,\varphi,\theta)$ are periodic functions in t of period T uniformly with respect to φ,θ. In the case b/ $f(t,\Theta), F(t,\Theta)$ are also periodic functions in t of period T.

Then a sufficiently small positive ε' and a positive σ_0 $(\sigma_0<\rho)$ can be pointed out such that for any positive $\varepsilon<\varepsilon'$ equation /5.1/ has a integral manifold S_t lying in region U_σ for all real t this manifold satisfying the results of theorem II.

The behaviour of solutions lying on this manifold S_t i.e.

$$x(t) = f(t,\Theta(t),\varepsilon), \qquad /5.4/$$

where $\Theta(t)$ is determined from the equation

$$\frac{d\Theta}{dt} = \varepsilon\overline{F}(t,\Theta,\varepsilon) \qquad /5.5/$$

($f(t,\Theta,\varepsilon), F(t,\Theta,\varepsilon)$ being periodic functions in t of period T independent of Θ /, is characterized by the number $\bar{\nu}$ and can represented in the form:

$$x(t) = f(t,\Theta(t),\varepsilon) = \Phi(\alpha_e\tau, \alpha_p t+\psi), \quad \psi = const, \qquad /5.6/$$

where $\alpha_e = \frac{2\pi}{T}$ is the "external" frequency, $\alpha_p = \frac{2\pi}{T}\bar{\nu}$ - the natural frequency, $\Phi(\varphi,\psi)$ - a continuous function of the angular variables φ,ψ of period 2π.

If $\bar{\nu}$ is irrational, then each of the solutions $x(t)$ /5.6/ on the integral manifold S_ε /lying on the torus/ is a quasi-periodic function of t with two fundamental frequencies α_e and α_p. Again, if $\bar{\nu}$ is rational, then there exist periodic solutions with the same fundamental frequencies on the integral manifold S_ε and every non-periodic solution lying on the integral manifold S_ε approaches one of such periodic solutions as $t\to\infty$.

If, in addition to the superposed conditions, all the $n-1$ characteristic exponents of the variational equations /3.5/ which passes at $t=t_0$ through some point of region U_σ approaches as $t\to\infty$ one of the stationary solutions lying on manifold S_ε /the quasi-periodic solution for $\bar{\nu}$ irrational or the periodic solution for $\bar{\nu}$ rational/.

A similar result can be obtained in considering a weak periodic effect on the relaxation oscillations.

Considering in this, as mentioned above, the system of equations

$$\frac{dx}{dt} = X(x) + \varepsilon X^*(\nu t, x, \varepsilon), \qquad /5.7/$$

where the functions $X^*(\nu t, x, \varepsilon)$ are periodic in νt of period 2π and assuming as fulfilled the conditions of theorem III /p. 59/, we arrive at the following theorem [7]:

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Theorem XI. - Suppose that for system of equations /5.7/ the conditions of theorem III are fulfilled and, in addition, that $X^{*}(\vartheta t, x, \varepsilon)$ are periodic functions in ϑt of period 2π.

Then positive numbers ε' and σ $(\varepsilon' < \varepsilon_0, \sigma < \rho)$ can be pointed out such that for any positive $\varepsilon < \varepsilon'$ the system possesses an integral manifold S_ε /represented by a torus /which satisfies the conditions of theorem III.

The solutions of system /5.7/ lying on this integral manifold can be represented by

$$x(t) = x^{\circ}(\psi(t)) + \frac{1}{2}\left\{A(\psi(t))\bar{h}(\vartheta t, \psi(t), \varepsilon) + \bar{A}(\psi(t))\bar{h}_1(\vartheta t, \psi(t), \varepsilon)\right\} = \Phi\left(\alpha_0 t, \alpha_\rho t + \psi_0\right), \quad \psi_0 = const \qquad /5.8/$$

and, consequently, as in the case of the previous theorem, are characterized by a certain number $\tilde{\nu}$, i.e.they are quasi-periodic or periodic depending on whether the number $\tilde{\nu}$ is irrational or rational.

In the latter case, every non-periodic solution lying on this integral manifold approaches one of such periodic solutions.

And, finally, any solution of equations /5.7/, which passes at $t = t_0$ through some point of region U_σ, approaches as $t \to \infty$ one of the stationary solutions mentioned lying on the manifold S_ε. This tendency of approach is sufficiently rapid if the $n-1$ -th negative real part of the characteristic exponents is large enough in absolute value. /The fundamental property of relaxation oscillations lies in that stationary conditions set in rapidly/.

In considering the behaviour of integral curves lying on a manifold, one cannot mentioning the classical memoir

by N.M.Krylov and N.N.Bogoliubov [13] who were the first to study the behaviour of integral curves for the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(t, x, \frac{dx}{dt}, \varepsilon\right), \qquad /5.9/$$

where ε is a small positive parameter $f\left(t, x, \frac{dx}{dt}, \varepsilon\right)$ - an analytical function of ε expanding into a power series of ε with coefficients which are whole polynomials with respect to

$$x, \quad \frac{dx}{dt}, \quad \cos t, \quad \sin t.$$

For equations of type /5.9/ from very complicated considerations based on the proof of the existence of an invarian curve of some special point transformation, as well as on th above mentioned results of Poincaré-Danjoys concerning the behaviour of the integral curves of equation /4.17/ on the torus, very interesting and important result have been obtained, of which the following will be mentioned here:

Theorem XII. - If the algebraic equation

$$F(a) = 0, \qquad /5.10/$$

where

$$F(a) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f_0(\theta, a\sin\varphi, a\omega\cos\varphi)\cos\varphi\, d\theta\, d\varphi \qquad /5.11/$$

has a simple nonzero root a_0, then the differential equation /5.9/ under consideration, for sufficiently small ε in the case of non-resonance /i.e. $\omega \neq \frac{p}{q}$ / possesses a family of accurate stationary solutions of the form

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$$x = z(t, \nu t), \qquad /5.12/$$

where $\nu = \nu(\varepsilon)$ is some continuous function satisfying Lipschitz's condition, $z(\theta, \varphi)$ is a continuous periodic function of θ, φ with period 2π, which in general depends on ε.

The initial values x_0, x_0' of these stationary solutions lie in the ε-neighbourhood of region Γ of the initial values of the approximate stationary solutions.

For irrational values of ν, the accurate stationary solutions /5.12/ are quasi-periodic functions in Bole's sense with two fundamental frequencies and form a set which depends continuously on an arbitrary constant ψ.

For rational values of ν the stationary solutions under consideration will be ordinary periodic functions and form, generally speaking, a discrete set.

If, in addition

$$F'(a_0) < 0,$$

then any solution of the differential equation /5.9/, whose initial values x_0, x_0' lie in the neighbourhood of region Γ, tend for $t \to \infty$ to one of the stationary solutions /5.12/.

The same results will hold for the case of resonance.

The results of this memoir, as well as those of the theorem given at the beginning of this section, can be generalized for systems of equations of more general form which has been done by O.B.Lykova in [14], [15].

In these papers criteria are established for which systems of equations of type.

$$\frac{dx}{dt} = X(x) + \varepsilon X^*(t, x, \varepsilon) \qquad /5.13/$$

fold and possessing the property of attracting any solutions of system /5.13/ whose initial values belong to some neighbourhood of the integral manifold.

We give one of the theorems which characterize this property of curves lying on the manifold of system /5.13/ [14].

Theorem XIII. - Let the conditions of the theorem of the existence and properties of a local integral manifold /p. 64/ be fulfilled for equations /5.13/; also assume that

1/ the right-hand sides of the system of equations

$$\frac{da}{dt} = \varepsilon Q^{(1)}(t, \psi, a, \varepsilon),$$
$$\frac{d\psi}{dt} = \omega(a) + \varepsilon P^{(1)}(t, \psi, a, \varepsilon) \qquad /5.14/$$

to which system /5.13/ on the manifold reduces, are analytic functions of a, ε;

2/ relation

$$\omega(a) \approx \frac{p}{q}$$

does not hold for any values of $a \in \mathcal{A}$ where p, q are whole mutually prime numbers;

3/ equation

$$\frac{da}{dt} = F(a, \varepsilon)$$

has an isolated static solution

$$a = a_0$$

for which

$$F'(a_0, \varepsilon) < 0.$$

Then positive constants $\varepsilon_1, \delta_1, \delta_2$ with $\delta_1 < \delta_0$ can be given such that for any positive $\varepsilon < \varepsilon_1$, $\delta < \delta_1$

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the system of equations /5.I3/ will have a family of exact stationary solutions

$$x(t)=x^{0}(\psi,a)+\frac{1}{2}\left\{A(\psi,a)f(t,\psi,a,\varepsilon)+\bar{A}(\psi,a)\bar{f}(t,\psi,a,\varepsilon)\right. \qquad /5.I5/$$

where ψ and a are determinal from system /5.I4/, lying on a two-dimensional integral manifold of the toroidal type, the behaviour of these solutions being characterized by $\omega(a_0,\varepsilon)$.

For irrational values of $\omega(a_0,\varepsilon)$ the exact stationary solutions are quasi-periodic functions with two fundamental frequencies 1 and $\omega(a_0,\varepsilon)$.

For rational values of $\omega(a_0,\varepsilon)$ the stationary solutions under consideration are periodic functions. In this case, any quasi-stationary solutions of system /5.I3/, lying on the manifold, gradually tend to one of these stationary periodic solutions.

Any solutions of system /5.I3/ not belonging to the manifold whose initial values lie in the δ_2-neighbourhood of the initial values of the approximate stationary solutions /in the δ_2-neighbourhood of the integral manifold/, tend as $t\to\infty$ to one of the exact stationary solutions of type /5.I!/: quasi-periodic or periodic depending on whether the value of $\omega(a_0,\varepsilon)$ is irrational or rational.

An analogous theorem has been proved for the case of resonance /i.e. when $\omega(a)\approx\frac{p}{q}\nu$, where ν is the frequency of the external perturbation/ [I5].

§ 6. Some Applications of the Theory of Integral Manifolds to Nonlinear Mechanics

This section deals with some applications of the theorems, given in the previous two sections, to the theory of nonlinear oscillations. Let us first consider systems with one degree of freedom.

Let the free oscillations of the system be characterized by a differential equation of the form

$$\frac{d^2x}{dt^2}+\omega^2x=\varepsilon f\left(x,\frac{dx}{dt}\right) \qquad /6.I/$$

with a small positive parameter ε.

Then the equation of the first approximation for the amplitude of oscillations becomes

$$\frac{da}{dt}=\varepsilon A_1(a), \qquad /6.2/$$

where

$$A_1(a)=-\frac{1}{2\pi\omega}\int_0^{2\pi}f(a\cos\psi,-a\omega\sin\psi)\sin\psi\,d\psi.$$

Using theorem I, it is not difficult to establish for the solutions of system /6.I/ the same results on the periodicity and stability, which are obtained when considering approximate equations.

For this purpose, we introduce new variables into equation /6.I/ in accordance with

$$x=a\cos\psi,$$
$$\frac{dx}{dt}=-a\omega\sin\psi. \qquad /6.3/$$

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and we obtain:

$$\frac{da}{dt} = -\frac{\varepsilon}{\omega} f(a\cos\psi, -a\omega\sin\psi)\sin\psi,$$
$$\frac{d\psi}{dt} = \omega - \frac{\varepsilon}{\omega a} f(a\cos\psi, -a\omega\sin\psi)\cos\psi \qquad /6.4/$$

from which will result:

$$\frac{da}{d\psi} = -\frac{\varepsilon}{\omega^2}\frac{f(a\cos\psi, -a\omega\sin\psi)\sin\psi}{1-\frac{\varepsilon}{a\omega^2} f(a\cos\psi, -a\omega\sin\psi)\cos\psi}. \qquad /6.5/$$

The corresponding equation of the first approximation assumes the form:

$$\frac{da}{d\psi} = \frac{\varepsilon}{\omega} A_1(a). \qquad /6.9/$$

Let the equation

$$A_1(a) = 0$$

have a trivial solution

$$a = a_0, \quad a_0 \neq 0$$

for which

$$A_1'(a_0) \neq 0.$$

Assume also that the function $f(x, x')$ on the (x, x') plane is continuous together with its partial derivatives of the first order in the neighbourhood of the ellipse

$$x^2 + \frac{x'^2}{\omega^2} = a_0^2. \qquad /6.7/$$

Then, from theorem I it follows that for sufficiently small ε the exact equation /6.5/ will have a periodic solution

$$a = a_0(\psi)$$

with period 2π close to a_0. This solution will be stable in the case

$$A_1'(a_0) < 0 \qquad /6.8/$$

and possesses the property of attracting close solutions.

In the case

$$A_1'(a_0) > 0 \qquad /6.9/$$

it is unstable and possesses the property of repulsion.

Taking into account the change of variables /6.3/ and the second equation of system /6.4/, it is seen that equation /6.4/ under consideration has, for sufficiently small ε, a limiting cycle corresponding to the periodic solution close to the ellipse /6.7/. This cycle will be stable under condition /6.8/ and unstable under condition /6.9/

Consider also an oscillatory system as described by the more general equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(\vartheta t, x, \frac{dx}{dt}\right) \qquad /6.10/$$

in which $f(\theta, x, \frac{dx}{dt})$ is a periodic function in θ of period 2π.

For the case of resonance when

$$\omega^2 = \left(\frac{p}{q}\vartheta\right)^2 + \varepsilon\Delta, \qquad /6.11/$$

where p, q are mutually prime numbers, the corresponding numbers of the first approximation will be

$$\frac{da}{dt} = \varepsilon A_1(a, v),$$
$$\frac{dv}{dt} = \varepsilon B_1(a, v), \qquad /6.12/$$

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where

$$A_1(a,v)=-\frac{1}{2\pi\rho}\int_0^{\frac{2\pi q}{p}} f_0(a,\gamma t,\frac{p}{q}\gamma t+v)\sin\left(\frac{p}{q}\gamma t+v\right)dt,$$

$$B_1(a,v)=\frac{\Delta q}{2p\gamma}-\frac{1}{2\pi\rho a}\int_0^{\frac{2\pi q}{p}} f_0(a,\gamma t,\frac{p}{q}\gamma t+v)\cos\left(\frac{p}{q}\gamma t+v\right)dt. \qquad /6.I3/$$

To make use of the theorems, we introduce new variables in equation /6.I0/, viz.

$$x=\xi\cos\frac{p}{q}\gamma t+\eta\sin\frac{p}{q}\gamma t,$$

$$\frac{dx}{dt}=-\xi\frac{p}{q}\gamma\sin\frac{p}{q}\gamma t+\eta\frac{p}{q}\gamma\cos\frac{p}{q}\gamma t \qquad /6.I4/$$

and reduce it to the standard form:

$$\frac{d\xi}{dt}=\varepsilon X(t,\xi,\eta),\quad \frac{d\eta}{dt}=\varepsilon Y(t,\xi,\eta), \qquad /6.I5/$$

where the right-hand sides appear to be periodic functions in t of period $\frac{2\pi q}{\gamma}$.

Note also that the equations of the first approximation corresponding to system /6.I5/

$$\frac{d\xi}{dt}=\varepsilon\frac{\gamma}{2\pi q}\int_0^{\frac{2\pi q}{\gamma}} X(t,\xi,\eta)dt,$$

$$\frac{d\eta}{dt}=\varepsilon\frac{\gamma}{2\pi q}\int_0^{\frac{2\pi q}{\gamma}} Y(t,\xi,\eta)dt \qquad /6.I6/$$

are equivalent to equations /6.I2/ and are transformed into these by a change such as

$$\xi=a\cos\vartheta,\quad \eta=-a\sin\vartheta.$$

Assume that equations /6.I2/ have a constant solution

$$a=a_0,\quad \vartheta=\vartheta_0 \qquad /6.I7/$$

and that in the neighbourhood of the ellipse

$$x^2+\frac{x'^2}{\left(\frac{p}{q}\gamma\right)^2}=a_0^2 \qquad /6.I8/$$

the function $f(t,x,x')$ is continuous together with its partial derivatives of the first order with respect to x, x'.

Again, suppose that both roots of the characteristic equation corresponding to the variational equations for solution /6.I7/ have negative real parts.

Then, it follows from theorem I that in the case under consideration for sufficiently small ε, equation /6.I0/ has a periodic solution with period $\frac{2\pi q}{\gamma}$, close to the harmonic solution

$$a_0\cos\left(\frac{p}{q}\gamma t+\varphi_0\right). \qquad /6.I9/$$

Any solution passing through a point of some neighbourhood of the ellipse /6.I8/ will approach asymptotically the period solution for $t\to\infty$.

Now, let the equations of the first approximation have a periodic solution with a characteristic exponent whose real part is nonzero, and let the function $f(t,x,x')$ in some neighbourhood of the orbit of this solution possess continuous partial derivatives with respect to x,x' up to the second order.

In this case, from theorems II and X the statment may follow, for example, that for sufficiently small values of ε in some neighbourhood of the orbit mentioned above, there are stationary solutions possessing two fundamental frequencies: "natural" and "forced".

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For irrational ratios of these frequencies the stationary solutions are quasi-periodic, for rational ratios- periodic.

In the case when the real part of the characteristic exponent is positive, the stationary solutions are unstable .

On the contrary, if the real part of the characteristic exponent is negative, then any solution of equation /6.10/, for which at some $t = t_0$ the point $\xi(t), \eta(t)$ lies sufficiently close to the orbita, approaches asymptotically the stationary solution for $t \to \infty$.

The case of non-resonance could be considered in a similar manner.

Let us now consider the application of the methods of the theory of integral manifolds to the investigation of the single-frequency states of oscillatory systems with many degrees of freedom.

It is known that for oscillatory systems with many degrees of freedom, methods of investigating single-frequency states [2], [3], [16] have recently come into general use, which consist in constructing approximate solutions depending, for instance, on two arbitrary constants and corresponding to some single-frequency oscillatory process.

Since the principle of superposition does not hold for nonlinear oscillatory, it is not possible to construct directly a general solution on the basis of such particular solutions. However, in a number of particular cases this family of particular solutions possesses a special property of stability: every solution of the initial differential equations, which passes through the starting point sufficiently close to the points of the family /of the manifold/, tends to the solutions belonging to the family /lying on the manifold/ with increasing t . The manifold under consideration seems to attract all the solutions which are close to it. And it is to bring out a number of criteria for which the oscillatory system in question has a stable manifold that the theorems of section 4 can be applied. Only if the system under investigation possesses a stable integral manifold /two-dimensional, for instance/, the construction of an approximate two-parameter solution is of physical interest, for in this case solely we may expect the oscillatory system being studied to possess a one-frequency state which can be investigated by considering the manifold /in general terms, since every solution tends only to the manifold as a whole, but does not tend to any concrete curves of the manifold/.

For illustrative purposes, we discuss here the conclusions following from theorem II.

Assume that oscillations of some system with N degrees of freedom are characterized by a system of equations in the standard form:

$$\frac{dx}{dt} = \varepsilon X(t,x), \qquad /6.20/$$

where x, X are points of an n- dimensional $(n \geq 2N)$ Eucledian space E_n , $X(t,x)$ – an almost periodic function of t bounded together with the partial derivatives of the first order respect to x on set E_n and uniformly continuous with respect to x .

Consider the case when the equations of the first approximation

$$\frac{df}{dt} = \varepsilon X_0(f) \qquad /6.21/$$

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has a periodic solution

$$\xi = \xi(\omega t), \quad \xi(\varphi + 2\pi) = \xi(\varphi) \qquad /6.22/$$

whose orbit together with its ρ -neighbourhood belongs to set E_n.

It is quite evident that equation /6.21/ has a family of solutions

$$\xi = \xi(\omega t + \chi) \qquad /6.23/$$

depending on one arbitrary constant χ and corresponding to the same orbit as /6.22/. It is known that for the investigation of the stability of the periodic solutions of /6.21/ under consideration, it is necessary to form the variational equation:

$$\frac{d\delta x}{dt} = \varepsilon X'_{of}(\xi(\omega t))\,\delta x. \qquad /6.24/$$

Suppose that the real parts of $n-1$ characteristic exponents for this system are negative. Then the periodic orbit under investigation

$$\xi = \xi(\Theta) \qquad /6.25/$$

for equations /6.21/ the equations of the first approximation/ is known to be stable: the trajectory of every other solution of these equations, which passes through a point sufficiently close to orbit /6.25/,-will tend to it as $t \to \infty$.

Note that the orbit under consideration is an integral manifold for equation /6.21/. The individual solutions lying on this manifold correspond to the particular solutions of the equation

$$\frac{d\Theta}{dt} = \omega. \qquad /6.26/$$

Let us now indicate similar properties of the solutions of the exact equation /6.20/, which follow from theorem II.

On the basis of this theorem it is possible to state that in the case under consideration equation /6.20/, for sufficiently small values of the parameter, possesses a stable integral manifold representable by

$$x = f(t,\Theta), \quad f(t,\Theta+2\pi) = f(t,\Theta), \qquad /6.27/$$

where $f(t,\Theta)$ is a function almost periodic in t.

Also, a positive $\sigma < \rho$ can be indicated such that every solution $x(t)$ of equation /6.20/, for which the point $x(t)$ lies in the σ -neighbourhood of orbit /6.25/ at the initial moment $t = t_o$ will tend to the integral manifold /6.27/ as $t \to \infty$.

Moreover, according to this same theorem, the integral manifold /6.27/ lies for all real t in the $\delta(\varepsilon)$ -neighbourhood of orbit /6.25/

$$|f(t,\Theta) - \xi(\Theta)| \leq \delta(\varepsilon), \qquad /6.28/$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

And, finally, for the solutions of equations /6.20/ lying on manifold S_t /6.27/, the variable Θ in the formulas parametrically representing manifold /6.27/, satisfies the equation

$$\frac{d\Theta}{dt} = \omega + \varepsilon F(t,\Theta) \qquad /6.29/$$

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ir which

$$|F(t,\Theta)| \leqslant \delta^*(\varepsilon) \to 0$$

as $\varepsilon \to 0$ /6.30/

Thus, for the solutions $x^*(t)$ of equations /6.20/, which lie on the integral manifold S_ε under consideration, the following asymptotic estimate is obtained:

$$|x^*(t) - \xi\{\Theta(t)\}| \leqslant \delta(\varepsilon) \to 0 \ , \ \varepsilon \to 0,$$

/6.31/

where

$$\left|\frac{d\Theta}{dt} - \omega\right| \leqslant \varepsilon\, \delta^*(\varepsilon).$$

/6.32/

Hence, in the case being considered, it is ready possible to judge of the properties of the solutions of system /6.20/, and a number of conclusions can be obtained by considering the equations of the approximations /6.21/, whose investigation is an incomparably simpler problem than that exact equations /6.20/.

Note that a similar analysis and application to concrete cases could be effected also for all the subsequent theorems given above.

§ 7. Application of the Method of Problems of Stability and Relaxation Oscillations

This section deals with some results not included in the previous sections, which were recently obtained in connection with the application of the method of integral manifolds to problems of stability and oscillations of special form. We shall describe the results in the same way as before, briefly and without proof, referring those interested to primary sources and concertating mainly on ideas offered and results obtained.

First of all consider some interesting results recently obtained by J.K.Hale and A.P.Stokes [17] in investigating the behaviour of a solution in the vicinity of integral manifolds.

The problem as considered in [17], is stated as follows:

Let a system of autonomous differential equations be given

$$\frac{dx}{dt} = X(x),$$

/7.1/

where x, X are n-vectors of the Euclidian space E_n.

Assume that $x = x^\circ(t)$ is a periodic solution of system /7.1/. Then, this periodic solution is known to be asymptotically orbitally stable /i.e. the closed curve C_o in the x-space as determined by equation $x = x^\circ(t)$ is asymptotically stable/, if $n-1$ characteristic exponents of the system of variational equations

$$\frac{dy}{dt} = X'_x(x^\circ(t))\, y,$$

/7.2/

corresponding to the solution $x = x^\circ(t)$ have negative real parts.

If $x^\circ(t)$ is a solution of equation /7.1/, then $x^\circ(t+\varphi)$, where φ is an arbitrary constant, is also a solution and the function $x^\circ(t+\varphi)$ thus determines a cylinder C in space (x, t). Supposing that a closed curve in x-space in

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uble, it is evident that the cylinder C in (x,t) space s also stable. Under the same assumptions concerning the aracteristic exponents of the variational system of equaions /7.2/, there is known to exist not only cylindrical symptotic stability, but also the so-called asymptotic rbital stability with asymptotic phase, consisting in that or every solution $x(t)$ of equation /7.I/, with $x(0)$ ufficiently close to C, there exists a φ_0 such that $x(t) \to x^0(t+\varphi_0)$ as soon as $t \to \infty$.

In the reference above a case in considered when sysm /7.I/ has a $\kappa+1$ -parametric family of periodic soutions

$$x = x^0[\omega(a)t + \varphi, a],$$

here $a = (a_1, \dots, a_\kappa)$ and φ is a real constant. In pace (x,t) such a family of solutions determines a κ -parametric family of cylinders C_a or a $\kappa+1$ -dimensional manifold $\mathcal{M}$ of solutions. For a fixed values of a the variational equation /7.2/ can be formed and it will have in the general case $\kappa+1$ characteristic exponents equal to zero.

Assuming that $n-(\kappa+1)$ characteristic exponents of system /7.2/ have negative real parts /for every value of the vector a /, the authors above show that under sufficiently general conditions superposed on manifold $\mathcal{M}$ for every solution $x(t)$ of equation /7.I/ with $x(0)$ sufficiently close to $\mathcal{M}$, there exist φ_0, a_0 such that $x(t) \to x^0[\omega(a_0)t + \varphi_0, a_0]$ as $t \to \infty$, *i.e.* $x(t)$ approach cylinder C_{a_0} with asymptotic phase.

Another case of interest is also considered by the uthors. Suppose that system /7.I/ has an asymptotic orbilly stable periodic solution $x^0(t)$ and that the funcn $X^*(t,x)$ which expresses the perturbation acting

on system /7.I/ tends to zero as $t \to \infty$ uniformly for all $t \geq T$ and $x \in U$ where U is a bounded set in x -space. Will the solution of the perturbed system $x^*(t)$ tend to the cylinder determined by the function $x^0(t+\varphi)$ in the case when $x^0(t)$ does not satisfy the perturbed equation at all?

Now, to make clear the ideas as explained above, let us give the main results and some theorems due to J.K.Hale and A.P.Stokes [17].

Consider two systems of differential equations:

$$\frac{dx}{dt} = X(x), \quad t \geq 0 \qquad /7.3/$$

$$\frac{dx}{dt} = X(x) + X^*(t,x), \quad t \geq 0, \qquad /7.4/$$

where x, X, X^* are real n -dimensional vectors and X, X^* satisfy the conditions which guarantee the uniqueness of the solutions of these systems.

Assuming that system /7.3/ has a $\kappa+1$ -parametric family of periodic solutions

$$x = x^0[\omega(a)t + \varphi, a],$$

$$x^0(s+2\pi, a) = x^0(s,a), \qquad /7.5/$$

where $a = (a_1, \dots, a_\kappa)$, φ are constants, $\omega(a) > 0$ as indicated by introduction of new local coordinates (s,a,h) where s is a scalar, a — κ -vector, h an $n-(\kappa+1)$ -vector, and introducing new variables of the type

$$x = x^0(s,a) + Q(s,a)h, \qquad /7.6/$$

where $Q(s,a)$ is a matrix on n rows and $n-(\kappa+1)$ columns, systems /7.3/ and /7.4/, can be transformed, ree-

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tively, as follows:

$$\frac{ds}{dt} = \omega(a) + S_1(s, a, h),$$

$$\frac{da}{dt} = A_1(s, a, h),$$

$$\frac{dh}{dt} = H(s, a)h + H_1(s, a, h), \qquad /7.7/$$

$$\frac{ds}{dt} = \omega(a) + S_1(s, a, h) + S_2(s, a, h, t),$$

$$\frac{da}{dt} = A_1(s, a, h) + A_2(s, a, h, t),$$

$$\frac{dh}{dt} = H(s, a)h + H_1(s, a, h) + H_2(s, a, h, t), \qquad /7.8/$$

where each of the functions S_1, A_1, H_1 is of order $O(\|h\|^2)$ as soon as $\|h\| \to 0$ while the functions S_2, A_2, H_2 behave essentially in the same manner as the functions $X^*(t, x)$. One of the conditions for this is $S_1 = A_1 = H_1 = 0$ with $h = 0$ and, consequently, system /7.7/ has a solution

$$h = 0, \quad a = C_a, \quad \frac{ds}{dt} = S_1(s, C_a, 0) + \omega(C_a),$$

where C_a, $s(0) = s_0$ are arbitrary constants.

Moreover S_2, A_2, H_2 do not necessarily become zero for $h = 0$.

The investigation of systems /7.7/ and /7.8/ comes down to establishing the existence, and comparing the properties of the manifolds for these systems. It is established for which types of the "perturbing" functions the properties of the manifold of system /7.8/ are the same as those of the manifold of system /7.7/

Investigating the behaviour of the solutions of systems similar to /7.7/ and /7.8/ involves, generally speaking, certain difficulties, for each of the functions $s(t)$ $a(t)$, $h(t)$ can be expected to behave as functions of time in quite a different manner. Moreover, difficulties arise due to the fact that as $t \to \infty$ also $s(t) \to \infty$.

In studying system /7.8/ /or /7.7/ /, the authors choose the most natural approach to the solution of the problem and consider each of the equations separately, by which the problem is somewhat simplified as compared to a different approach when the solutions of system /7.8/ are being investigated all at the same time.

By considering the last equation of system /7.8/, where $s(t)$ and $a(t)$ are given functions, the problem is solved of how great the classes of functions $S, \mathcal{A}$ might be in order that for every $s(t) \in S$, $a(t) \in \mathcal{A}$ and given H, H_1, H_2 the solution $h(t)$ as determined from this equation possess definite properties /approach zero, for instance/ as $t \to \infty$.

Thereafter, the second equation of system /7.8/ is considered, from which a is found and it is assumed that $\mathcal{A}$ and S are known functions belonging to classes H, $S - h(t) \in H$, $s(t) \in S$.

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Then it is shown that $a \in \mathcal{O}$ where $\mathcal{O}$ is a given class of functions. After that $s(h) \in S$ is found from the first equation and, finally, substituting $s(h) \in S$, $a(h, s(h)) \in \mathcal{O}$ in the third equation, it is obtained $\bar{h} \in H$.

The there functions $\bar{s} = s(\bar{h})$, $a(\bar{h}, \bar{s})$, $\bar{h}$ provide a family of solutions which determine for system /7.8/ the manifold belonging to $S \times \mathcal{O} \times H$.

The fundamental requirement in this approach is that the functions $a(h, s)$ and $s(h)$ must be continuous in topologies of different functional spaces.

By using the results as obtained for systems /7.7/ and /7.8/, a number of theorems was proved by the authors of which only two will be shown here [17].

Theorem XIV. - Assume that the system of equations

$$\frac{dx}{dt} = X(x), \qquad /7.9/$$

where x, X are real n-dimensional, $X \in C^2$ has a $\kappa+1$ - parametric family of real periodic solutions:

$$x = x^\circ[\omega(a)t + \varphi, a], \qquad /7.10/$$

$$x^\circ(s + 2\pi, a) = x^\circ(s, a),$$

where $a = (a_1, \ldots, a_\kappa)$, φ are real constants $x^\circ(s,a) \in C^1_{(s)}$, $x^\circ(s,a) \in C^2_{(a)}$, $\omega(a) > 0$, $\omega(a) \in C^2$, $a \in U$ in which U is one of the open sets E_κ or a unique point, in the latter case $\kappa = 0$.

If $n-(\kappa+1)$ characteristic exponents $\lambda_1(a), \ldots, \lambda_{n-(\kappa+1)}(a)$ of a system of linear variational equations corresponding to the periodic solution /7.10/ have negative real parts for all $a \in U$ and

$$\operatorname{rank} \left\| \frac{\partial x^\circ(s,a)}{\partial s}, \frac{\partial x^\circ(s,a)}{\partial a} \right\| = \kappa + 1 \qquad /7.11/$$

for all $a \in U$, $s \in R$ then the manifold $\mathcal{M}$ as determined by relations /7.10/, is asymptotically stable with asymptotic phase and amplitude.

Consider the system of equations:

$$\frac{d^2x}{dt^2} + g(x) = f(x, y),$$

$$\frac{dy}{dt} + \gamma(x)y = h(x, y), \qquad /7.12/$$

where x, y are scalar $g(x), \gamma(x), f(x,y), h(x,y) \in C^2$ for all x, y; $f(x,0) = h(x,0) = \frac{\partial f(x,0)}{\partial y} = \frac{\partial h(x,0)}{\partial y} = 0$, $g(-x) = -g(x)$, $xg(x) > 0$, $x \neq 0$, $\gamma(x) > 0$ for all x.

Under these conditions system /7.12/ has a two-parametric family of periodic solutions

$$y = 0, \quad x = x^\circ(\omega(a)t + \varphi, a),$$

$$x^\circ(s + 2\pi, a) = x^\circ(s, a), \qquad /7.13/$$

where $x^\circ(\omega(a)t + \varphi, a)$ is a solution of the equation

$$\frac{d^2x}{dt^2} + g(x) = 0.$$

Let $\det \left| \frac{\partial x^\circ}{\partial \varphi}, \frac{\partial x^\circ}{\partial a} \right| \neq 0$ /7.14/

for all a.

The variational equations corresponding to solution /7.13/ for a fixed a are of the form

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$$\frac{d^2\delta x}{dt^2} + \frac{\partial g(x^0)}{\partial x}\delta x = 0,$$

$$\frac{d\delta y}{dt} + \gamma(x^0)\,\delta y = 0. \qquad /7.15/$$

From the theorem of Floquet one of the characteristic exponents of system /7.15/ is determined by

$$\lambda(a) = \frac{1}{T(a)}\int_0^{T(a)} \gamma\left[x^0(\omega(a)t + \varphi, a)\right]dt, \qquad /7.16/$$

where $T(a) = \frac{\pi}{\omega(a)}$. Since $\gamma(x) > 0$ for all x, $\lambda(a) < 0$ for all a. The condition of the previous theorem is fulfilled and, consequently, the manifold of solutions /7.13/ is asymptotically stable with asymptotic phase and amplitude.

Theorem XV.- Consider the system equations

$$\frac{dx}{dt} = X(x) + X^*(t, x) \qquad /7.17/$$

for which the system

$$\frac{dx}{dt} = X(x)$$

satisfies the conditions of the previous theorem and assume that there exist continuous functions $h(y)$, $g(t)$ and $T > 0$ such that $g(t) \to 0$ as $t \to \infty$ and

$$|X^*(t, x)| \leq g(t)h(\|x\|)$$

for all x, $t \geq T$. If, in addition to these conditions

$$\int_0^\infty g(t)dt < \infty$$

then the manifold $\mathfrak{M}$ as determined by relations /7.10/ is an asymptotically stable manifold of system /7.17/ with asymptotic amplitude. Finally, if

$$\iint_{T\,t}^{\infty\,\infty} g(u)\,du\,dt < \infty \qquad /7.20/$$

then this asymptotically stable manifold of system /7.17/ has an asymptotic amplitude and phase.

Note, however, that notwithstanding the fact that the small parameter ε does not enter in explicit form in the proof of the above theorems, it is assumed for systems, as it is usually done in considering equations in problems of the stability of motion, that a number of terms is either order compared to the main terms or possesses coeficients tending to zero as $t \to \infty$.

Let us now consider some results obtained by K.V.Zadiraka [18], [19] on the application of the method of integral manifolds to the investigation of complex oscillatory phenomena in relaxation oscillatory systems.

In many cases, relaxation oscillations in systems with many degrees of freedom are known to be described by systems of differential equations with a small parameter in some derivatives. In this connection, the author of [18] considers a system of equations

$$\frac{dx}{dt} = f\left(t, x, z, \frac{t}{\varepsilon}\right), \quad \varepsilon\frac{dz}{dt} = F(t, x, z), \qquad /7.21/$$

where x, f are n-dimensional vectors and α, z, F — m-dimensional vectors of Euclidean space E_{n+m}.

Assuming that there exists

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$$\lim \frac{1}{T}\int_0^T f(t,x,z,\lambda)\,d\lambda = f_0(t,x,z) \qquad /7.22/$$

uniformly with respect to x, $z \in E_{n+m}$ the degenerated system

$$\frac{dx}{dt} = f_0(t,x,z), \quad z = \varphi(t,x) \qquad /7.23/$$

is considered simultaneously with system /7.2I/ $z = \varphi(t,z)$ being determined as the root of equation

$$F(t,x,z) = 0. \qquad /7.24/$$

It is assumed that in region

$$-\infty < t < \infty,\ x \in U_n,\ \|z - \varphi(t,x)\| \le \rho,\ 0 < \varepsilon < \varepsilon^* \qquad /7.25/$$

the following conditions are fulfilled in addition to /7.22/:

a/ the vector-function $f(t,x,z,\frac{t}{\varepsilon})$ is continuous, bounded and satisfies Lipschitz's condition with respect to x, z;

b/ the vector-functions $F(t,x,z)$, $\varphi(t,x)$ and their derivatives with respect to all the arguments up to the second order are continuous and bounded;

c/ the second mixed derivatives of the vector-function $F(t,x,z)$ with respect to x and z satisfy Lipschitz's condition with respect to z;

d/ the roots of the charcteristic equation

$$\operatorname{Det} \|\lambda_m \rho - A(t,x)\| = 0, \qquad /7.26/$$

where $A(t,x) = F_z'(t,x,0)$ is a square matrix of m-th order, satisfy the condition

$$Re\{\rho_i(t,x)\} \le -\alpha < 0. \qquad /7.27/$$

Under these conditions the theorem is proved, which establishes the existence and a number of properties of the integral manifold of system /7.2I/ close to the integral manifold $z = \varphi(t,x)$ [18].

Theorem XVI. - If the above conditions /a/ to /d/ are fulfilled, then a positive ε_0 can be given such that for any positive $\varepsilon < \varepsilon_0$ the system /7.2I/ has a unique local integral manifold of the form

$$z(t,x) = \varphi(t,x) + \psi(t,x,\varepsilon) \qquad /7.28/$$

in which the function $\psi(t,x,\varepsilon)$ is defined in region $-\infty < t < \infty$, $x \in U_n$ and satisfies the inequalities

$$|\psi(t,x,\varepsilon)| \le D(\varepsilon),\ |\psi(t,x',\varepsilon) - \psi(t,x'',\varepsilon)| \le \Delta(\varepsilon)|x' - x''|, \qquad /7.29/$$

where $D(\varepsilon) \to 0$, $\Delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\varphi(t,x)$ is an isolated root of equation /7.24/.

Moreover, manifold /7.28/ has bounded and uniformly continuous derivatives with respect to x, whose order is determined by the degree of smoothness of the vectors f and F.

By using this theorem, the structure of the solutions of system /7.2I/ can be analyzed. Thus, for examp-

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ie, if the vector-functions $f(t, x, z, \frac{t}{\varepsilon})$, $F(t, x, z)$ are periodic in t with period 2π and system /7.23/ has an isolated periodic solution $x = x(t)$, $z = z(t)$ with period 2π, then using theorem XVI, it is not difficult to prove [19] the existence of a unique solution of system /7.21/ with period 2π: $x = x(t, \varepsilon)$, $z = z(t, \varepsilon)$ which uniformly with respect to t tends to the periodic solution of system /7.23/ as $\varepsilon \to 0$.

The autonomous relaxation systems can be studied in the same manner.

Thus, for example, considering the system of differential equations

$$\frac{dx}{dt} = f(x, z),$$
$$\varepsilon \frac{dz}{dt} = F(x, z), \qquad /7.30/$$

where ε is a small positive parameter, x, f are n-dimensional vectors, z, F are m-dimensional vectors, and assuming that the variational equations corresponding to the periodic solution $x = x(t)$, $z = z(t)$ of the degenerated system

$$\frac{dx}{dt} = f(x, z), \quad F(x, z) = 0, \qquad /7.31/$$

have one characteristic exponent equal to zero, it is possible to prove for sufficiently general the existence and uniqueness of the integral manifold

$$z(x) = \varphi(x) + \psi(x, \varepsilon)$$

for system /7.30/, where $\varphi(x)$ is an isolated root of

equation $F(x, z) = 0$ and $\psi(x, \varepsilon) \to 0$ as $\varepsilon \to 0$.

From this readily follows the proof of the existence of the periodic solution of system /7.30/, tending to the periodic solution of the degenerated system /7.31/ as $\varepsilon \to 0$.

The same question is dealt with by D.V.Anosov in [20] where a number of theorems has been proved which establish the properties of the solutions of system /7.30/. It has been proved, in particular, that the solutions with initial values from the compact subregion of region D / D being the region in which the unique solution $z = \varphi(x)$ of equation $F(x, z) = 0$ is defined /approach the surface $z = \varphi(x)$ during a time of order ε and then pass at a distance $\omega(\varepsilon)$ from the curves $z = \varphi(x(t))$, $x = x(t)$, where $x(t)$ is the solution of the degenerated system /7.31/ with the same initial value of x_0 as that of the solution of system /7.30/ under consideration. /The same result was previously obtained by A.N.Tikhonov [21] as well as by J.J.Levin and N.Levinson [22] /A number of theorems has been also obtained generalizing the well known theorems due to L.Flatto and N.Levinson [23].

Of great interest is also D.V.Amosov's contribution [24] which deals with the system of equations

$$\varepsilon \dot{x} = f(x, y) + \varepsilon X(x, y, \varepsilon),$$
$$\dot{y} = Y(x, y, \varepsilon), \qquad /7.32/$$

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where $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ are vectors, $f(x,y)$ is a smooth function in some region $\mathcal{D}$ of the space of variables (x,y) and functions $X(x,y,\varepsilon)$, $Y(x,y,\varepsilon)$ are defined in region $\mathcal{D}$ $0\le\varepsilon\le\varepsilon_0$ continuous with respect to x,y,ε and have derivatives with respect to x,y which are also continuous with respect to x,y,ε. System

$$\dot{x}=f(x,y) \qquad /7.33/$$

is supposed to be Hamiltonian and it is assumed that the surfaces of constant energy are closed surfaces on which system /7.33/ induces metrically transitive dynamical systems. The more general case is also studied when system /7.33/ has an integral invariant and some first integrals

$$H_1(x,y)=h_1,\dots,H_k(x,y)=h_k \quad (k<n) \qquad /7.34/$$

the $y, h_1,\dots,h_k$ manifolds /7.34/ for almost all y being closed manifolds on which system /7.33/ indices metrically transitive dynamical systems.

Closely associated with these questions are the results as obtained by V.M.Volosov [25], [26] and G.S.Makayeva [27], as well as the fundamental contributions by L.S.Pontriaghin [28] and E.F.Mischenko [29], in which an effective approach to systems of type /7.30/ was first formulated, based on the fact that investigation starts with systems of rapid motions /7.33/.

Let us also deal briefly with some considerations connected with the study of some problems of the stability of motion by applying the method of integral manifolds.

This question has not yet been sufficiently investigated, however some consideration have been brought forward by O.B.Lykova in [30] where the system of different equations is considered

$$\frac{dx}{dt}=X(x)+\varepsilon X^*(t,x,\varepsilon), \qquad /7.35/$$

ε being a small positive parameter X, X^* – n-dimensional vector-functions defined for $-\infty<t<\infty$, $0<\varepsilon<\varepsilon_0$, and for x from $\mathcal{D}\in E_n$.

The functions $\varepsilon X^*(t,x,\varepsilon)$ are characterized by small perturbations acting on the system

$$\frac{dx}{dt}=X(x). \qquad /7.36/$$

It is assumed that system /7.36/ has a statical solution $x_0=0$, while $x_0=0$ is not a solution of system /7.35/.

System /7.35/ can be taken to be the equations of perturbed motion, which correspond to some problem of the stability of motion, $\varepsilon X^*(t,x,\varepsilon)$ characterizing the permanently acting perturbations.

Assume that the characteristic equation corresponding to the variational equations as formed for system /7.36/ and for the solution

$$|J_n\lambda-X'_x(0)|=0 \qquad /7.37/$$

has two purely imaginary roots.

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while the other roots have negative real parts. /This is in accordance with the fact that with permanently acting perturbations the critical case is considered in addition to stability.

Assuming that the conditions of theorem IV hold for system /7.35/ /p.62/, it is not difficult to prove the existence of the solution $x_0 = 0$ of a two-dimensional local integral manifold S in some $U_{\sigma'}(0)$ -neighbourhood, any solutions $x(t)$ of system /7.35/, whose initial values belong to region $U_{\sigma_1}(0)$ $(\sigma_1 < \sigma')$, are attracted to S in accordance with

$$\left| x(t) - \Phi(t, \xi, \xi^*, \varepsilon) \right| \leq C(\varepsilon, \rho_0^2) e^{-\delta(t-t_0)} \qquad /7.38/$$

/so long as $|\xi| < \rho_1$, $|\xi^*| < \rho_1$, $\rho_1 < \rho_0$ /, where $\Phi(t, \xi, \xi^*, \varepsilon)$ is the manifold S, $C(\varepsilon, \rho_0^2) \to 0$ as $\varepsilon \to 0$, $\rho_0^2 \to 0$, δ-a positive constant.

However, in addition to the above, for certain conditions imposed on system /7.35/ a sufficiently great $\bar{t}$ and a sufficiently small $\bar{\varepsilon}$ can be pointed out such that for any $t > \bar{t}$ and $0 < \varepsilon < \bar{\varepsilon}$ any solutions of system /7.35/ whose initial values lie in $U_{\sigma_1}(0)$, will belong to some δ -neighbourhood of the solution $x_0 = 0$ which can be made arbitrarily small for arbitrarily great $\bar{t}$ and arbitrarily small $\bar{\varepsilon}$.

In concluding this review which is far from being complete, it should be noted that many interesting results recently obtained and largely concerned with questions of the qualitative investigation of differential equations /such as the investigation of invariant surfaces, the behaviour of the solutions on manifolds, etc./ have been entirely left out of consideration.

At this point, it is well to mention, for example, S.P. Diliberto's results on invariant surfaces [31]; M.D.Marcus's [32] results on the existence of a family of tori for a three-dimensional holonomic system of differential equations of the form:

$$\frac{d\theta}{dt} = 1 + z H(\theta, \varphi, z) + \varepsilon P(\theta, \varphi, z),$$
$$\frac{d\varphi}{dt} = 1 + z K(\theta, \varphi, z) + \varepsilon Q(\theta, \varphi, z),$$
$$\frac{dz}{dt} = A(\theta, \varphi) z + z^2 \mathcal{L}(\theta, \varphi, z) + \varepsilon R(\theta, \varphi, z), \qquad /7.39/$$

where $H, K, \mathcal{L}, P, Q, R \in C^3$ are periodic in θ and φ with periods ω_1 and ω_2, respectively; P.Koosis' results [33] on the existence and properties of invariant surfaces for systems of type /7.39/; B.L.Reinhart's paper [34] on periodic orbits on two-dimensional manifolds; S.P.Diliberto's fundamental memoir [35], etc.

In conclusion, let us consider briefly some aspects of the further development of the theory of integral manifolds and its application to the investigation of numerous important problems of nonlinear mechanics, in general, and the investigation of the differential equations, containing a small parameter, herein encountered.

Of great interest is the extension of the methods of investigation of integral manifolds to infinite systems of differential equations. This is an important problem in connection with the investigation of oscillations in systems with distributed parameters /vibrations of beams, shafts, turbine rotors, plates and various kinds of shells, etc./ described by partial differential equations, and in particular, in connection with substantiating the application of asymptotical methods of investigating single frequency oscillatory processes to the solution of this problem.

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Here, however, we are likely to meet some great difficulties concerned, for instance, with the reducibility of infinite systems with periodic coefficients, etc. Therefore, effective results may be expected to obtain only in particular cases for sufficiently rigorous conditions superposed on the oscillatory systems with distributed parameters under investigation.

Of promising significance is the extension of the methods of integral manifolds to systems with retarded argument. In this connection, most of the above theorems of the existence and properties of integral manifolds can be generalized for sufficiently wide class of differential equations with retarded argument, or at any rate to the case or slow retardation, slowly changing retardation, etc. The results to be obtained here will no doubt be of great interest in investigating oscillatory phenomena and stability problems of complex control systems, in servomechanisms, etc.

Of practical importance, though, rather complex is the problem of the development and application of the theory of integral manifolds to systems of differential equations containing a small parameter in the higher derivatives and describing relaxation oscillatory processes. For the case when it is rather simple to find the analytic expression of the limiting cycle for a degenerate oscillatory system, it readily follows from some of the above theorems that the method is comparatively easy to apply and a number cycle of the degenerate system is not sufficiently smooth, quite a number of great difficulties may arise. At the same time, the solution of this problem even for some particular cases is of great interest in connection with numerous problems in physics and engineering.

Of some interest are also the problems involving the topological study of the structures of integral manifolds as obtained for different types of differential equations which describe rather complex oscillating phenomena associated impacts, instantaneous changes of speed, etc.

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О ПЕРИОДИЧЕСКИХ РЕШЕНИЯХ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ n-ГО ПОРЯДКА С МАЛЫМ ПАРАМЕТРОМ

In this paper we consider a differential equation of the n-order

$$\frac{d^n x}{dt^n} = \mu f\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}, \mu\right), \qquad (1)$$

where f is a periodic function of period 2π in t, μ-small parameter.

By an ordinary change of variables the equation can be transformed to the so-called standard form

$$\frac{dx_k}{dt} = \varepsilon X(t, x, x_1, \ldots, x_{n-1}; \varepsilon), \qquad (2)$$

which becomes in our case

$$\frac{dx}{dt} = \varepsilon x_1, \quad \ldots, \quad \frac{dx_{n-2}}{dt} = \varepsilon x_{n-1},$$

$$\frac{dx_1}{dt} = \varepsilon x_2, \quad \ldots, \quad \frac{dx_{n-1}}{dt} = \varepsilon f(t, x, \varepsilon x_1, \ldots, \varepsilon^{n-1} x_{n-1}), \qquad (3)$$

where $\varepsilon = \mu^{\frac{1}{n}}$.

To construct the approximate solution of equations in standard form we could use the method of averaging which has been worked out by N. Bogoliubov and Yu. Mitropolski [1].

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As it is known, this method of averaging allows by a change of variables

$$x = \bar{x} + \varepsilon u_1(t,\bar{x}) + \varepsilon^2 u_2(t,\bar{x}) + \cdots , \qquad (4)$$

where $u_1(t,\bar{x}), u_2(t,\bar{x}), \ldots$, ... are periodic function of period 2π in t , to transform our fundamental equations into the form

$$\frac{d\bar{x}_k}{dt} = \varepsilon \overline{X}(\bar{x}, \bar{x}_1, \ldots, \bar{x}_{n-1}; \varepsilon) . \qquad (5)$$

In general case these equations are also very complicated but as $\bar{X}$ does not depend on t , we can find important particular solutions e.g. the equilibrium solutions for which $\overline{X}(\bar{x})=0$. We can notice directly from (4), that equilibrium solutions of averaged equations correspond to periodic solutions of our equations.

Investigating the variational equation for (5) in the neighbourhood of equilibrium solutions, we can obtain the equilibrium conditions.

The method of averaging uses the asymptotical power representation in the small parameter ε in our case. But for our problem

$\varepsilon \sqrt[n]{\mu}$, and hence the order of the approximation deteriorates comparatively with the asymptotic formulas for the power expansion in μ.

Moreover we find the same difficulty in the other variant of averaging which was considered by authors [2], when instead of (5) we get immediatly only one indepenedent from t equation of n -order.

According to this we have stated the problem to construct such an asymptotical method in which only the expansions in the integer μ are used. In solving this problem we restrict ourselves to find only the periodic solutions and to investigate their stability.

It is to say that in the method of averaging the investigating of the periodic solutions plays often the main role. But as it is, in this paper we not only obtain the asymptotic developments in the integer powers of μ , but we also prove their convergence.

§ 1.

Let us consider the differential equation (1) and assume that $f(t, x, \ldots, x^{(n-1)}; \mu)$

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is an analytical function of the complex variables $x, \dots, x^{(n-1)}; \mu$ in the domain

$$|x - x_0| \le \rho, |x'| \le \rho, \dots, |x^{(n-1)}| \le \rho, |\mu| \le \mu^*, \quad (D)$$

where ρ and μ^* are some positive, non-zero numbers.

Let us note, that if the equation (1) has a periodic solution, we can always represent this solution as

$$x = \xi + y(t) \qquad (6)$$

where $\langle y(t) \rangle = 0$ (7)

and where $\xi = \langle x(t) \rangle$;

here always $\langle f(t) \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt$.

The equation for $y(t)$ will be

$$\frac{d^n y}{dt^n} = \mu f(t, \xi + y, y', \dots, y^{(n-1)}, \mu) \qquad (8)$$

and since $y(t)$ is periodic

$$\langle f(t, \xi + y, y', \dots, y^{(n-1)}, \mu) \rangle = 0. \qquad (9)$$

Thus (8) may be done in the form

$$\frac{d^n y}{dt^n} = \mu \{ f - \langle f \rangle \}. \qquad (10)$$

Let us note, that if $y(t)$ is a periodic solution with period 2π of the integro-differential equation (10) with the supplementary conditions (7) and (9), then (6) is the periodic solution of the original equation (1).

Thus, it is natural to split our problem into two parts:

1°. To find the periodic solution $y = y(t, \xi, \mu)$ of the equation (10) under the condition (7) as the function of parameter ξ.

2°. To define ξ from the equation (9) which will be an ordinary equation with one unknown

$$\Phi(\xi, \mu) = 0 \qquad (11)$$

Let us proceed to the first part of our problem. We construct the periodic solution $y = y(t, \xi, \mu)$ with period 2π of the equations (10), (7) and show that $y(t, \xi, \mu)$ is an analytical function of the complex variables μ, ξ in the region

$$|\xi - x_0| \le \rho_0 = \frac{\rho}{2}, \quad |\mu| \le \mu_0, \quad 0 \le t \le 2\pi, \quad (T)$$

where μ_0 is sufficiently small and positive.

To this end we use the ordinary method of successive approximations

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As the "zero" approximation we take $y_0 = 0$. The $(m+1)^{th}$ approximation is defined from the previous one as a periodic solution (of period 2π) of equation

$$\frac{d^n y_{m+1}}{dt^n} = \mu\left\{f(t, \xi + y_m, y_m', \ldots, y_m^{(n-1)}; \mu) - \right.$$
$$\left. - \langle f(t, y_m + \xi, y_m', \ldots, y_m^{(n-1)}; \mu)\rangle\right\} \quad (12)$$

with the supplementary condition

$$\langle y_{m+1} \rangle = 0.$$

First of all we shall show that

$$x_m = \xi + y_m, \; x_m' = y_m', \ldots, \; x_m^{(n-1)} = y_m^{(n-1)} \quad (13)$$

will be within the analytic domain defined by the inequalities ($\mathcal{D}$), if we choose correctly the sufficiently small number μ_0.

We shall base on the following evident statement:

if $z(t)$ is a periodic solution of period 2π of the equations

$$\frac{d^n z}{dt^n} = F(t), \quad \langle z(t) \rangle = 0,$$

where $F(t)$ is a periodic function of period 2π such that

$$\langle F(t) \rangle = 0, \quad |F(t)| \le M,$$

the following estimates hold:

$$|z^{(n-1)}| \le 3\pi M, \; |z^{(n-2)}| \le (3\pi)^2 M, \ldots, |z| \le (3\pi)^n M.$$

By means of these estimates it is easy to see, that if

$$\mu_0 \le \mu^*,$$
$$\mu \le \frac{\rho}{4(3\pi)^n M},$$

where

$$M = \max \left| f(t, x, x', \ldots, x^{(n-1)}; \mu) \right|, \quad (14)$$
$$\left(0 \le t \le 2\pi, \; |x - x_0| \le \rho, \; |x'| \le \rho, \ldots, |x^{(n-1)}| \le \rho, \right.$$
$$\left. |\mu| \le \mu^* \right)$$

the arguments (13) will satisfy always to the inequalities (D) with any m.

Let us proceed now to the proof of the uniform convergence of

$$y_m(t, \mu, \xi), \; y_m'(t, \mu, \xi), \ldots, y_m^{(n)}(t, \mu, \xi) \quad (15)$$

in the domain (T).

Since the function f is regular in the domain D and the arguments (13) are in (D) we have done at any "m"

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$$|f(t;\xi+y_m, y_m', \dots, y_m^{(n-1)};\mu) - f(t;\xi+y_{m-1}, y_{m-1}', \dots, y_{m-1}^{(n-1)};\mu)| \leq$$
$$\leq \lambda \left\{ |y_m - y_{m-1}| + |y_m' - y_{m-1}'| + \dots + |y_m^{(n-1)} - y_{m-1}^{(n-1)}| \right\},$$

where $\lambda = \sup_{(D)} \left| \frac{\partial f}{\partial x^{(\kappa)}} \right| \quad (0 \leq \kappa \leq n-1)$.

Then we put in the same equation $m-1$ instead of m and subtruct it from (I2); we shall have:

$$\left| \frac{d^n (y_{m+1} - y_m)}{dt^n} \right| \leq \lambda |\mu| D_{m-1},$$

where

$$D = \max \left\{ |y_{m+1} - y_m| + |y_{m+1}' - y_m'| + \dots + |y_{m+1}^{(n-1)} - y_m^{(n-1)}| \right\},$$
$$(0 \leq t \leq 2\pi).$$

Hence with our previous estimates we shall find

$$D_m \leq \lambda \left\{ 3\pi + \dots + (3\pi)^n \right\} |\mu|.$$

Now we take μ_0 to satisfy both the inequalities (14) and the following inequality

$$\gamma = \left\{ 3\pi + \dots + (3\pi)^n \right\} \lambda \mu_0, \quad \gamma < 1. \qquad (16)$$

Now we have to note that

$$D_m \leq \gamma^m D_0, \quad |y_{m+1}^{(n)} - y_m^{(n)}| \leq \dots |\mu| D_0 \gamma^m$$

and we prove in this way that the sequences (15) are uniformly convergent in the regi...

$$y_m(t, \xi, \mu) \to y(t, \xi, \mu), \quad y_m'(t, \xi, \mu) \to y'(t, \xi, \dots$$
$$\dots \quad y_m^{(n)}(t, \xi, \mu) \to y^{(n)}(t, \xi, \mu),$$
$$(m \to \infty).$$

Therefore we may make the transition to the limit ($m \to \infty$) in the equations (12) and corresponding supplementary conditions and in this way it is proved the function

$$y = y(t, \xi, \mu) \qquad (17)$$

is the desired periodic solution (of period 2π) of integro - differential equation (10), (7) for ξ, μ, satisfying the inequalities (I').

As by the some process of construct... $y_m(t, \xi, \mu)$ ... in (...), and the sequences ... uniformly convergent ...

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...tion (17) is analytic function of the complex variables ξ, μ, in the same region

To solve the problem (1^0) we have used the method of successive approximations. For the real construction of approximate solutions of this problem one may use the power development in μ.

Let us put in the equations of our problem

$$y(t,\xi,\mu) = \mu z_1(t,\xi) + \mu^2 z_2(t,\xi) + \cdots \qquad (18)$$

and equate in both parts the coefficients of equal powers of μ. Thus we obtain

$$z_1(t,\xi) = J^n\{f(t,\xi,0,\ldots,0,0)\},$$

$$z_2(t,\xi) = J^n\{f'_\mu(t,\xi,0,\ldots,0,0) + f'_x(t,\xi,0,\ldots,0,0)\, z_1(t,\xi) + f'_{x^{(n-1)}}(t,\xi,0,\ldots,0,0)\, z_1^{(n-1)}(t,\xi)\}, \qquad (19)$$

where

$$J\{F(t)\} = \int_0^t \{F(t) - \langle F\rangle\}\, dt - \int_0^t \{F(t) - \langle F\rangle\}\, dt\rangle$$

and J^n denote the operation J n-times

repeated. Let us remark it is not necessary to prove the convergence of this algorithm, for we know the expansion (18) is convergent in (T), since the analicity of (17) is proved above.

From the practical point of view the method of the development (18) in many cases is more convenient than the method of successive approximations, which in its turn is much better to determine the general qualitative properties e.g. the property of analiticy.

Let us proceed to the second part. We have to find the unknown ξ from the equation

$$\Phi(\xi,\mu) = \langle f(t,\xi+y,y',\ldots,y^{(n-1)},\mu)\rangle = 0 \qquad (20)$$

which is obtained by substituting (10) in (9)

As it has been established in the previous discussion, it is clear that the function (Φ) is analytic in the domain (T) For its construction we may substitute the expansion (18) in (20) and we obtain the series

$$\Phi(\xi,\mu) = \Phi_0(\xi) + \mu\,\Phi_1(\xi) + \cdots. \qquad (21)$$

The series is necessary convergent in our

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domain.

In this way the problem to find the periodic solution of equation (21) is reduced to the equation (20). There are different methods to investigate it. Here we shall briefly review some sufficient criterions for the equation (20).

Let the equation $\Phi_0(\xi)=0$ for instance have a root $\xi=\xi_0$, satisfying following conditions

$$\Phi_0(\xi_0)=0, \quad \Phi_0'(\xi_0)\neq 0.$$

Then according to the well- known theorem on the implicit function the equation (20) will have a solution ξ developable in power series in μ in the region

$$|\xi-\xi_0|<\rho_0, \quad |\mu|<\mu_1. \tag{22}$$

A similar situation arises also in the case

$$\Phi_0(\xi)=0, \quad \Phi_0'(\xi_0)\neq 0,$$
$$\Phi_0(\xi_0)=0, \quad (0\leq s\leq 2).$$

For these cases the periodic solution under consideration

$$x=\xi(\mu)+y(t,\xi(\mu),\mu) \tag{23}$$

can be represented as power series in μ and to construct it one may use the classical Poincaré method [1], based on representation of $x(t)$ as the power series of μ.

Now let us treat the case, when

$$\Phi_0(\xi_0)=0, \quad \Phi_0'(\xi_0)=0, \quad \Phi_0''(\xi_0)=0,$$
$$\Phi_0^{(\alpha-1)}(\xi_0)\neq 0.$$

Then it is easy to see that the solution $\xi(\mu)$ of (20) can be developed in power series in $\mu^{\frac{1}{\alpha}}$. Therefore the periodic solution (23) can be represented by the power series proceeding in fractional powers of μ.

We shall not to discuss here the equation (20) any more. We shall suppose only that it has a solution $\xi=\xi(\mu)$ contained in the domain (22) for sufficiently small μ.

We proceed now to the problem of stability of obtained periodic solution

$$\dot{x}=f(x,\mu)+\varphi(t,f(x,\mu),\mu)$$

and to this end we write down the variational eq.

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$$\delta x^{(n)} = \mu \sum_{\kappa=0}^{n-1} \frac{\partial f(t, \xi + y(t,\xi,\mu), y'(t,\xi,\mu), \ldots, y^{(n-1)}(t,\xi,\mu); \mu)}{\partial x^{(\kappa)}} \delta x^{(\kappa)}.$$

If we put

$$f'_{x^{(\kappa)}}(t, \xi + y(t,\xi,\mu), y'(t,\xi,\mu), \ldots, y^{(n-1)}(t,\xi,\mu); \mu) = A_\kappa(t,\xi,\mu), \quad (\kappa = 0, 1, \ldots, n-1), \tag{24}$$

we can write such eq. in the form

$$z^{(n)} = \mu \sum_{q=0}^{n-1} A_q(t, \xi(\mu), \mu) z^{(q)}. \tag{25}$$

To obtain effective criterions of stability it would be more suitable to reduce this eq. with periodic coefficients, to the eq. whose coefficients do not contain explicitly the variable t.

In the process of reducing it is more convenient to have expressions in integer powers of μ, but the functions $A_q(t, \xi(\mu), \mu)$ may not be developable in power series, in spite of the analycity of the expressions (24) as functions of (ξ, μ) in the region (Ψ), since $\xi(\mu)$ defined by eq. $(\Phi(\xi,\mu)=0)$ may have singularities in the vicinity of $(\mu = 0)$.

To avoid these difficulties we shall take the equation

$$z^{(n)} = \mu \sum_{q=0}^{n-1} A_q(t, \xi, \mu) z^{(q)} \tag{26}$$

in which (ξ, μ) are independent parameters;

we reduce it to the eq., whose coefficients do not contain explicitly variable t (for sake of brevity we shall name this eq. the eq. with constant coefficients); we use the expansion in the powers of μ and only in the last formulas we "remember" that

$$\xi = \xi(\mu).$$

To this end we shall apply to the ideas of Kreus articles [4].

Let us take the change of variables

$$z^{(\kappa)} = y^{(\kappa)} + \mu \sum_{z=0}^{n-1} u_{\kappa z}(t, \xi, \mu) y^{(z)}, \tag{27}$$

in which $(u_{\kappa z})$ are the periodic functions of t (with period 2π) and the conditions

$$\langle u_{\kappa z} \rangle = 0 \tag{28}$$

are satisfied.

We consider the problem to determine both such functions $(u_{\kappa z}(t, \xi, \mu))$ [illegible] ons independent of t

$(\lambda(\xi, \mu))$, so that

[illegible] of formulas ( ) eq. with [illegible]

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coefficients (26) has been reduced to the one whose coefficients do not depend explicitly on t

$$y^{(n)} = \mu \sum_{k=0}^{n-1} \mathcal{L}_k(\xi, \mu) y^{(k)} \qquad (29)$$

Taking into consideration the next eqs. (26, 27, 28) thus we obtain the following "recursive" chain of equations for $(U_{\kappa z}, \mathcal{L}_n)$:

$$\frac{\partial U_{n-1,2}}{\partial t} + U_{n-1,2-1}(1-\delta_2) + \mathcal{L}_2 + \mu U_{n-1,n-1}\mathcal{L}_2 = A_2 + \mu \sum_{j=0}^{n-1} A_j U_{j2}, \qquad (30)$$

$$\frac{\partial U_{\kappa 2}}{\partial t} + U_{\kappa,2-1}(1-\delta_2) + \mu U_{\kappa,n-1}\mathcal{L}_2 = U_{\kappa+1,2},$$

$$0 \le 2 \le n-1, \quad 0 \le \kappa \le n-2, \quad \delta_2 = \begin{cases} 1, & 2=0 \\ 0, & 2 \ne 0 \end{cases} \qquad (31)$$

with the supplementary condition of periodicity.

From the practical point of view we may find this expressions approximately in the form

$$U_{\kappa 2}(t, \xi, \mu) = U_{\kappa 2}^{(0)}(t, \xi) + \mu U_{\kappa 2}^{(1)}(t, \xi) + \mu^2 U_{\kappa 2}^{(2)}(t, \xi) + \dots,$$

$$\mathcal{L}_\kappa(\xi, \mu) = \mathcal{L}_\kappa^{(0)}(\xi) + \mu \mathcal{L}_\kappa^{(1)}(\xi) + \mu^2 \mathcal{L}_\kappa^{(2)}(\xi) + \dots \qquad (32)$$

Inserting these expansions into eqs (30, 31) and equating the coefficients of the same powers of μ we obtain relations for the unique determination of the series coefficients (32).

Here arises the problem of convergence for these series. It is clear that the convergence of these series is proved if we can show, that functions ($\mathcal{L}_n(\xi, \mu)$) defined by eqs. (30, 31) are analytic functions of complex variables (ξ, μ) in the region $|\mu| \le \mu_1$, $|\xi - \xi_0| \le \rho_0$, (T') where $\mu_1 \le \mu_0$ is sufficiently small.

To prove this last property let us make use of the method of successive approximations. As the "zero" approximations let us take ($U_{\rho 2}^{(0)} = 0$, $\mathcal{L}_\kappa^{(0)} = 0$, $0 \le \rho \le n-1$), the $(m+1)^{th}$ approximation is determined from the previous one by eqs.

$$\frac{\partial U_{n-1,2}^{(m+1)}}{\partial t} + U_{n-1,2-1}^{(m+1)}(1-\delta_2) = A_2 - \mu U_{n-1,n-1}^{(m)} \mathcal{L}_2^{(m)} + \mu \sum_{q=0}^{n-1} A_q U_{q2}^{(m)} - \mathcal{L}_2^{(m)},$$

$$\frac{\partial U_{\kappa 2}^{(m+1)}}{\partial t} + U_{\kappa,2-1}^{(m+1)}(1-\delta_2) - U_{\kappa+1,2}^{(m+1)} = -\mu U_{\kappa,n-1}^{(m)} \mathcal{L}_\kappa^{(m)},$$

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In this case in expansion (19)

$$z_1(t,\xi) + J^2\left\{f(t,\xi,0)\right\} = \sum_{\kappa\geq 1} \frac{f_\kappa(\xi,0)\cos\kappa t + g_\kappa(\xi,0)\sin\kappa t}{\kappa^2}, \quad (35)$$

$$\frac{d}{dt} z_1 = z_1' = \sum_{\kappa\geq 1} \frac{f_\kappa(\xi,0)\sin\kappa t - g_\kappa(\xi,0)\cos\kappa t}{\kappa}, \quad (36)$$

$$z_2(t,\xi) = J^2\left\{f_x'(t,\xi,0)z_1 + f_{x'}'(t,\xi,0)z_1'\right\}. \quad (37)$$

Let us notice that ξ is determined from the eq. (9, n = 2), which at the present case may be represented in the form

$$\Phi(\xi,\mu) = f(\xi,0) + \mu\sum_{\kappa\geq 1}\left\{\frac{g_{\kappa x}'(\xi,0)f_\kappa(\xi,0) - f_{\kappa x}'(\xi,0)g_\kappa(\xi,0)}{2\kappa} - \frac{f_{\kappa x}'(\xi,0)f_\kappa(\xi,0) - g_{\kappa x}'(\xi,0)g_\kappa(\xi,0)}{2\kappa^2}\right\}. \quad (38)$$

From this eq. is not difficult to determin ξ if the function in eq. [1] has an explicit expression. We put the expression (18) into the variational equation (25, n = = 2).

Take into consideration (35-37) and write down the coefficients to the second order of μ, we obtain

$$A_0^\circ = f_x'(t,\xi,0), \quad A_1^\circ = f_{x'}'(t,\xi,0),$$
$$A_0' = \left\{f_{xx}''(t,\xi,0)z_1 + f_{xx'}''(t,\xi,0)z_1'\right\},$$
$$A_1' = \left\{f_{xx'}''(t,\xi,0)z_1 + f_{x'x'}''(t,\xi,0)z_1'\right\}. \quad (39)$$

Now we determine $(\mathcal{X}_0^\circ, \mathcal{X}_0', \mathcal{X}_1^\circ, \mathcal{X}_1')$ in the equation with constant coefficients (29, n = 2).

To this end we use "recursive relations" (30, 31) and previously substituting the expression (32) and take into account (39, 40); equating the coefficients for "zero" power of μ, we get

$$\frac{\partial u_{02}^{(0)}}{\partial t} + u_{0,2-1}(1-\delta_2) = u_{12}^{(0)},$$
$$\frac{\partial u_{12}^{(0)}}{\partial t} + u_{1,2-1}(1-\delta_2) + \mathcal{X}_2^{(0)} = A_2 \quad (2=0,1)$$

By averaging the last eq., we find

$$\mathcal{X}_0^\circ = \langle A_0^\circ \rangle = \langle f_x(t,\xi,0)\rangle, \quad (41)$$

$$\mathcal{X}_1^\circ = \langle A_1^\circ \rangle = \langle f_{x'}'(t,\xi,0)\rangle. \quad (42)$$

Integrating we thus obtain

$$u_{10} = J\left\{A_1^\circ - \mathcal{X}_0^\circ\right\},$$
$$u_{00} = J^2\left\{A_0^\circ - \mathcal{X}_0^\circ\right\},$$

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$$u_{11} = J\left\{A_1^\circ - \mathcal{X}_1^\circ - u_{10}\right\},$$
$$u_{01} = J\left\{u_{11}\right\} - J^2\left\{u_{00}\right\} \qquad (43)$$

Analogously we write down "recursive relation" (30-31) for the first power of μ and we have

$$\mathcal{X}_0' = \langle A_0' + \mu\,(A_0^\circ u_{00}^\circ + A_1^\circ u_{10}^\circ)\rangle,$$
$$\mathcal{X}_1^{(1)} = \langle A_1' + \mu\,(A_0^\circ u_{01}^\circ + A_1^\circ u_{11}^\circ)\rangle. \qquad (44)$$

It is clear that the expressions (44) will enter in to the eq. with "constant coefficient" (29, n = 2), within the order μ^3 and we restrict us to the order μ^2.

There is not necessery to use the generalized Fourier series to calculate these functions.

After some simple calculations we obtain

$$\mathcal{X}_0' = \langle A_0'\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left\{f_{xx}^\circ(t,\xi,0)Z_1(t,\xi) + f_{xx'}^{\circ\circ}(t,\xi,0)Z_1^{(1)}(t,\xi)\right\}dt =$$
$$= \sum_{\kappa\geq 1}\left\{\frac{g_{\kappa xx'}^\circ(\xi,0)f_\kappa(\xi,0) - f_{\kappa xx'}^\circ(\xi,0)g_\kappa(\xi,0)}{2\kappa} - \frac{f_{\kappa xx}^\circ(\xi,0)f_\kappa(\xi,0) + g_{\kappa xx}^\circ(\xi,0)g_\kappa(\xi,0)}{2\kappa^2}\right\},$$

$$\mathcal{X}_1^{(1)} = \langle A_1'\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left\{f_{xx'}^\circ(t,\xi,0)Z_1(t,\xi) + f_{x'x'}^\circ(\xi,t,0)Z_1^{(1)}(t,\xi)\right\}dt =$$
$$= \sum_{\kappa\geq 1}\left\{\frac{g_{\kappa x'x}^\circ(\xi,0)f_\kappa(\xi,0) - f_{\kappa xx'}^\circ(\xi,0)g_\kappa(\xi,0)}{2\kappa} - \frac{f_{\kappa xx'}^\circ(\xi,0)f_\kappa(\xi,0) + g_{\kappa xx'}^\circ(\xi,0)g_\kappa(\xi,0)}{2\kappa^2}\right\}.$$

So the condition of periodicity is defined by (38), and the Stability criterion follows from the investigation of the roots of eq.

$$\rho^2 - \mu\,(\mathcal{X}_1(\xi,\mu)\rho - \mathcal{X}_0(\xi,\mu)) = 0,$$

where

$$\mathcal{X}_1(\xi,\mu) = \mathcal{X}_1^\circ(\xi) + \mu\,\mathcal{X}_1^{(1)}(\xi),$$
$$\mathcal{X}_0(\xi,\mu) = \mathcal{X}_0^\circ(\xi) + \mu\,\mathcal{X}_0^{(1)}(\xi).$$

In our article we suppose that the functions $f(t, x, x', \ldots, x^{(n-1)}, \mu)$ are analytical functions of $(x, x', \ldots, x^{(n-1)}; \mu)$ in the neighbourhood of $(x = x_0, x' = 0, \ldots, x^{(n-1)} = 0, \mu = 0)$.

One can consider the more general case, when. in the neighbourhood of this point the function f has continuous partial derivatives as to variables $(x, x', \ldots, x^{(n-1)}, \mu)$ to some order τ incl.

In this case one can show by the above

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method that the expansion (32) give us asimptotical formulas and the order of these formulas is the heigher, the heigher is γ.

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ОБ УСЛОВИЯХ СУЩЕСТВОВАНИЯ ПРЕДЕЛЬНОГО ПЕРИОДИЧЕСКОГО РЕЖИМА ДЛЯ НЕЛИНЕЙНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Let

1) $A = const$, $K(x)$, $B(x)$ be n-square matrix functions; u, Ψ, ψ – n-dimensional vector functions;

2) $K(x) \equiv \sum_{i=1}^{k} Q_i(x) e^{\beta_i x}$, where $Q_i(x)$ is a polynomial of m_i-th degree with n-square matrix coefficients;

3) $B(x) \to A$ for $x \to \infty$;

4) $\varphi(s)$ – the characteristic polynomial (s.[1]) of the system of integro-differential eqations

$$\frac{du}{dt} = Au + \int_a^t K(t-s)u(s)\,ds \qquad (I)$$

D e f i n i t i o n s

1. The curve $v(t) = [v_1(t), \ldots, v_n(t)]$ of the phase space $u(u_1, \ldots, u_n)$ is said to envelop around the curve $Y(t) = [Y_1(t), \ldots, Y_n(t)]$, if

$$\| v(t) - y(t) \| \equiv \sum_{k=1}^{n} | v_k(t) - Y_k(t) | \to 0,$$

either for $t \to \infty$ or for $t \to -\infty$

2. The curve Γ of the phase space is called the limiting state for the system of integro-differential equations

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$$\frac{dx}{dt} = B(t)x + \int_a^t K(t-s)x(s)\,ds + \Psi[t, x(t)] + \int_a^t \Psi[t, s, x(s)]\,ds, \qquad (2)$$

if all the solutions of the system envelop around Γ either for $t \to \infty$ or for $t \to -\infty$. The vector functions ψ and Ψ are assumed to satisfy Lipschitz condition.

Assuming that

1) $f(t) \equiv \Psi(t,0) + \int_a^t \Psi(t,s,0)\,ds$ is a periodic (almost periodic) function;

2) the real parts of all the roots of the polynomial are negative, the existence of a periodic (almost periodic) limiting state for the system of integro-differential equations (2) is proved.

R e f e r e n c e

Ya.V.Bykov, On some problems of the theory of integro-differential equations, Frunze, 1957.

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О НЕКОТОРЫХ НЕЛИНЕЙНЫХ СИСТЕМАХ С D-СВОЙСТВОМ ЛЕВИНСОНА

§ 1. Introduction.* The real system of ordinary differential equations is said to have D-property /dissipative/ [1], if every solution $x = x(t)$ /$x - n \times 1$ vector/ for $t > T[x]$, belongs to some fixed boundary domain. In this work the sufficient conditions of existence of D-property are given for some non-linear system of differential equations. In addition a behavior of solutions of such system is studied. The previous achivements of the author [2], [3], and a theorem of S.A.Samjedova [4] are generalized in the work.

§ 2. Basic lemma. Let $x = (x_1, \dots, x_n) \in E^n$, $\mathcal{K} \subset E^n$, $f(x) = [f_1(x), \dots, f_n(x)] \in C^1(\mathcal{K})$. Let us denote through $\lambda(x)$ and $\Lambda(x)$ respectively the smallest and the largest characteristic numbers of a symmetrized Jacoby's matrix

$$\mathcal{J}_s(x) = \frac{1}{2}\left[f'(x) + f'^*(x)\right] = \frac{1}{2}\left(\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}\right). \qquad (2.1)$$

Then for any points x and $x + h$ such that $\xi = x + th \in \mathcal{K}$ with $0 \leqslant t \leqslant 1$ an inequality[1]

$$\lambda_m(h,h) \leqslant \left(f(x+h) - f(x), h\right) \leqslant \Lambda_M(h,h), \qquad (2.2)$$

is fulfilled, where $\lambda_m = \min_{\xi} \lambda(\xi)$ and $\Lambda_M = \max \Lambda(\xi)$.

Proof. We have $f(x+h) - f(x) = \int_0^1 \frac{d}{dt} f(x+th)\,dt = \int_0^1 f'(\xi)h\,dt$, where $\xi = x + th$. Hence $\left(f(x+h) - f(x), h\right) = \left(\int_0^1 f'(\xi)h\,dt, h\right) = \int_0^1 \left(f'(\xi)h, h\right)dt = \int_0^1 \left(\mathcal{J}_s(\xi)h, h\right)dt = \left(\mathcal{J}_s(\bar{\xi})h, h\right)$. (2.3)

From the formula /2.3/ evidently the inequality /2.2/ follows.

---

* Basic results of the work were reported at a Conference of Moscow Mathematical Society on 21/III 1961.

[1] If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ then we denote $(x,y) = \sum_j x_j y_j$ the scalar product and $|x| = (\sum_j x_j^2)^{1/2}$ — the Euclide norm.

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§ 3. The sufficient conditions of a dissipativbility of the system.

Theorem I. Let $\frac{dx}{dt} = f(x,t)$, /3.I/

where $f(x,t) \in C(E^n \times I^+)$, $I^+ = (\underline{t} < t < +\infty)$ /$\underline{t}$ - a real number or a symbol $-\infty$/, $f(x,t_0) \in C'(E^n)$ for every $t_0 \in I^+$ and $\widetilde{\Lambda}(x,t)$ - the largest characteristic number of generalized symmetrized Jacoby's matrix

$$\widetilde{f}_J(x,t) = \frac{1}{2}\left\{A f'_x(x,t) + [A f'_x(x,t)]^*\right\},$$

where $A = (a_{ij})$ - some positively defined constant $n \times n$ matrix.

If:

I/ $\widetilde{\Lambda}(x,t) \leqslant -\alpha < 0$ for $|x| > R_0$, $t \in I^+$

and $\widetilde{\Lambda}(x,t) \leqslant \beta$ for $|x| < R_0$, $t \in I^+$

/α and β are positive numbers/;

2/ $|f(0,t)| \leqslant c < +\infty$ /3.2/

then the system /3.I/ has D-property.

Remark. The condition of type I/ is considered by N.N. Crasovsky [5] in connection with stability questions.

Proof. We put $v(x) = (Ax,x)$ /3.3/

and let $x_\rho = \frac{R_0}{|x|} x$.

For a solution $x = x(t)$ for $|x| > 2R_0$ we have $\frac{1}{2}\frac{dv}{dt} = (Af(x,t) - Af(x_\rho,t), x - x_\rho) + (Af(x_\rho,t) - Af(0,t), x_\rho)\cdot\frac{|x| - R_0}{R_0} + (Af(x,t) - Af(0,t), x)\cdot\frac{R_0}{|x|} + (Af(0,t), x)$.

Hence, using the basic lemma and taking into account the conditions I/ and 2/ we get

$$\frac{dv}{dt} \leqslant -\frac{\alpha}{2}|x|^2 + (4\beta R_0 + 2\rho c)|x|, \qquad /3.4/$$

where $\rho = \|A\| = \left(\sum_{i,j} a_{ij}^2\right)^{1/2}$

if only $|x| \geqslant R_1 > 2R_0$. From inequality /3.4/ and on the basis of elementary reasoning, or by force of Iosidzava's theory [6], it follows that for each solution $x = x(t)$ of the system /3.I/ where R is sufficiently large the inequality

$$|x(t)| < R \quad \text{by} \quad T_x < t < +\infty. \qquad /3.5/$$

will be constantly fulfilled

Corollary. If $f(0,t) \equiv 0$ and in place of the condition I/ an inequality is fulfilled

$$\widetilde{\Lambda}(x,t) \leqslant -\alpha(z) < 0 \quad \text{for} \quad |x| \geqslant z > 0, \quad t \in I^+, \qquad /3.6/$$

where $\alpha(z)$ - monotonuous nondecreasing function of z $(0 < z < +\infty)$, then a trivial solution $x \equiv 0$ of the system /3.I/ is assimptotically stable in large when $t \to +\infty$.

In reality, it is easy to check, that if the condition /3.6/ exists then the function v /3.3/ is an infinitely large Ljapunov's function of the second type.

The corollary is a generalisation of the Crasovsky's theorem [5].

§ 4. The sufficient conditions of the existence of the almost periodic solutions of the non-linear system with D-property.

Theorem 2. Let $\frac{dx}{dt} = f(x,t)$, /4.I/

where $f(x,t) \in C(E^n \times I)$, $I = (-\infty < t < +\infty)$, $f(x,t_0) \in C'(E^n)$ for any $t_0 \in I$; $\widetilde{\Lambda}(x,t)$ having the sense mentioned in the theorem I. If

I/ $\widetilde{\Lambda}(x,t) \leqslant -\alpha < 0$ for $(x,t) \in E^n \times I$ /α - a positive constant/;

2/ $|f(0,t)| \leqslant C < +\infty$ then I/ there exists a solution $\xi = \xi(t)$ of the system /4.I/ bounded on I; 2°/ $\xi(t)$ is exponentially stable in large if $t \to +\infty$ on each interval $t_0 \leqslant t < +\infty$ where $t_0 > -\infty$;

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3'/ if moreover $f(x,t)$ is an almost periodic vector-function, uniform with respect to x for $|x| \leq R$ where R is a such positive number that $|\xi(t)| \leq R$ then the bounded solution $\xi(t)$ is a almost periodic by t.

Proof. I'/ The existence of the solution $\xi = \xi(t)$ bounded for $-\infty < t < +\infty$ is proved similarly to that it has been made in a scientific author's work [2] and is based on Lusin's idea [7]. For this purpose the topological Vazevsky's principle also may be used [8].

2'/ Let $x = x(t)$ is a solution of the system /4.1/ which is determined by an initial condition $x(t_o) \in E^n$ We put $v = \left(A(x-\xi),\ x-\xi\right)$.

Hence applying the basic lemma and the condition I/ we get

$$\frac{1}{2}\frac{dv}{dt} = \left(Af(x,t) - Af(\xi,t),\ x-\xi\right) \leq -\alpha |x-\xi|^2 \qquad /4.2/$$

Let $\alpha = \lambda_{min}(A)$ and $\beta = \lambda_{max}(A)$.

From the inequality /4.2/ we have $\frac{dv}{dt} \leq -\frac{2\alpha}{\beta} v$ and consequently

$$|x(t)-x(\xi)| \leq \sqrt{\frac{\beta}{a}} |x(t_o)-\xi(t_o)| e^{-\frac{\alpha}{\beta}(t-t_o)} \qquad /4.3/$$

for $t \geq t_o$.

Thus the solution $\xi(t)$ is exponentially stable for $t \to +\infty$. In addition from the formula /4.3/ it follows that the bounded solution $\xi(t)$ is unique.

3'/ Let $|\xi(t)| \leq R$ and the function $f(x,t)$ is uniformly almost periodic by t for $|x| \leq R$. Denote by $\tau = \tau(\varepsilon^2)$ an uniformly almost-period of the function $f(x,t)$ to within ε^2 that is

$$|f(x,t+\tau) - f(x,t)| < \varepsilon^2 \qquad /4.4/$$

for $-\infty < t < +\infty$ and $|x| \leq R$.

We put $\Delta_\tau \xi(t) = \xi(t+\tau) - \xi(t)$ and let

$$u = \left(A\Delta_\tau \xi(t),\ \Delta_\tau \xi(t)\right).$$

By force system /4.1/ we have

$$\frac{1}{2}\frac{du}{dt} = \left(Af[\xi(t+\tau), t+\tau] - Af[\xi(t), t+\tau], A\Delta_\tau\xi(t)\right) + \left(A\{f[\xi(t), t+\tau] - f[\xi(t), t]\},\ \Delta_\tau \xi(t)\right).$$

Hence using the condition I/ and the basic lemma, and also the inequality /4.4/, we get

$$\frac{1}{2}\frac{du}{dt} \leq -\frac{\alpha}{\beta} u + 2\|A\| R \varepsilon^2.$$

Consequently

$$u(t) \leq u(t_o) e^{-\frac{2\alpha}{\beta}(t-t_o)} + \frac{2\beta R \|A\|}{\alpha} \varepsilon^2$$

for $t \geq t_o$

Putting $t_o \to -\infty$ from the inequality /4.5/ we find

$$u(t) \leq \frac{2\beta R \|A\|}{\alpha} \varepsilon^2$$

and so $|\Delta_\tau \xi(t)| = |\xi(t+\tau) - \xi(t)| \leq \kappa\varepsilon$, where $\kappa = (2\beta R\|A\|/\alpha\alpha)^{1/2}$ Thus the solution $\xi(t)$ is almost periodic.

This result also follows from a Halany's [9] theorem.

Remark. If $f(x,t)$ is periodic by t with a period T, then the bounded solution $\xi(t)$ is also periodic with period T.

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А.Н.Филатов.О применении рядов Ли к задачам нелинейной механики

The purpose of this paper is to show that it is suitable to apply Lie series /I/ in the asymptotic methods of nonlinear mechanics /2/; /3/ for the integration of differential equations of the first, second and following approximations.

I. We consider non-linear oscillatory systems with slowly altering parameters. Let the differential equation, describing tne oscillations of such systems, have the following form

$$\frac{d}{dt}\left[\alpha(\tau)\frac{dx}{dt}\right]+\beta(\tau)x=\varepsilon f\left(\tau, x, \frac{dx}{dt}\right) \qquad /I.I/$$

where $\tau=\varepsilon t$, $\varepsilon>0$ - is a small parameter, and the functions $\alpha(\tau)$ and $\beta(\tau)$ never vanish. /In the following we assume the functions α, β and f to be analytic with respect to all arguments/.

The asymptotic solution of the equation /I.I/ for the first approximation is to be found /2/, /3/ in the form

$$x = a\cos\psi \qquad /2.I/$$

where $a(t)$ and $\psi(t)$ are the solutions of the equations of the first approximation:

$$\frac{da}{dt}=-\frac{\varepsilon a}{2\alpha(\tau)\omega(\tau)}\frac{d\alpha(\tau)}{d\tau}+\frac{\varepsilon}{2\pi\alpha(\tau)\omega(\tau)}\int_0^{2\pi} f(\tau, a, \psi, -a\sin\psi)\sin\psi\, d\psi$$

$$\frac{d\psi}{dt}=\omega(\tau)-\frac{\varepsilon}{2\pi\alpha(\tau)\omega(\tau)}\int_0^{2\pi} f(\tau, a, \psi, -a\sin\psi)\cos\psi\, d\psi \qquad /3.I/$$

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In the system /3.I/ the notations

$$\omega(\tau)=\sqrt{\frac{\alpha(\tau)}{\beta(\tau)}} \qquad /4.I/$$

$$f_0(\tau,a,\psi)=f(\tau,a,\cos\psi,-a\omega\sin\psi)$$

are introduced.

Now we designate for the sake of brevity the right-hand sides of equations /3.I/ by $F(\tau,a)$ and $\Phi(\tau,a)$, and we write /3.I/ in the form

$$\frac{da}{dt}=F(\tau,a),\quad \frac{d\psi}{dt}=\Phi(\tau,a),\quad \frac{d\tau}{dt}=\varepsilon \qquad /5.I/$$

Then we show how the system /5.I/ may be integrated in its general form

Let for the moment $t=0$ be given

$$a(0)=a_0,\quad \psi(0)=\psi_0,\quad \tau(0)=\tau_0=0 \qquad /6.I/$$

We notice, that in the following the quantities a_0,ψ_0,τ_0 should be considered as parameters, all calculations should be carried out with arbitrary a_0,ψ_0,τ_0 and a_0,ψ_0,τ_0 should be replaced by their numerical values in the final result only. To solve the system /5.I/ at the start conditions /6.I/ we introduce the operator

$$D=\varepsilon\frac{\partial}{\partial\tau_0}+F(\tau_0,a_0)\frac{\partial}{\partial a_0}+\Phi(\tau_0,a_0)\frac{\partial}{\partial\psi_0} \qquad /7.I/$$

Then, as it will be shown in the following, the solutions of the system /5.I/ will have the form

$$a(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}D^n a_0,\quad \psi(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}D^n\psi_0$$

$$\tau(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}D^n\tau_0\equiv\varepsilon t. \qquad /8.I/$$

Computing the coefficients of series /8.I/ we have

$$D^0a_0=a_0,\ Da_0=F(\tau_0,a_0),$$
$$D^2a_0=D(Da_0)=\varepsilon\frac{\partial F}{\partial\tau_0}+F\frac{\partial F}{\partial a_0},$$
$$D^3a_0=D(D^2a_0)=\varepsilon^2\frac{\partial^2F}{\partial\tau_0^2}+2\varepsilon F\frac{\partial^2F}{\partial\tau_0\partial a_0}+$$
$$+\varepsilon\frac{\partial F}{\partial\tau_0}\frac{\partial F}{\partial a_0}+F^2\frac{\partial^2F}{\partial a_0^2}+F\left(\frac{\partial F}{\partial a_0}\right)^2,$$

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$$D^0\psi_0=\psi_0,\ D\psi_0=\Phi(\tau_0,a_0),$$
$$D^2\psi_0=D(D\psi_0)=\varepsilon\frac{\partial\Phi}{\partial\tau_0}+F\frac{\partial\Phi}{\partial a_0},$$
$$D^3\psi_0=D(D^2\psi_0)=\varepsilon^2\frac{\partial^2\Phi}{\partial\tau_0^2}+2\varepsilon F\frac{\partial^2\Phi}{\partial\tau_0\partial a_0}+$$
$$+\varepsilon\frac{\partial F}{\partial\tau_0}\frac{\partial\Phi}{\partial a_0}+F\frac{\partial^2\Phi}{\partial a_0^2}+F\frac{\partial F}{\partial a_0}\frac{\partial\Phi}{\partial a_0}.$$

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$$D^0\tau_0=\tau_0\ ,\ D\tau_0=\varepsilon\ ,\ D^2\tau_0=0,$$
$$D^3\tau=0,\ \ldots\ldots\ D^n\tau_0=0,\ \ldots\ldots \qquad /11.1/$$

Remark 1. As the function ψ is not contained in the right-hand sides of equations /5.I/, the first equation and the third one may be integrated independently of the second, i.e.

$$a(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}D^n a_0\ ,\ \tau=\sum_{n=0}^{\infty}\frac{t^n}{n!}D^n\tau_0\equiv\varepsilon t,$$

where

$$D=\varepsilon\frac{\partial}{\partial\tau_0}+F(\tau_0,a_0)\frac{\partial}{\partial a_0}$$

Then ψ will be determined from the equality

$$\psi(t)=\int_0^t\Phi(\varepsilon t,a(t))dt+\psi_0$$

Remark 2. The equations of the second approximation and the following ones may be integrated by the same method as the equations of the first one.

2. The proof of the method. The immediate substitution of series /8.I/ in the equation /5.I/ shows, that these equations are really satisfied by series /8.I/. The absolute and uniform convergence of the series /8.I/ may be proved by the same method as in [1].

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This paper discusses the main results of the examination of the effects of the forced oscillations in the systems with nonlinear characteristics of the resilient couplings and in the systems with periodical change parameters on analogous electronic machines. The forced oscillations are realized by a source of energy with predeterminated characteristics. The relation of the vibration system to the energy source characteristics and its influence on the stability stationary periodical oscillations are considered.

As long ago, as in 1904 year Zommerfeld [1] noticed some peculiarities of the resonance oscillations of the mechanical system containing a motor, as the energy source of forced vibrations. In recent years a number of theoretical works dedicated to the studing of the influence of the energy source on the forced vibrations were published with a modern prezentation of the problem. The interaction of the linear vibration system with the source of energy was studied by Rocard [2], Pust L. [3] Blehman I.I., Dganelidje G.U. [4, 5] and others.

The most extentive study of the problem of the interaction of the energy source with linear, nonlinear, parametrical and other systems was undertaken by Kononenko V.O. [6 - 9]. It appears, that in many cases the oscillations of the system influenced on the source of energy and this influence is most essential at resonance conditions. The examination of the oscillation system in jointly with the energy source brings to quite new conceptions of scientific and practical interest. So, for example, the idea about

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the stable and instable state of nonlinear vibration systems and about unlimited increase of the amplitudes of parametrical systems in resonance well known from classical scientific literature is correct only for particular cases and corresponds to the source of energy (motor) with unlimited capacity safety. With the corresponding choice of the characteristics of the source of energy it is possible to receive stable oscillations where they were supposed instable and vice versa; it is possible to receive limited amplitude of the parametrical oscillations in resonance, as well.

This work is dedicated to the subsequent development of the resonance effects of the oscillation systems depending on the source of energy of the forced oscillations. The equations of motion of nonlinear and parametrical systems was studed on analogous electronic machines and mechanical models, with the application of the main results, received in [6-9].

Two systems was examined: a)-the system with nonlinear characteristic of the resillent couplings and b)- the system with periodical changing parameters. We shall consider each of them separately.

I. N o n l i n e a r s y s t e m.

We have modeled a nonlinear system with one degree of freedom, excited by the inertia forces of the unbalanced

mass m_1, rotating at the distance r, from the rotating axis, and mounted on the rotor of the motor. The motor is the energy source of the forced oscillations of the system considered. The inertia moment of the rotor is I. The rotating moment M depends upon the rotating speed Ω. The vibrating system contains mass, m and the stiffener with nonlinear characteristic, that is expressed as following:

$$F(y) = Cy \pm f(y) \qquad \text{where } f(y) = \pm\gamma y^3$$

The sign "plus" corresponds to the "stiff" characteristic of the restorting elastic force. The sign "minus" corresponds to the "soft" characteristic .

The parameters of the system examined are indicated in the following table:

m	m_1	I	C	r	α	γ
$6{,}09\cdot10^{-3}$	$0{,}305\cdot10^{-3}$	$10\cdot10^{-3}$	77,7	0,75	34,4	0 ÷330
$kg\,sec^2.cm^{-1}$	$kg.sec^2.cm^{-1}$	$kg.cm.sec^2$	$kg.cm^{-1}$	cm.	kg.sec.cm.	$kg.cm^{-3}$

Note: Parameters m, C, γ were variated for different variants of modelation.

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The equations of motion such a system with the influence of the energy source characteristics taken into account will be:

$$m\ddot{y} + Cy + f(y) + d\dot{y} = m_1 r\ddot{\varphi}\sin\varphi + m_1 r\dot{\varphi}^2\cos\varphi$$
$$J\ddot{\varphi} - M(\dot{\varphi}) = m_1 r\ddot{y}\sin\varphi + m_1 g r\sin\varphi \quad \text{.. (I)}$$
$$\varphi = \Omega t$$

where $M(\dot{\varphi}) = M_0 + N\dot{\varphi}$ - static characteristics of the motor, N - its angular coefficient. For the convenience of modelation we shall reduce the system (I) to the following system of equations:

$$\frac{dx_1}{d\tau} = K_{11}x_2 ;$$

$$\frac{dx_2}{d\tau} = -K_{21}x_1 - K_{22}f(x_1) - K_{23}x_2 + K_{24}\frac{dx_4}{d\tau}\sin x_3 + K_{25}x_4^2\cos x_3 , \quad \cdots\cdots (2)$$

$$\frac{dx_3}{d\tau} = K_{31}x_4 ,$$

$$\frac{dx_4}{d\tau} = K_{41}M(x_4) + K_{42}\frac{dx_3}{d\tau}\sin x_3 + K_{43}\sin x_3$$

where $y = x_1$, $\dot{y} = x_2$, $\varphi = x_3$, $\Omega = x_4$, $\tau = 100t$;

τ - time of the modelation, t - natural time.

K_{11}, K_{21}, K_{22}.... and so on are coefficients of the modeling.

The main attention in the programme of modeling was paid to the influence of the source of energy on:

a) motion of the system in the stationary regimes with the formas: $x = a\cos(\Omega t + \xi) + u_1(\Omega t)$; $\varphi = \Omega t + u_2(\Omega t)$; where $a = Const$, $\xi = Const$, $\Omega = Const$;

$u_1(\Omega t), u_2(\Omega t)$ - small periodical functions of t, and also in the event of a substantial increase of the nonlinear parameter of γ and a substantial departure from the resonance, i.e. outside the limit taken into consideration in the works [7, 9]

b)-the character of nonstationary regimes of motion appearing near the boundaries of the stability region and correspond to the transition of the system from one stationary regime of motion to another.

c) - the peculiarities of the motion of the system when stable criterioums [8] are not satisfied.

Before the experiments the characteristics of resistance force to the oscillations motion:

$$S(\Omega) = \frac{d\omega}{4\Omega}(\omega + \Omega)\alpha^2$$ were constructed

with the use of method [8].

The expectly stationary regimes of motion were determined as points of intersections of graph $S(\Omega)$ with characteristics of source of energy M (Ω).

The characteristic of source energy was taken as linear function $M(\Omega) = Mo + N\dot{\varphi}$, that permited an easy change its steepness $N = \frac{dM(\Omega)}{d\Omega}$

The value of N was changed from the region $-\infty < N < 0$

For the value of N, that was given by the programme of modelation; two series of experiments were carred out:

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- the experiments with the gradually slow increase of the frequency Ω , by moving the characteristic of the energy source $M(\Omega)$ to the right side parallel to itself.

- the experiments with the gradually slow decrease of the frequency Ω , by moving the characteristics to the left side parallel to itself.

The initial conditions for X_{04} in the 1st series of experiments were above $0 \div 10$ volt and in the second - 100 volt, (100 volt corresponds to maximum of the frequency Ω).

The 1st series of experiments permitted us to recieve the data for the construction of the resonance curve under the increasing frequency Ω and the second series permitted us to recieve the data for the construction of the resonanse curve under the decreasing frequency Ω .

The recording of the frequency $X_4 = \Omega$ and amplitude $X_1 = A$ in the experiments was made simultaneous for the transient and stationary process. The transient process arose in the system at the moment of the switching and proceeded till the stationary regimes had set in the system.

The resonance curves were constructed with the use of oscillogram data of stationary regimes. Nonstationary regimes are not considered here. But such transient processes, that arise near the stable region and characterise to the passing of the system from one stationary regime to another - the so-called "frustrations" of motion - have a special interest, and some of them are considered here.

Besides, the transient non-stable processes which depend on the characteristics of the source of energy $M(\Omega)$ were considered.

The interval between the positions of characteristics $M(\Omega)$ in their moving was given before hand by the programme of modelation and by the calculations.

We shall state on paper only the main results without details for the system with "stiff" ($\gamma > 0$) and "soft" ($\gamma < 0$) characteristics of nonlinear restoring force in connection with the specifics of the resonance phenomenons in the nonlinear system separately.

The system with "stiff" characteristics of nonlinear restoring force ($\gamma > 0$)

The results of modelation of the "stiff" nonlinear system are represented in Fig. 1, 2.

The resonance curve is constructed using datas of oscillograms of stationary stable regimes of motion. The points in Fig. 1 correspond to a calculation datas and to intersection characteristic $M(\Omega)$ with graph $S(\Omega)$ for a rather wide region of values N (the shaded sector 1 in Fig. 2).

With the movement of the characteristic of energy source $M(\Omega)$ from left to right the point F (point of intersection of characteristics $M(\Omega)$ with graph $S(\Omega)$ moves along the graph $S(\Omega)$ from left to right in the direction.

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Point of the resonance curve moves in the same direction along branch BD.

Near by point D the stationary regime breaks and the system rather guickly passes to a new stationary regime of motion from the left to right branch of the resonance curve. This transition essentialy depends on incline N of characteristics $M(\Omega)$. Stationary regimes has set at points I, II,III (Fig. I.) These points correspond to the points of intersection of characteristics $M_1(\Omega)$, $M_2(\Omega)$, $M_3(\Omega)$ with graph $S(\Omega)$ (points I, II, III, Fig. 2).

With decreasing of /N / the points of stationary oscillations will move from left to right along the axis of Ω .

Oscillograms in Fig. 3a and in Fig. 3b give us an obvious idea about the transition process and stationary regimes at the points II, III, for $M_2(\Omega)$ and $M_3(\Omega)$

With the decreasing of Ω , when characteristic $M(\Omega)$ moves from right to left along the axis Ω , the points of stationary regimes move in the same direction up till point E.

At the point E the system passes to the points of the crossing characteristic $M_1(\Omega)$, and the peak of the resonanse curve is lost.

The changes of amplitude X_1 and velocity X_4 at the nonstationary pass of the system from the point E to points IV, V, VI are presented on the oscillograms in fig. 6, a,b.

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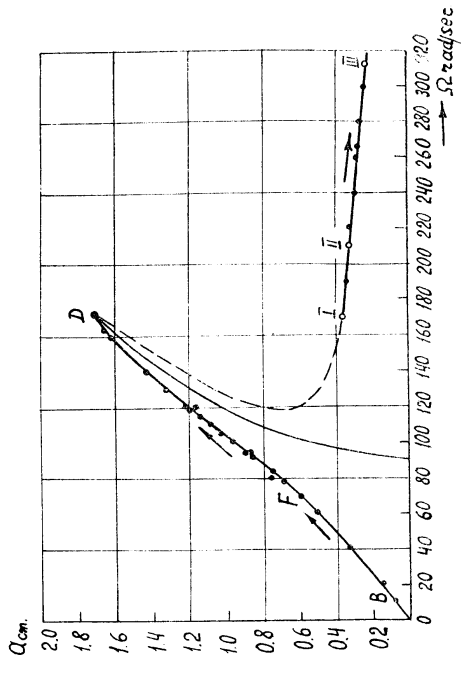

Fig. I Resonance curve of the system with $\gamma > \delta(\Omega$ is increasing).

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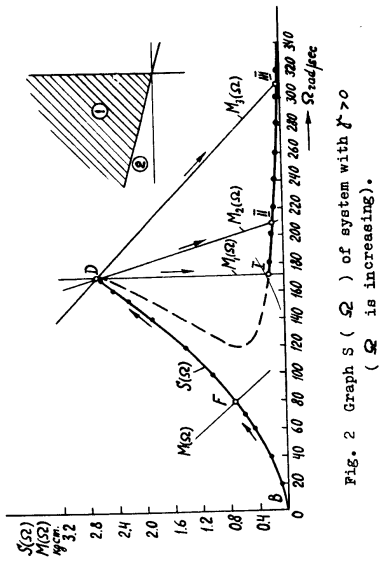

Fig. 2 Graph S (Ω) of system with $\gamma > 0$ (Ω is increasing).

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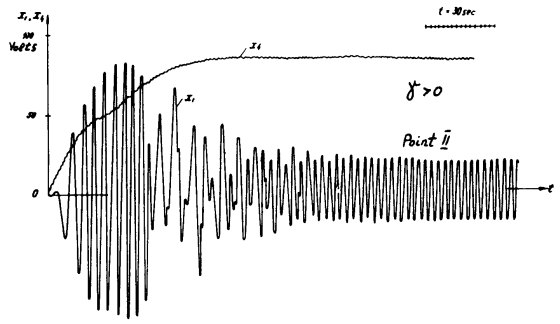

Fig. 3a

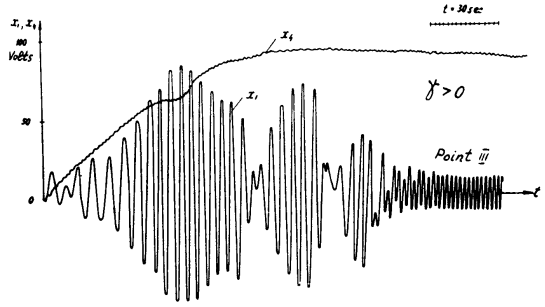

Fig. 3b

Fig. 3 Oscillograms of nonstationary processes at frustrations of oscillations in the system with $\gamma > 0$ a) for the characteristic M_2 (Ω) and b) for the characteristic M_3 (Ω), (Ω is increasing).

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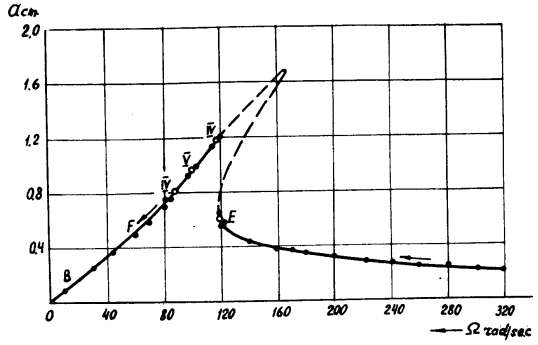

Fig. 4 Resonance curve of the system with $\gamma > 0$ (Ω is decreasing)

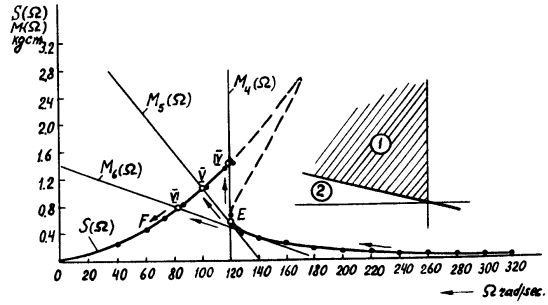

Fig. 5 Graph S (Ω) of system with $\gamma > 0$ (Ω is decreasing)

It was impossible to succeed in creating a stationary oscillation in the model for very slight slope and horizontal characteristi c.

Hatched sectors 2 in fig. 2 and fig. 5 represent the regions of these charactersitics.

The oscillograms shaping the changes in magnitude of amplitude and frequency of oscillations for the characteristics M (Ω) within sector 2 /fig. 2 and fig. 5/are presented in fig. 7.

The system with soft characteristic of nonlinear restoring force ($\gamma < 0$)

The results of modeling of this system which was carried out as well as the obove mentioned are represented in fig.8.

Four regions of values of N with qualitatively differentiating regimes of system's motion were established when the incline of characteristic was varied from N = 0 to N = $-\infty$. These regions pictured in the shape of sectors 1-4 are presented in fig. 8.

The region I contains "steep" characteristics, which are placed near the vertical straght line. The system with such characteristics of energy source behaves itself at the resonant point in the same way as it known for the system with given frequency of exciting force.

Stationary regimes of motion received in the model for

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the characteristics with incline in the limits of sector I, and drawn on the oscillograms data are shown knots at the curve S (Ω) in fig. 8. Here the corresponding branch of resonanse curve and i.e. graph $S(\Omega)$ is shown by a dotted line.

The point I corresponds to the frustraction (the breakdown) of oscillations with the characteristic's motion from left to right and its reaching the point E.
The point II. corresponds to the frustraction of oscillations from point T_1.

Characteristics M (Ω) with incline in the limits of sector 2 are tangents to the branches of graph S (Ω) and stable stationary oscillations with them are impossible for the greater part of resonant curve.

Characteristics M (Ω) with an incline in the limits of sector 3 provide the possibility to create stationary oscillation regimes at the rising branch of the resonant curve(sections Q T_1, Q T_2, Q T_3 in fig. 8) by the increasing of frequency i.e. where the regimes of oscillations with characteristic incline in the limits of sector I were unstable).

The oscillogram of stable oscillation at the point T_2 at M (Ω) placed in sector 3 is presented in fig. 10, b.

At small values of $|\gamma|$ the left branch of the resonant curve is realized up till to the summit T_1.

Maximum availible magnitude of stationary oscillation

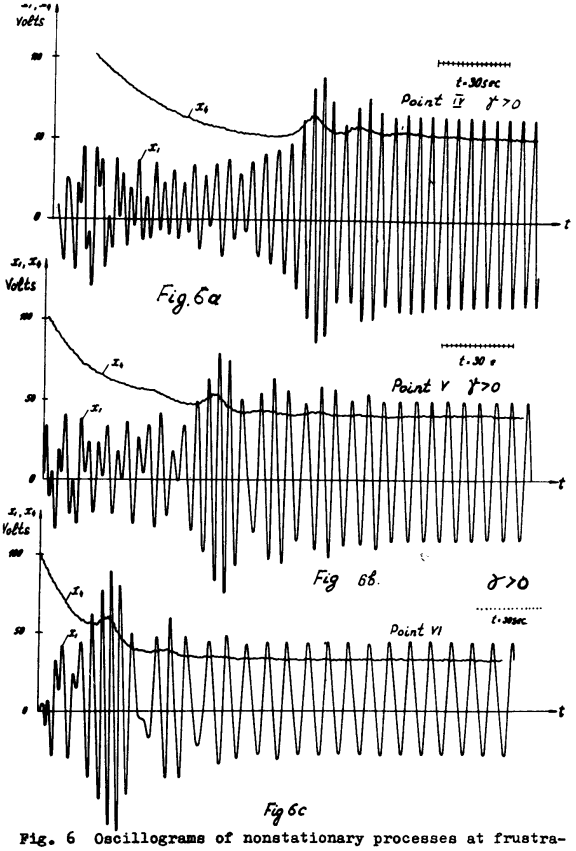

Fig. 6 Oscillograms of nonstationary processes at frustrations of oscillations in the system with $\gamma > 0$, a) for the characteristic M_4 (Ω), b) for charadteristic M_5 (Ω) c) for characteristic M_6 (Ω). (Ω is

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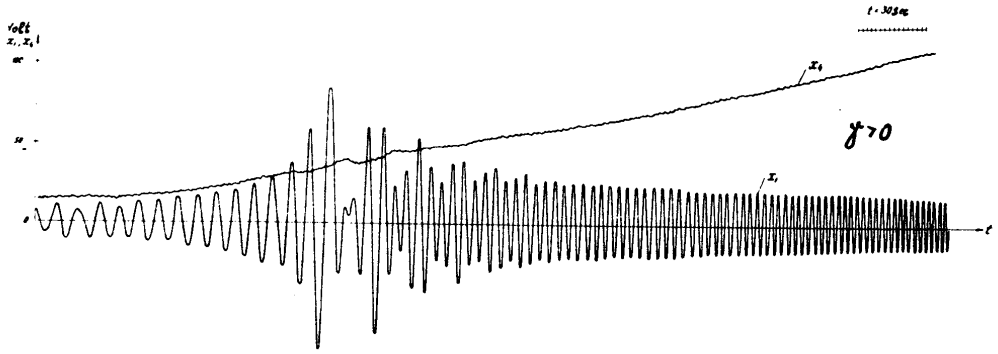

Fig. 7 Oscillograms of nonstationary processe for characteristic M (Ω) within sector 2, Fig. 2,5 , in the system with $\gamma > 0$

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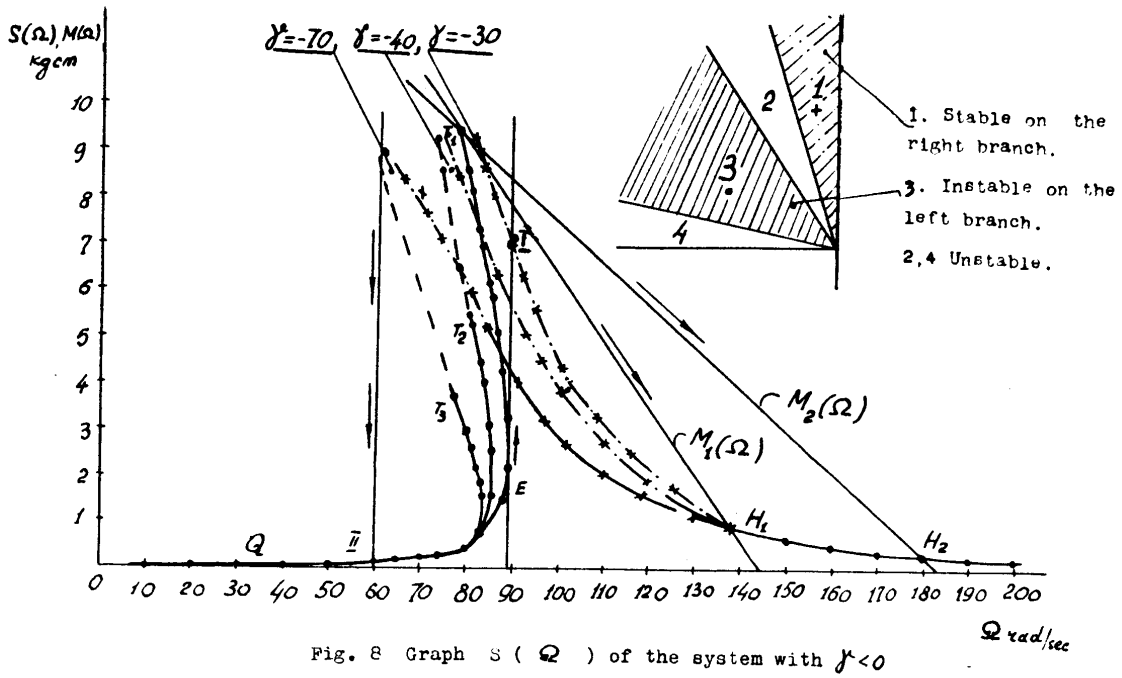

Fig. 8 Graph S (Ω) of the system with $\gamma < 0$

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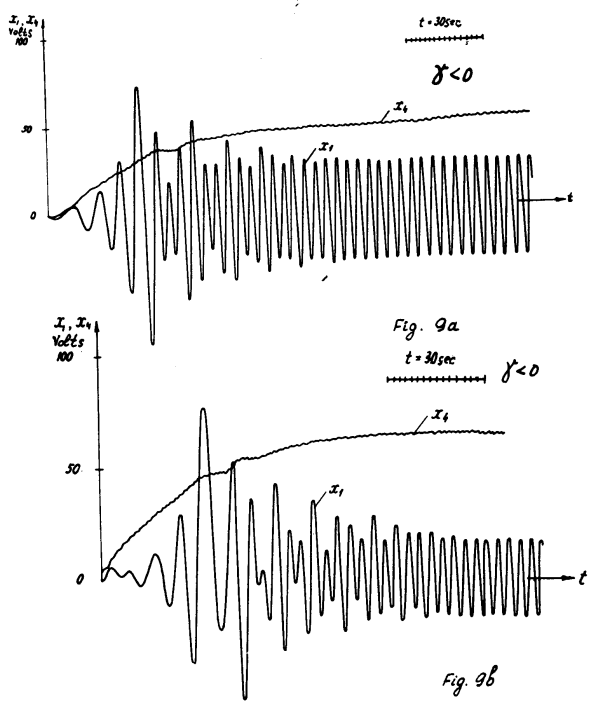

Fig. 9 Oscillograms of nonstationary processes at fruistrations of oscillations in the system with $\gamma<0$ a) from point T_1 to point H_1 and b) from fpoint T_1 to point H_2 .

at the left branch of resonant curve / points T_2, T_3 in fig. 8/ decreases with the increasing of nonlinearity /γ/

When the charateristics M (Ω) with incline in the limits of sector 3 reach the point T_1 the motion regime of system changes from T_1 to points near-by H_1, H_2 for the characteristics M_1 (Ω), and M_2 (Ω) respectively. The break-down of oscillation at the points T_2 and T_3 occur by similar manner.

When the frequency Ω is decreasing / by the sucesessive replacement of characteristic M (Ω) to the left / stable oscillations are available at the right branch of the resonant characteristic till M (Ω) has reached point R.

In the region near of point R the system looses stability and does an unstationary passing to the region near point P.

Stable oscillations corresponding to the crossing of curve M (Ω) by graph S (Ω) are realized at section P Q of the left branch of the resonant curve by the following replacement of characteristic M (Ω) to the left.

The disposition of unstationary motion at the break-down of oscillation for characteristics incline within sector 3 is presented in oscillograms in fig. 9.

I - is the passing from point T_1 to H_1 ; point II - is the passing from point T_2 to H_1 point.

With the characteristics of a very slight slope, which pass near the horizontal line and have incline within

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sector 4 / fig. 8 / it is impossible to obtain stable stationary regimes of oscillation at the resonant point.

Typical regimes motion of system with inclines of characteristics within sector 4 are shown in fig. 10, α.

II. The parametrical system

The parametrical system represented by the resilient bar was modeled. With the bar under action of periodical force in the direction of axis of the bar and therefore the bending stiffness of the bar experiencing the periodical changes. The source of energy (motor) with definite characteristic M (Ω) realizes these periodical changes. When the frequency of periodical force is twice above that of the bar itself the parametrical resonance, corresponding to the main instablility region, will be in the system.

The motion equations of such a system, taking into account the dependence between the rotating motion of the rotor of motor and oscillating motion will be:

$$m\ddot{y} + d\dot{y} + (c + c_3 \sin 2\varphi) y + m\gamma y^3 = 0 ; \qquad \ldots (3)$$

$$J\ddot{\varphi} - M(\dot{\varphi}) = -\frac{1}{2} c_2 y^2 \cos 2\varphi - \frac{1}{2} c_3 \sin 2\varphi - c_4 \cos \varphi ,$$

where $M(\dot{\varphi}) = M_0 + N\dot{\varphi}$ - is the static characteristic of the motor, N - is its angular coefficient, and $d\dot{y}$ - is the small resistance force to the oscillation motion.

$$c_2 = \frac{\pi^2 \eta}{2\ell} c_1 , \quad c_2 = \eta^2 c_1 , \quad c_4 = f_0 \eta c_1 , \quad \gamma = \frac{\pi^4 c_1}{8\ell^2 m} ,$$

$$\omega^2 = \frac{\pi^4}{\ell^4 m_1} \left(1 - \frac{P_0}{P_1}\right) E J_x , \quad P_0 = f_0 c_1 , \quad P_1 = \frac{\pi^2}{\ell^2} E J_x ;$$

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The following parameters of the system were accepted:

Parameters	Sygn	Value	Terms
Length of bar	L	125	cm.
Specific weight of material of the bar.	γ_0	$7,85 \cdot 10^{-3}$	$kgcm^{-3}$
Area of cross section of the bar.	F	0,67	cm^2
Elasticity modulus of the bar material	E	$2 \cdot 10^6$	$kgcm^{-2}$
The mass of half the longitude of the bar.	m	$0,334\ 10^{-3}$	$kg.sec^2\ cm^{-1}$
The stiffness of spring which transmits the excitation from the motor to the bar	C_1	26	$kg.cm^{-1}$
Static spring deformation	f	0,2	cm
The raduis of crank, that deformated the spring	r	0,2 ; 0,5	cm
The moment of inertia of the cross section of the bar.	I_x	0,049	cm^4
The moment of inertia of the motor's rotor.	I	$10\ 10^{-3}$	kg.cm.sec.

Comment: Values L,m, C_1 were changed in the process of modelations correspondingly to the different alternates.

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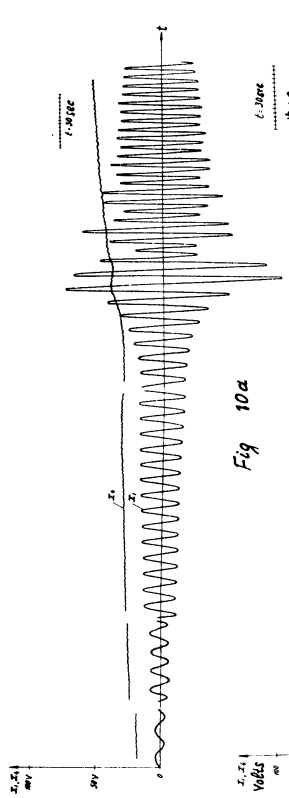

Fig 10a

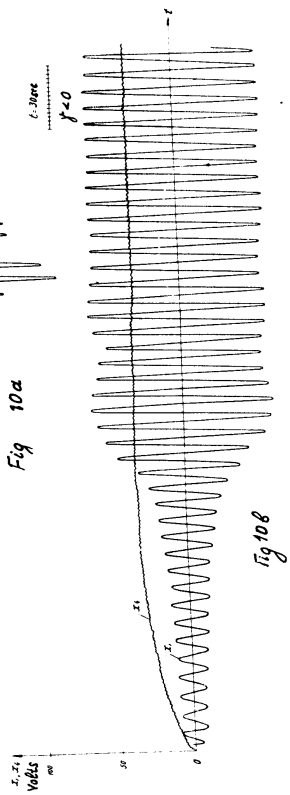

Fig 10b

Fig. 10 Oscillograms of oscillations in the system with $\gamma<0$
a) for characteristic M (Ω) within sector 4, and b)
within sector 3, fig. 8.

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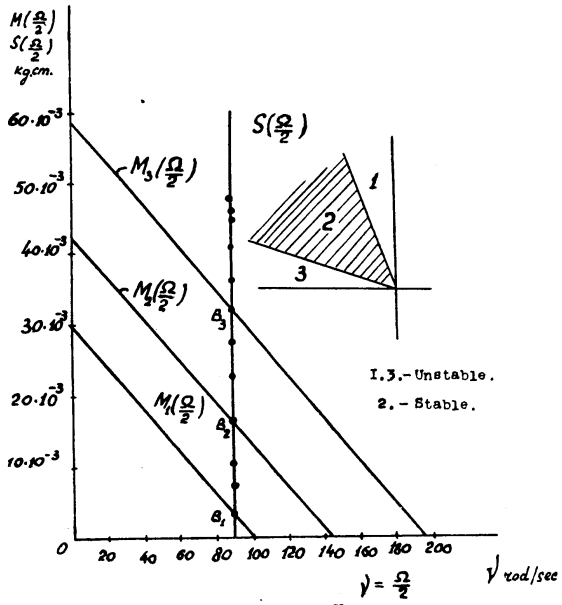

Fig. 11 Graph S (Ω) of linear parametrical system ($\gamma = 0$) with characteristics M (Ω).

The system of electronic machine equations which is an equivalent to the system (3) has the following shape:

$$\frac{dx_1}{d\tau} = K_{11} x_2 ;$$

$$\frac{dx_2}{d\tau} = -K_{21} x_2 - K_{22} x_1 - K_{23} x_1 \sin x_3 - K_{24} x_1^3 ; \qquad \ldots(4)$$

$$\frac{dx_3}{d\tau} = K_{31} x_4 ;$$

$$\frac{dx_4}{d\tau} = K_{41} M(x_4) - K_{42} x_1^2 \cos x_3 - K_{43} \sin 2x_3 - K_{44} \cos x_3 ;$$

Here the exchange of the variables is made and the time of the modelation was introduced $\tau = 100t$, t - is the natural time. K_{11}, K_{21}, K_{22}, K_{23} ... - and so on - are the coefficients, which were obtained with the modeling scales taken into account.

By the above mentioned programme of modelation two series experiments were made for three magnitudes of the nonlinear parameter - $\gamma = \frac{\pi^4 C_1}{8\ell^2 m}$;

1) $\gamma = 0$ - linear parametrical systeme.
2) $\gamma > 0$ - "stiff" nonlinear restoring force.
3) $\gamma < 0$ - "soft" nonlinear resorting force.

The first series of experiments was made, as it was above mentioned for the nonlinear system, with a slowly increase of the frequency Ω - the second series was made with a decrease of the frequency Ω .

The results of the modelation of parametrical system with a linear resilient force are represented in fig. II. The graph of the force of resistance to the oscillation

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motion is drawn with the use of data of oscillograms. Three specific regimes of the motion of the system under resonance were discovered which were depended on the incline of the characteristic of an energy source M (Ω). These three regions of the characteristic M (Ω) inclines are shown in the fig. II as three sectors. Sector I corresponds to the "steep" characteristics M (Ω) which is placed near the vertical straight line. For the characteristics with inclines within the sector I the amplitude of resonance vibrations increases without limit.

The stationary stable oscillations with constant magnitude of amplitude has become settled in the system for all characteristics within the sector 2. The size of the established amplitude of the resonance oscillations is determined by the crossing of the characteristic of source of energy M (Ω) with the curve S (Ω). By the variating the characteristic of the source of energy M (Ω) the magnitude of amplitude of resonance oscillations is controlled in wide limits at a constant resonance frequency. The oscillograms represented in fig. 12 show us the process of stationary oscillations installing in the parametrical system at the resonance with magnitudes of amplitudes determened by the points B_1 , B_2 , B_3 for the characteristic M_1 (Ω) , M_2(Ω) , M_3(Ω) correspondigly (Fig. 11).

It is necessary to remark that in the problems about parametrical oscillations in the linear systems ($\gamma = 0$) studied without accounting the energy source properties the unlimited increase of magnitude of amplitude at the

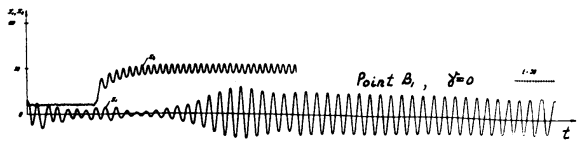

Fig. 12 a

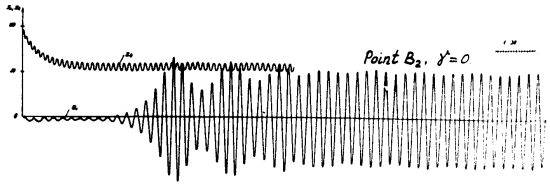

Fig 12 b

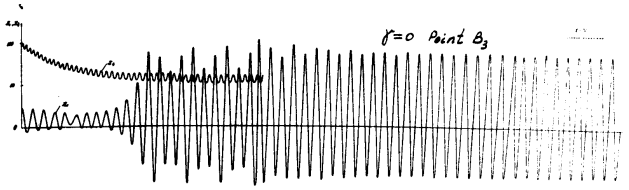

Fig 12 c

Fig. 12 Oscillograms of stationary oscillations in the linear parametrical system ($\gamma = 0$)

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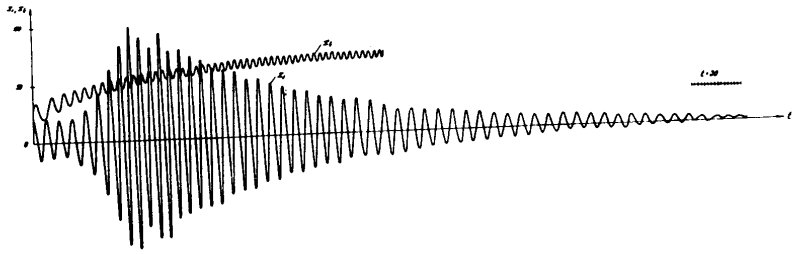

Fig. 13. Oscillogram of nonstationary processe in the nonlinear parametrical system ($\gamma < 0$) (Ω is increasing)

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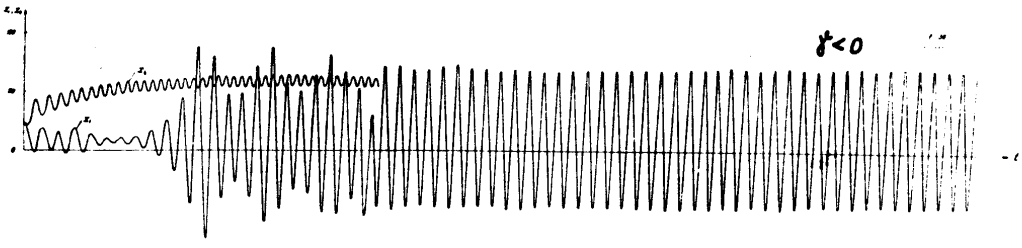

Fig. 14 Oscillogram of stationary oscillations in the nonlinear parametrical system ($\gamma < 0$), corresponding to the left branch of the resonance curve.

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resonance is pointed out. But indeed when the energy source of the forced oscillations is taken into account it is true only for the characteristic with incline within sector I, Fig. II.

Horisontal characteristics and other characteristics lying at a short distance from it do not give stationary regimes of motion under resonance. The oscillations reached a small amplitude and rather quickly disappear.

2. The results of the modelation of the parametrical system with "stiff" ($\gamma > 0$) and "soft" ($\gamma < 0$) nonlinear characteristics of resilient force were quite analogous to the above mentioned results for nonlinear system.

Only one difference was apparent: - after the "frustrations" oscillations quite disappeared with the increase of frequency for $\gamma > 0$ 0 and the decrease of frequency for $\gamma <$ 0.

The oscillogram in Fig. 13 illustrates the "frustrations" of oscillations of the nonlinear parametrical system with "stiff" characteristics of resilient at increase of frequency.

The oscillogram in Fig. 14, represents the stable stationary oscillations the stable stationary oscillations under resonanse in the ascending (left) branch of the resonanse curve for the characteristic with inclines within sector 3, (Fig. 8) in the parametrical system with $\gamma < 0$. For the characteristic with inclines within sector I (Fig. 8) in the asecending branch there were instable regimes of oscillations.

The stable oscillations in the slumping (right) branch and instable in the ascending (left) branch of the resonance curve for the characteristic with inclines within sector I, (Fig. 8) were discovered in the parametrical system.

In conclusion we shall note that preliminarily, before the electronic modelation, we made experiments with the use of specially developed mechanical models of the nonlinear and parametrical systems. The results of these experiments were limited by the characteristics of motor and parameters of system but they permitted to determine the general dependences between the motor and the vibration system.

The experiments were implemented in the Mechanical Engineering Research Institute of the Academy of Sciences of the USSR with use of the electronic machine of the Mechanical Institute of Moscow University. The author wishes to convey his acknowledgment to the scientific workers of the Laboratory of the Oscillations of the Mechanical Institute of Moscow University who have actively helped the author in these experiments.

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И. В. ГЛАТЕНОК

К ВОПРОСУ ОБОСНОВАНИЯ МЕТОДА ГАРМОНИЧЕСКОГО БАЛАНСА

To find periodic solutions in an analysis of automatic control systems one usually uses the approximate method of harmonic balance [1, 2]. About the permissibility of this method one judges comparing obtained results with results of exact analysis /see, for example, [3]/. But such comparison one carries out only for the systems with piecewise linear functions, because only for these systems one succeeds to find /for example by the method of pasting together /припасовывания/ / the exact solution.

In the author's thésis /see [4] / there is a foundation of the method of harmonic balance in finding stable periodic solutions of the differential equation

$$\ddot{y} = f(y, \dot{y}) \qquad /1/$$

for some classes of functions $f(y, \dot{y})$. Herewith one understands the foundation of this method not in the sense that one defines sequentially higer approximations according to the method of harmonic balance and one shows that they describe the true solution more and more full. Here one suppose that it is found only the first approximation

$$y = a \sin \omega t \qquad /2/$$

and shows that by some restrictions on $f(y, \dot{y})$ /not connected with its piecewise linear approximation/ the true periodic solution of /1/ exists, it is stable and is in some neighbourhood of the approximate solution. When proving there are obtained estimates of size of this neighbourhood.

Suppose that

$$\frac{1}{2\pi} \int_0^{2\pi} f(a \sin u, a\omega \cos u)\, du = 0 .$$

In this case the equations of harmonic balance have following form:

$$\left.\begin{aligned} \frac{1}{\pi} \int_0^{2\pi} f(a \sin u, a\omega \cos u) \sin u \, du &= -a\omega^2, \\ \frac{1}{\pi} \int_0^{2\pi} f(a \sin u, a\omega \cos u) \cos u \, du &= 0 . \end{aligned}\right\} \qquad /3/$$

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Let the system /3/ has a solution a, ω, i.e. the method of harmonic balance yields the first approximation /2/.

For twice continuously differentiable functions $f(y,\dot{y})$ sufficient conditions of the existence of a stable periodic solution of /1/ in some neighbourhood of the approximate solution /2/ in essence mean that the Fourier coefficients of $f(a\sin u, a\omega\cos u)$, $f'_y(a\sin u, a\omega\cos u)$, $f'_{\dot{y}}(a\sin u, a\omega\cos u)$ must be sufficiently small. Takes place

Theorem 1.

If $f(y,\dot{y})$ is a twice continuously differentiable function whith respect to $y,\dot{y}$ and

1/ $f''_{yy}(\cdot)$, $f''_{y\dot{y}}(\cdot)$, $f''_{\dot{y}\dot{y}}(\cdot)$ [1] have a bounded variation in $[0, 2\pi]$,

2/ $\sigma < \dfrac{(Q-\psi)^2\omega^2}{8m}\varepsilon^2$,

3/ $Q>0$, $\psi<Q$,

4/ $P>0$, $\varphi<P$, $\varphi < \dfrac{(Q-\psi)\omega}{2}\varepsilon$,

5/ $-\varphi < \omega^2 - P < \varphi$,

6/ $\varepsilon \le \min\left\{\dfrac{1}{10}, \dfrac{ma}{4(Q-\psi)\omega}, \dfrac{ma}{5(Q-\psi)^2}, \dfrac{ma\omega}{6(Q-\psi)P}, \dfrac{2ma\omega}{(Q-\psi)^3}, \dfrac{(Q-\psi)^4}{100m^2a^2}\right\}$,

where

$$P = -\frac{1}{2\pi}\int_0^{2\pi} f'_y(\cdot)\,du, \qquad Q = -\frac{1}{2\pi}\int_0^{2\pi} f'_{\dot{y}}(\cdot)\,du,$$

$$\sigma = \sum_{k=2}^{\infty}\left[\left|\frac{1}{\pi}\int_0^{2\pi} f(\cdot)\sin ku\,du\right| + \left|\frac{1}{\pi}\int_0^{2\pi} f(\cdot)\cos ku\,du\right|\right],$$

$$\varphi = \sum_{k=1}^{\infty}\left[\left|\frac{1}{\pi}\int_0^{2\pi} f'_y(\cdot)\sin ku\,du\right| + \left|\frac{1}{\pi}\int_0^{2\pi} f'_y(\cdot)\cos ku\,du\right|\right],$$

$$\psi = \sum_{k=1}^{\infty}\left[\left|\frac{1}{\pi}\int_0^{2\pi} f'_{\dot{y}}(\cdot)\sin ku\,du\right| + \left|\frac{1}{\pi}\int_0^{2\pi} f'_{\dot{y}}(\cdot)\cos ku\,du\right|\right],$$

$\left(\frac{1}{2}\right)\left|f''_{zz}(\varkappa)\right| < m$ [2], $\quad 0 \le \Theta \le 1, \quad |z| \le a,$

---

[1] $c_1^{\flat}(\cdot)$ denotes $c_1^{\flat}(a\sin u, a\omega\cos u)$

[2] $c_1^{\flat}(\varkappa)$ denotes $c_1^{\flat}[a\sin u, (a+\omega z)\sin u, (a+\omega z)\omega\cos u]$.

then the equation /1/ has the stable periodic solution, the amplitude A of which satisfies inequalities

$$a - \frac{(Q-\psi)\omega}{m}\varepsilon < A < a + \frac{(Q-\psi)\omega}{m}\varepsilon.$$

For continuously differentiable functions $f(y,\dot{y})$ we introduce in consideration a function

$$F(u) = \left[\omega^2 + f'_y(\varkappa)\right]\sin u + f'_{\dot{y}}(\varkappa)\omega\cos u, \quad 0 \le \Theta \le 1, \ |z| \le a, \qquad /4/$$

and a value

$$\sigma = \max_{0 \le u \le 2\pi} \left| f(a\sin u, a\omega\cos u) + a\omega^2 \sin u \right|$$

and put some restrictions on the sign of $F(u)$ and $\min|F(u)| = q$ on some set of values of a and σ. Takes place

Theorem 2.

If $f(y,\dot{y})$ is a continuously differentiable function with respect to $y,\dot{y}$ and

1/ $F(u)\cos u < 0$ for $u \ne \frac{\pi}{2}, \frac{3\pi}{2}$, $F\left(\frac{\pi}{2}\right) = F\left(\frac{3\pi}{2}\right) = 0$,

2/ $\sigma < \min\left\{\dfrac{aq}{8}, \dfrac{a\omega^2\sqrt{4-\varepsilon^2}}{8}, \dfrac{aq\omega^2\sqrt{1-\varepsilon^2}}{q+\omega^2(\sqrt{1-\varepsilon^2}+\varepsilon}\right\}$,

3/ $\varepsilon < \min\left(\dfrac{1}{2}, \dfrac{q}{4\omega^2}\right)$,

where

$$q = \min_{\substack{\frac{\pi}{2}+\beta \le u \le \frac{3\pi}{2}-\beta \\ \frac{\pi}{2}+\beta \le u \le \frac{3\pi}{2}-\beta}} |F(u)| \qquad \left(\beta = \arcsin\frac{\varepsilon}{2}\right),$$

then the equation /1/ has the stable periodic solution, the amplitude A of which satisfies inequalities

$$\left(a - \frac{\sigma' + a\omega^2\varepsilon}{q - \omega^2\varepsilon}\right)\frac{1}{\sqrt{1-\varepsilon^2}} \le A \le \left(a + \frac{\sigma' + a\omega^2\varepsilon}{q - \omega^2\varepsilon}\right)\frac{1}{\sqrt{1-\varepsilon^2}}.$$

For continuous functions $f(y,\dot{y})$ takes place

Theorem 3.

If $f(y,\dot{y})$ is a continuous function with respect

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to $y,\dot{y}$ and

1/ $f(y,0)<0$, $-\sigma_2/q_2 \le y-a \le \sigma_1/q_1$, $y>0$,

$f(y,0)>0$, $-\sigma_2/q_2 \le y-a \le \sigma_1/q_1$, $y<0$,

2/ f'_y, $f'_{\dot{y}}$ have a break of the first kind on Oy-axe and on some line Λ and they are continuous in other parts of $(y,\dot{y})$-plane,

3/ $F(u)\cos u<0$ for $u \neq \frac{\pi}{2}+\frac{k}{2}\pi$,

$F(\frac{\pi}{2}+0)\neq 0$, $F(\frac{3\pi}{2}+0)\neq 0$, $F(\frac{\pi}{2}-0)\neq 0$, $F(\frac{3}{2}\pi-0)\neq 0$,

4/ $\sigma < \min\{aq_1, aq_2\}$,

where

$$\sigma = \max(\sigma_1,\sigma_2),$$

$$\sigma_1 > \max\left\{ \begin{matrix} \max \sum(u), & \max|\sum(u)| \\ -\pi/2 \le u \le \pi/2 & \pi/2 \le u \le 3\pi/2 \\ \sum(u)>0 & \sum(u)<0 \end{matrix} \right\},$$

$$\sigma_2 > \max\left\{ \begin{matrix} \max|\sum(u)|, & \max \sum(u) \\ -\pi/2 \le u \le \pi/2 & \pi/2 \le u \le 3\pi/2 \\ \sum(u)<0 & \sum(u)>0 \end{matrix} \right\},$$

$$\sum(u) = f(\cdot) + a\omega^2 \sin u,$$

$F(u)$ is defined by formula /4/,

$$q_1 = \min\left\{ \begin{matrix} \min|F(u)|, & \min F'(u) \\ -\pi/2 \le u \le \pi/2 & \pi/2 \le u \le 3\pi/2 \\ \sum(u)>0 & \sum(u)<0 \end{matrix} \right\},$$

$$q_2 = \min\left\{ \begin{matrix} \min|F(u)|, & \min F'(u) \\ \pi/2 \le u \le 3\pi/2 & -\pi/2 \le u \le \pi/2 \\ \sum(u)<0 & \sum(u)>0 \end{matrix} \right\},$$

then the equation /1/ has the stable periodic solution, the amplitude A of which satisfies inequalities

$$a - \frac{\sigma_2}{q_2} \le A \le a + \frac{\sigma_1}{q_1}.$$

The principal idea of the proofs of these theorems is the following one. The approximate solution $y = a\sin\omega t$ of /1/ is an ellipse

$$\frac{x^2}{\omega^2} + y^2 = a^2 \qquad /5/$$

$x = \dot{y}$

in (y,x) plane. Around this ellipse we build an annular region Γ which does not contain singular points of the system

$$\frac{dy}{dt} = x, \quad \frac{dx}{dt} = f(y,x) \qquad /6/$$

and has such property that integral lines of /1/ /or - on (y,x) plane - of /6/ / beginning on the bound of Γ, further do not leave Γ. Hence by the so_called principle of ring /see [5], p.52, 148/ one can come to the conclusion that in Γ there is a periodic solution of the equation /1/.

Example. Consider an automatic control system described by the differential equation

$$T\ddot{y} + \dot{y} = -KG \qquad /7/$$

where G - the characteristic of a non-linear link showed on fig. 1 and defined by the formulae

$$G(y,\dot{y}) = \begin{cases} y-\mathcal{B} \text{ for } \dot{y}>0 \text{ or } \dot{y}=0 \text{ and } y-G=\mathcal{B}, & (I) \\ \bar{y}-\mathcal{B} \text{ for } \dot{y}<0 \text{ and } |G-y|<\mathcal{B}, & (II) \\ y+\mathcal{B} \text{ for } \dot{y}<0 \text{ or } \dot{y}=0 \text{ and } G-y=\mathcal{B}, & (III) \\ y+\mathcal{B} \text{ for } \dot{y}>0 \text{ and } |G-y|<\mathcal{B}, & (IV) \end{cases} \qquad /8/$$

where $\bar{y}$ and $\underline{y}$ are constants equal to values of y for $\dot{y}=0$ when $G>0$ and $G<0$, respectively.

Further studies of the nature of $G(y,\dot{y})$ shows that on $(y,\dot{y})$- plane there exists a line Λ which with Oy- axis divides it into four regions $\mathcal{D}_I - \mathcal{D}_{IV}$ /see fig.2/ that

$$G(y,\dot{y}) = \begin{cases} y-\mathcal{B} \text{ for } (y,\dot{y}) \in \mathcal{D}_I \\ \bar{y}-\mathcal{B} \text{ for } (y,\dot{y}) \in \mathcal{D}_{II} \\ \underline{y}+\mathcal{B} \text{ for } (y,\dot{y}) \in \mathcal{D}_{III} \\ y+\mathcal{B} \text{ for } (y,\dot{y}) \in \mathcal{D}_{IV}. \end{cases}$$

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The equations of Λ is obtained as following: the integral lines of /7/ or - on phase plane - of the system

$$\left.\begin{aligned} \frac{dy}{dt} &= x, \\ \frac{dx}{dt} &= -\frac{1}{T}x - \frac{K}{T}G(y,x) \end{aligned}\right\}$$

are being built on steps /supposing that these steps are numbered according to Roman figures in /8//.

Then, for example, the left branch of Λ is the bound between II and III step and we find it as the geometric place of the ends of the integral lines of II step. Its equation is

$$x = \frac{\varepsilon\beta}{T} + K(y+\beta)\ln\left(1+\frac{x}{K(y+\beta)}\right). \qquad /9/$$

Therefore in our case

$$f(y,\dot{y}) = -\frac{1}{T}\dot{y} - \frac{K}{T}\begin{cases} y-\beta & \text{for } (y,\dot{y}) \in \mathcal{D}_I, \\ \dot{y}-\beta & \text{for } (y,\dot{y}) \in \mathcal{D}_{II}, \\ y+\beta & \text{for } (y,\dot{y}) \in \mathcal{D}_{III}, \\ \dot{y}+\beta & \text{for } (y,\dot{y}) \in \mathcal{D}_{IV}, \end{cases}$$

where $\bar{y}$, y are defined as intersection points of the integral lines of II and IV step with Oy- axis respectively.

The equations of harmonic balance have the following form

$$\left.\begin{aligned} \frac{K}{T\pi}\left(\frac{\pi}{2}+u_1+\frac{1}{2}\sin 2u_1\right) &= \omega^2, \\ \frac{K}{\pi}\cos^2 u_1 &= \omega, \end{aligned}\right\} \qquad /10/$$

where $u_1 = \arcsin\left(1-\frac{2\beta}{a}\right)$.

Following expressions for $\Sigma(u)$ and $F(u)$ are obtained:

$$\Sigma(u) = -\frac{a\omega}{T}\cos u + \begin{cases} -\left(\frac{K}{T}-\omega^2\right)a\sin u + \frac{K}{T}\beta, & \pi+u_1 \le u \le \pi/2, \\ a\omega^2\sin u - \frac{K}{T}(\bar{y}-\beta), & \pi/2 \le u \le u_2, \\ -\left(\frac{K}{T}-\omega^2\right)a\sin u - \frac{K}{T}\beta, & u_2 \le u \le 3\pi/2, \\ a\omega^2\sin u - \frac{K}{T}(y-\beta), & 3\pi/2 \le u \le \pi+u_1, \end{cases}$$

$$F(u) = \frac{a}{T}\cos u + \begin{cases} -\left(\frac{K}{T}-\omega^2\right)\sin u, & \pi+u_1 \le u \le \pi/2, \\ \omega^2\sin u, & \pi/2 \le u \le u_2, \\ -\left(\frac{K}{T}-\omega^2\right)\sin u, & u_2 \le u \le 3\pi/2, \\ \omega^2\sin u, & 3\pi/2 \le u \le \pi+u_1, \end{cases}$$

u_2 denotes the polar angle of intersection point of line /9/ with ellipse /5/, i.e. the root of the equation

$$a\omega\cos u = \frac{\varepsilon\beta}{T} + K(a\sin u+\beta)\ln\left(1+\frac{a\omega\cos u}{K(a\sin u+\beta)}\right). \qquad /11/$$

Let us perform the numerical calculation for the case

$$T=0{,}1, \quad K=50, \quad \beta=10.$$

The system /10/ has following solution:

$$a=20{,}22, \quad \omega=15{,}91.$$

The equation /11/ has the solution $u_2 = \pi + 25'$,

$$\Sigma(u) = -321{,}7\cos u + \begin{cases} -4992\sin u + 5000, & 25' \le u \le \pi/2, \\ 5118\sin u - 500(\bar{y}-10), & \pi/2 \le u \le \pi+25', \\ -4992\sin u - 5000, & \pi+25' \le u \le 3\pi/2, \\ 5118\sin u - 500(y+10), & 3\pi/2 \le u \le 25', \end{cases}$$

$$F(u) = \begin{cases} -159{,}1\cos u - 246{,}9\sin u, & \begin{matrix} 25' \le u \le \pi/2, \\ \pi+25' \le u \le 3\pi/2, \end{matrix} \\ -159{,}1\cos u + 253{,}1\sin u, & \begin{matrix} \pi/2 \le u \le \pi+25', \\ 3\pi/2 \le u \le 25'. \end{matrix} \end{cases}$$

As the values of $\Sigma(u)$ and $F(u)$ for u and $u+\pi$ differ only by sign, so to find σ_1, σ_2, q_1, q_2^* it is sufficient to consider their values, for example, for $25' \le u \le \pi+25'$

For $25' \le u \le \pi/2$ by elementary means one obtains

$$\max_u \Sigma(u) = \Sigma(25') = 1747,$$
$$\min_u \Sigma(u) = \Sigma(57°12') = -959.$$

For $\pi/2 \le u \le \pi+25'$ one finds an estimate for $\max \Sigma(u)$ in the following way. The consideration of values of $\bar{y}$ corresponding to points of ellipse /5/ for $u \in \left[\frac{\pi}{2}, \pi+25'\right]$ shows that $\bar{y}$ increases from $\bar{y}\left(\frac{\pi}{2}\right) = a = 20{,}22$ to some $\bar{y}\max$ corresponding to point in which an integral line of

$$\frac{dy}{dx} = \frac{x}{\frac{x}{T}+\frac{K}{T}(\bar{y}-\beta)}$$

touches ellipsis /5/ /in our case $\bar{y}\max$ equals to 21,06 for $u = 34°$) and afterwards decreases from $\bar{y}\max$ to $\bar{y}(\pi+25') = 19{,}85$.

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Hence $\max \Sigma(u) < \Sigma_1(u_m) - 500(a-10) = 936,$

where $\Sigma_1(u_m) = \max_{\pi/2 \le u \le \pi + 25'} (-3217\cos u + 5118 \sin u).$

As result we obtain

$$\sigma_1 = 1748, \quad \sigma_2 = 940.$$

By elementary means one obtains

$$q_2 = \min\left(|F(u_1^\circ)|, |F(\tfrac{\pi}{2}-0)|, F(\tfrac{\pi}{2}+0), F(u_2^\circ)\right) = |F(\tfrac{\pi}{2}-0)| = 246,9,$$

where u_1°, u_2° – roots of the equation $\Sigma(u) = 0$ included in intervals $[0, \pi/2]$, $[\pi/2, \pi + 25']$, respectively,

$$q_1 = F(u_2 - 0) = 157,3.$$

The conditions of the theorem 3 are satisfied. For conditions 1-3 this immediately follows from the definition of the functions G and F and also from the fact that in our case we have

$$a - \frac{\sigma_2}{q_2} > 6 \quad (20,22 - 3,81 > 10),$$

$$F'(u_2 - 0) > 0.$$

Further one has

$$aq_1 = 3181, \quad aq_2 = 4992.$$

Therefore the condition 4 is also satisfied:

$$\sigma = 1748, \quad \min\{aq_1, aq_2\} = 3181$$

By the theorem 3 we have

$$20,22 - 3,81 \le A \le 20,22 + 11,11$$

$$\left(\frac{\sigma_1}{q_1} = 11,11, \quad \frac{\sigma_2}{q_2} = 3,81\right).$$

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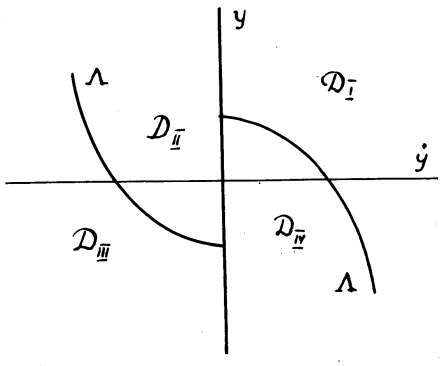

Fig 2.

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В. А. ГРОБОВ

НЕЛИНЕЙНЫЕ ЗАДАЧИ ДИНАМИКИ РОТОРОВ ТУРБОМАШИН

In view of the fact that power and velocity of turbines are constantly growing, the dynamic design problems of rotor become more and more important.

All the attempts to minimize the weight of construction lead to unavoidable decrease of shaft and supporting elements stiffness. Because of this, turbine rotors often work at high angular velocities, which overcome one or two critical speeds.

For turbines of this type the dynamic design of rotors involves the investigation of two main problems:

I. Nonstationary vibrations of the shaft during acceleration through critical speed region.

2. Stability of rotor stationary movement in the beyond-critical speed region.

The results of author's investigations are described concerning mainly the design of nonstationary turbine rotor vibrations, when one takes into account nonlinear elasticity of the support; gyroscopic action of the discs, situated on the shaft; external and internal friction, and some other factors.

The author analyze partly the problems of stability of rotor stationary movement in the beyond-critical speed region. The problems are solved by using the asymptotic methods of N.M.Krilov, N.N.Bogoljubov and Ju.A.Mitropolskij /I,2,3/. This allowed to develop relatively simple design schemes for the determination of amplitudes of nonlinear vibration in multidegree of freedom systems. As a result, the shemes of analyzed objects become much more real, than before.

§I. Bending vibrations of the shaft together with gyroscopic action of the discs, situated on this shaft, can be described for a general case by the differential equations:

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$$\sum_{j=1}^{n}[a_{ij}\ddot{q}_j + b_{ij}(\tau,\omega)\dot{q}_j + c_{ij}q_j] = \mu Q_i(q_1, q_2, \dots q_N, \dot{q}_1, \dot{q}_2, \dots \dot{q}_N, \varphi, \dot{\varphi}, \ddot{\varphi}) \quad /1/$$

where $N=4n$, n -number of discs, situated on the shaft.

μ - small parameter, denoting the relationship between this system and a linear conservative one.

$\tau = \mu t$ - slow time.

Functions $\mu Q_i(q_1, \dots q_N, \dot{q}_1, \dot{q}_2, \dots \dot{q}_N, \varphi, \dot{\varphi}, \ddot{\varphi})$, in the right side of equations /1/ indicate small dissipative forces; nonlinear incriments of restoring forces; and perturbations, caused by statical and dynamical disbalance of the discs. Frequency of the latter $\dot{\varphi} = \omega(\tau)$, by a nonsteady-state regime is defined from the equation

$$J_0 \frac{d\omega}{dt} = \mu F(q_1, \dots, q_N, \dot{q}_1, \dots, \dot{q}_N, \varphi, \dot{\varphi}) + \Delta M(\dot{\varphi}) \quad /2/$$

where J_0 - axial moment of rotor inertia;

$\mu F(q_1, \dots, q_N, \dot{q}_1, \dots \dot{q}_N, \varphi, \dot{\varphi})$ - function, showing the interaction between bending vibration and rotation, that is the possibility of pumping over the oscillation energy into the rotation movement and vice versa.

$\Delta M(\dot{\varphi}) = M_{dz}(\dot{\varphi}) - M_{fz}(\dot{\varphi})$ - excessive torque moment, which ensures speeding of the rotor.

The acymptotic solution technique for systems of the type /1/ is developed by Ju.A.Mitropolskij [3].

If there is no internal resonance in the oscillating system /1/, then the equations /1/ and /2/ can be solved by a single-frequency method and the averaging principle proposed by N.N.Bogoljubov [2].

Assume for the system /1/ as a first approximation

$$q_j^{(k)} = a_k \sigma_j^{(k)}(\tau) e^{i(\gamma_k + \chi_k)} + a_k \sigma_j^{(k)*}(\tau) e^{-i(\gamma_k + \chi_k)},$$

where a_k and γ_k - amplitude and phase of a single-frequency vibration, regarded as slowly varied time function;

$\sigma_j^{(k)}(\tau)$ fundamental functions of undisturbed system, corresponding to oscillation of the k-th form;

$\sigma_j^{(k)*}(\tau)$ their conjugated functions.

Angular velocity ω in the equation /2/ may be considered as the superposition of slowly varied member Ω and the sum of small vibration terms

$$\omega = \Omega + \text{small vibration terms} \quad /4/$$

Then, in the first approximation

$$\omega = \Omega \quad /5/$$

According to [4], where the determination of cross-conjugated quantities a_k, γ_k and Ω are given, we shall get in the first approximation the differential equations:

$$\frac{da_k}{dt} = \mu A_k^{(k)}(\tau, \gamma_k, a_k, \Omega),$$

$$\frac{d\gamma_k}{dt} = \lambda_k(\tau) - \Omega(\tau) + \mu B_k^{(k)}(\tau, a_k, \dots),$$

$$\frac{d\Omega}{dt} = \frac{\mu}{J_0} \tilde{F}(\tau, a_k, \gamma_k, \Omega) + \Delta M(\dots),$$

where $\tilde{F}$ - averaged function.

Sufficient power supply of the turbine and small amplitude of vibrations in most cases decrease to a negligible value the influence of the shaft deformation on the law of angular velocity change at nonstationary movement. In view of this the equation of torque moment

[illegible]

[illegible]

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$$m\ddot{x} + p\dot{x} - q_1\alpha = -\rho_1(Ex + F\alpha)^3 + m\varepsilon\omega^2(\tau)\cos\varphi + R_1$$
$$m\ddot{y} + p_2\dot{y} - q_2\beta = -\rho_2(Ey + F\beta)^3 + m\varepsilon\omega^2(\tau)\sin\varphi + R_2$$
$$J_x\ddot{\alpha} + J_0\omega(\tau)\dot{\beta} + s_1\alpha - z_1 x = -\sigma_1(Ex + F\alpha)^3 + (J_0 - J)\delta\omega^2(\tau)\sin(\varphi - \chi) + R_3$$
$$J_y\ddot{\beta} - J_0\omega(\tau)\dot{\alpha} + s_2\beta - z_2 y = -\sigma_2(Ey + F\beta)^3 - (J_0 - J)\delta\omega^2(\tau)\cos(\varphi - \chi) + R_4$$

/9/

where

$\rho_1, \rho_2, q_1, q_2, z_1, z_2, s_1, s_2, \rho_1, \rho_2, \sigma_1, \sigma_2, \ldots$

... quasi-elastic constants, which depend on elastic qualities of the shaft and supporting elements, are found from the formulas given in [4]: R_1, R_2, R_3, R_4 - dissipative forces, which involve components of external and internal friction.

The forces of external friction are assumed to be proportional to the deformation rate of the shaft, while the coefficients of proportionality are the even functions of saggings and angles of roll [5]. The forces of external friction in corresponding coordinates may be approximately defined as:

$$R_1^{(e)} = -\chi_1\dot{x} - \chi_{21}\dot{x}^3 - \chi_{22}\dot{x}\dot{y}^2, \quad R_2^{(e)} = -\chi_1\dot{y} - \chi_{21}\dot{y}^3 - \chi_{22}\dot{x}^2\dot{y},$$
$$R_3^{(e)} = \chi_3\dot{\alpha} - \chi_4(\dot{\alpha}^2 + \dot{\beta}^2)\dot{\alpha}, \quad R_4^{(e)} = -\chi_3\dot{\beta} - \chi_4(\dot{\alpha}^2 + \dot{\beta}^2)\dot{\beta}. \qquad /10a/$$

The forces of internal friction are assumed to be proportional to the deformation rate in the co-ordinate system, rotating with the shaft, while coefficient of proportionality are defined by Bohk's hypothesis [5]. Then linear parts of internal friction forces, related to the co-ordinates x, y, α, β, may be written as:

If in a case of stationary movement we'll take $\Delta\varphi$ =const and assume, that the right sides of the first two equations /6/ are equal to zero, then we shall get relations, which will be used to define amplitude and phase of stationary vibrations.

Designing the right sides of the first and the second equations /6/ as $S(a,\psi)$ and $\Phi(a,\psi)$ correspondingly, and developing variation equations for amplitude and phase, we can write conditions for stationary vibrations stability:

$$\frac{\partial S(a,\psi)_0}{\partial a} + \frac{\partial \Phi(a,\psi)_0}{\partial \psi} < 0$$
$$\frac{\partial S(a,\psi)_0}{\partial a}\cdot\frac{\partial \Phi(a,\psi)_0}{\partial \psi} - \frac{\partial \Phi(a,\psi)_0}{\partial a}\cdot\frac{\partial S(a,\psi)_0}{\partial \psi} > 0 \qquad /8/$$

In accordance with the scheme given above one can analyze stationary and unstationary oscillation of shafts as well as their stationary rotation stability.

Let us illustrate this with examples.

§ 2. As an example let us take bending vibrations of the shaft with the unbalanced disc, which is asymmetric to its supports. One of those supports is elastic and has a nonlinear characteristic, /fig.1/.

The equations of bending oscillations of the shaft, caused by static and dynamic unbalance of the disc, have the form:

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[4] See for the development of these equations

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$$R_1^{(i)} = -\sum \nu_\kappa' (\dot{x}_\kappa - \omega(\tau) y_\kappa);\ R_2^{(i)} = -\sum \nu_\kappa' (\dot{y}_\kappa + \omega(\tau) x_\kappa);$$
$$R_3^{(i)} = -\sum \nu_\kappa'' (\dot{\alpha}_\kappa - \omega(\tau) \beta_\kappa);\ R_4^{(i)} = -\sum \nu_\kappa'' (\dot{\beta}_\kappa + \omega(\tau) \alpha_\kappa); \quad /10b/$$

where index κ = the number of natural vibrations form - $\nu_\kappa', \nu_\kappa''$ coefficients of proportionality defined by Bohk's hypothesis /5/. The law of angular velocity change ω in the right sides of the equations /9/ is defined from the drive characteristic according to /7/.

For the determination of amplitude and phase of single - frequency nonstationary vibrations, when the critical zone /let us say, the low one/ is being crossed over, we have the following equations /6/:

$$\frac{da}{dt} = \dots \frac{E(\tau)\omega^2(\tau)\cos(\psi - 45^\circ)}{m_1(\tau)[\lambda_1(\tau) + \omega(\tau)] + m_2(\tau)} + \frac{H(\tau)\omega^2(\tau)\sin(\psi + \chi - 45^\circ)}{m_1(\tau)[\lambda_1(\tau) + \omega(\tau)] + m_2(\tau)},$$

$$\frac{d\psi}{dt} = \dots \frac{1}{a} \frac{E(\tau)\omega^2(\tau)\sin(\psi - 45^\circ)}{m_1(\tau)[\lambda_1(\tau) + \omega(\tau)] + m_2(\tau)} \dots \frac{1}{a} \frac{H(\tau)\omega^2(\tau)\cos(\psi + \chi - 45^\circ)}{m_1(\tau)[\lambda_1(\tau) + \omega(\tau)] + m_2(\tau)}$$

where

$$\dots = \frac{\mathcal{X}_1(\sigma_1^{(1)2} + \sigma_3^{(1)2}) + \mathcal{X}_3(\sigma_3^{(1)2} + \sigma_4^{(1)2})}{2[m_1(\tau)\lambda_1(\tau) + m_2(\tau)]} \lambda \dots$$

$$\dots \frac{1}{3} \frac{(\sigma_1^{(1)2} - 6\sigma_3^{(1)2}\sigma_4^{(1)2} + \sigma_2^{(1)4})\mathcal{X}_2 + (\sigma_3^{(1)4} + 6\sigma_3^{(1)2}\sigma_4^{(1)2} + \dots)\mathcal{X}_4}{[m_1(\tau)\lambda_1(\tau) + m_2(\tau)]}$$

$$\dots \frac{[\dots]\lambda - 2[\dots]}{[m_1(\tau)\lambda_1(\tau) + m_2(\tau)]}$$

... coefficient of proportionality constant.

$\dots = \frac{3}{8}\mathcal{X}_3(\tau)a^2$ - equivalent frequency of nonlinear system

Interpretation of $E(\tau), H(\tau), m_1(\tau), m_2(\tau), \mathcal{L}(\tau)$ and other quantities in equations /II/ are given in /4/.

Suppose that the right sides of the equations /II/ are equal to zero. Then, omitting from the resulting expressions the phase ψ, we shall find the dependence of stationary vibrations amplitude upon the angular velocity

$$\omega = \lambda_e \sqrt{\frac{1}{\kappa^2}\left(1 - \frac{2\delta_e^2}{\lambda_e^2}\right) \pm \sqrt{\frac{E^2 + H^2 - 2EH\sin\chi}{m^2 a^2} - \frac{4\delta_e^2}{\lambda_e^2}}}$$

$$\kappa = 1 - \frac{E^2 + H^2 - 2EH\sin\chi}{m^2 a^2}$$

/12/

This relation allows to draw the resonance curves of stationary vibrations.

In order to build the resonance curves of nonstationary vibrations the equations /II/ must be numerically integrated. It is convenient to use for this purpose electronic analog, which enables to observe oscillation process in progress with the value of damping constant, the value of nonlinearity and the rate of crossing over the resonance zone. The recorded curves, which show the dependence of nonstationary vibrations amplitude upon the value of ... [illegible] ... angle, by means of ... [illegible] ... from the rule of ... [illegible] ... the ... [illegible] ... Fig. 2.

... [illegible] ... example we have ... [illegible] ... relations of the ... [illegible] ... for the ... [illegible] ... supporting elements is stiff and the other is elastic with non-linear characteristic (Fig. 3).

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When one doesn't account for gyroscopic action of discs but take into consideration external and internal friction, which linearly depends on deformation rate, the equations of rotor bending vibrations have the form:

$$\frac{d^2x_i}{dt^2}+\sum_{k=1}^{n}c_{ik}x_k=\mp n_i\Big(\sum_{k=1}^{n}A_kx_k\Big)^3-\sum_{k=1}^{n}\big[\varkappa_{ik}x_k+\rho_{ik}(x_k-\omega(\tau)y_k)\big]+m_i\varepsilon_i\omega^2(\tau)\cos\varphi_i(t)$$

$$\frac{d^2y_i^2}{dt^2}+\sum_{k=1}^{n}c_{ik}y_k=\mp n_i\Big(\sum_{k=1}^{n}A_ky_k\Big)^3-\sum_{k=1}^{n}\big[\varkappa_{ik}\dot{y}_k+D_{ik}(\dot{y}_k+\omega(\tau)x_k)\big]+m_i\varepsilon_i\omega^2(\tau)\sin\varphi_i(t).$$

$$\varphi_i(t)=\varphi(t)+x_i \qquad /I1/$$

The [illegible] of coefficients c_{ik}, n_i, A_k and other quantities, involved in the equations /I1/, is given in [4].

Having in mind, that vibrations in horizontal and vertical planes are for a quater of period out of phase, we shall write as the first approximation:

$$x_i^{(k)}=a_k\sigma_i^{(k)}\cos(\varphi+\psi_k),$$

$$y_i^{(k)}=a_k\sigma_i^{(k)}\sin(\varphi+\psi_k), \qquad /I4/$$

where a_k and ψ_k are defined by the following equations

$$\frac{da}{dt}=-\delta_e^{(k)}a_k-\frac{\sum_{i=1}^{n}E_i(\tau,\omega)\sin(\psi_k-x_i)}{M^{(k)}[\lambda_k+\omega(\tau)]},$$

$$\frac{d\psi_k}{dt}=\lambda_e^{(k)}(a,\tau)-\omega(\tau)-\frac{1}{a_k}\frac{\sum_{i=1}^{n}E_i(\tau,\omega)\cos(\psi_k-x_i)}{M^{(k)}[\lambda_k+\omega(\tau)]}, \qquad /I5/$$

where

$$\delta_e^{(k)}=\frac{\sum_{i=1}\sum_{j=1}[\varkappa_{ij}\lambda_k+\rho_{ij}(\omega-\lambda_k)]\sigma_i^{(k)}\sigma_j^{(k)}}{2M^{(k)}\lambda_k},$$

$$\lambda_e(a,\tau)=\lambda_k\pm\frac{\frac{3}{8}\sum_{i=1}^{n}n_i\sigma_i^{(k)}\Big(\sum_{j=1}^{n}A_j\sigma_j^{(k)}\Big)^3}{M^{(k)}\lambda_k};$$

$$E_{i1}^{(k)}=m_i\varepsilon_i\omega^2(\tau)\sigma_i^{(k)};\quad M^{(k)}=\sum_{i=1}^{n}m_i\sigma_i^{(k)2}. \qquad /I6/$$

$\lambda_k(k=1,2,\dots,n)$ – natural frequencies of the system. Setting the right-hand sides of /I5/ to be zero and eliminating the phase from the resulting relations, we develop the formula for determination of stationary vibrations amplitude of multidisc rotor:

$$\omega=\lambda_e\sqrt{\frac{1}{K}\left\{\left(1-\frac{2\delta_e^2}{\lambda_e^2}\right)\pm\sqrt{\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}E_i^{(k)}E_j^{(k)}\cos(x_i-x_j)}{M^{(k)2}a_k^2}-\frac{4\delta_e^{(k)2}}{\lambda_e^{(k)2}}}\right\}}$$

where $K=1-\dfrac{\sum_{i=1}^{n}\sum_{j=1}^{n}E_i^{(k)}E_j^{(k)}\cos(x_i-x_j)}{M^{(k)2}a_k^2}$ /I7/

In the case of electronic analog approach it is advisable by changing the variables to reduce the equations /I5/, which characterize nonstationary vibrations amplitude and phase of the multidisc rotor, to the form:

$$a_k=a_k\cos\psi_k,\quad b_k=a_k\sin\psi_k$$

[illegible] denotes the number of the resonance zone

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$$\frac{dU_\kappa}{dt} = -\delta_e^{(\kappa)} U_\kappa - [\lambda_\kappa - \omega(t)]V \mp \mathcal{L}V(U^2+V^2) + \sum_{i=1}^{n} \mathcal{E}_i^{(\kappa)}(t)\sin x_i,$$

$$\frac{dV_\kappa}{dt} = -\delta_e^{(\kappa)} V_\kappa + [\lambda_\kappa - \omega(t)]U \pm \mathcal{L}U(U^2+V^2) - \sum_{i=1}^{n} \mathcal{E}_i^{(\kappa)}(t)\cos x_i,$$

$$\mathcal{E}_i^{(\kappa)}(t) = \frac{E_i(\tau,\omega)}{M^{(\kappa)}[\lambda_\kappa + \omega(\tau)]}.$$

/18/

For the two-disc rotor the equations /18/ have been integrated on the electronic analog МН-7 . The curves in fig.6, 7 and 8 show the development of shaft-saggings in the resonance zone as a function of discs unbalance location.

§ 4. Let us consider applicability of the technique, described in chapter I, to the investigation of stability of the shaft stationary movement. For comparison of results with those, published in literature [7], we shall show stability of elastic shaft movement, assuming stiff support and counting for gyroscopic action, as well as for external and internal friction. All the assumptions, made in [8], will be resolved. The use of complex displacements $\alpha + i\varphi = z_1$, $\alpha + i\beta = z_2$ will give us for the case of straightforward precession of the shaft following set stationary oscillation equations:

$$-\frac{\delta_e\lambda_1 - \rho_c(\lambda_1 - \omega)}{2m_1\lambda_1 + m_2} U - E\sin\varphi - H\cos(\varphi + \chi) = 0$$

$$(\lambda_1 - \omega)U - E\cos\varphi + H\sin(\varphi + \chi) = 0$$

where $\delta_e = \sum_{i=1}^{n}\sum_{j=1}^{n} \varkappa_{ij}\sigma_i^{(1)}\sigma_j^{(1)}$; $\rho_c = \sum_{i=1}^{n}\sum_{j=1}^{n} \kappa_{ij}\sigma_i^{(1)}\sigma_j^{(1)}$;

$$E = \frac{m\varepsilon\omega^2}{m_1(\lambda_1+\omega)+m_2}; \quad H = \frac{(J_0 - J)\delta\omega^2\sigma_2^{(1)}}{m_1(\lambda_1+\omega)+m_2}.$$

Finding from /19/ the value of a and ψ for the shaft stationary oscillations and deriving the variation equations for amplitude and phase, we shall define, in accordance with /8/, the condition of stability for the shaft stationary movement in the beyond - critical speed region:

$$\omega = \dot{\varphi} < \omega_\kappa\left(1 + \frac{\delta_e}{\rho_c}\right). \qquad /20/$$

The condition /20/ is similar to one obtained by F.M.Dimentberg /6/, who investigated stability against perturbations of the generalized coordinates, but the technique proposed here is much simpler.

In conclusion it is necessary to note that the asymptotic technique enables us to solve broader range of nonlinear problems of turbine rotor dynamics, than it was possible to show in this report. In [4] particularly the following problems are discussed: vibration of coaxial rotors, connected by supporting elements and subjected to double frequency perturbation; coupled vibrations of rotor and the case; vibration of rotor, the mass of which is distributed along the length, and so on. In [7] the author analyzed oscillation of the shaft in bearings, taking into account nonlinearity of the hydrodynamic forces. It is shown that nonlinearity of the hydrodynamic forces, which arise in the oil film of the bearing, leads to fraction resonance at angular velocities equal to double critical speed.

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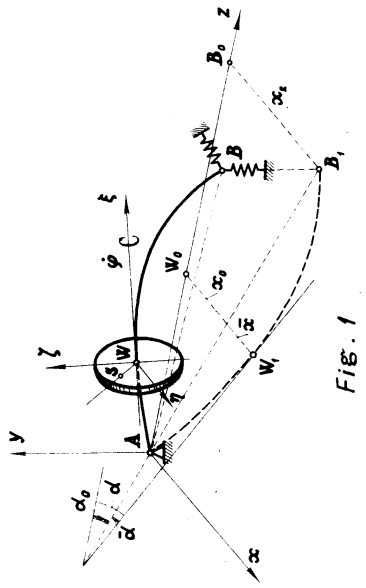

Fig. 1

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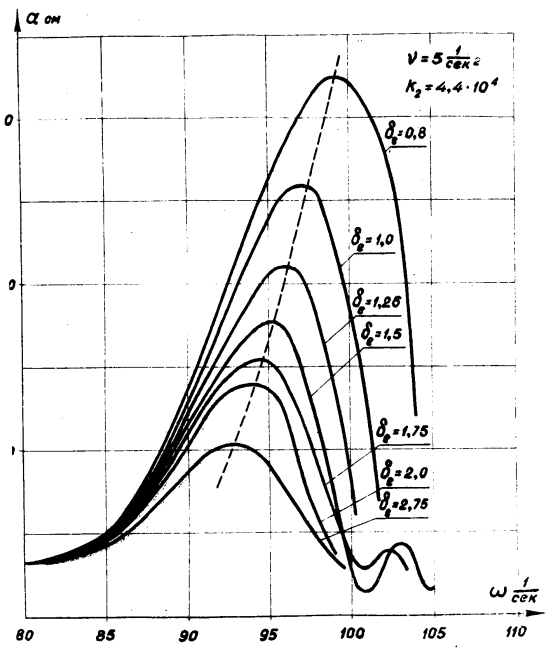

Fig. 2

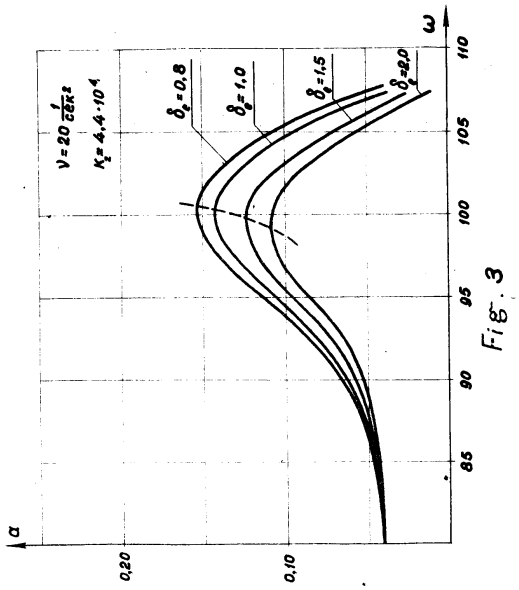

Fig. 3

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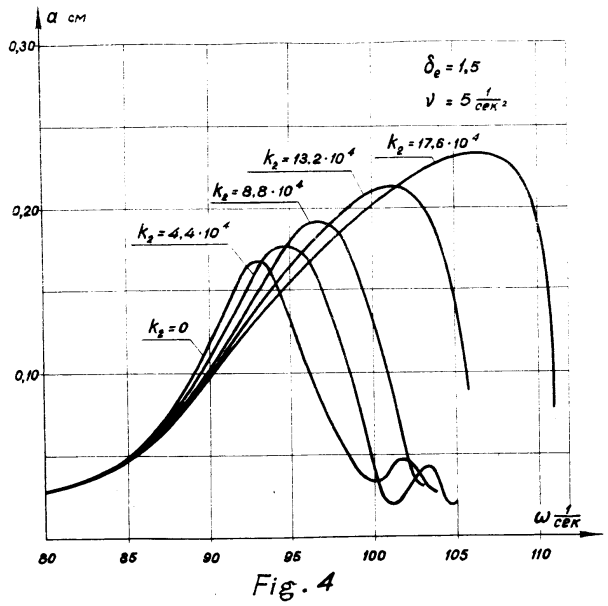

Fig. 4

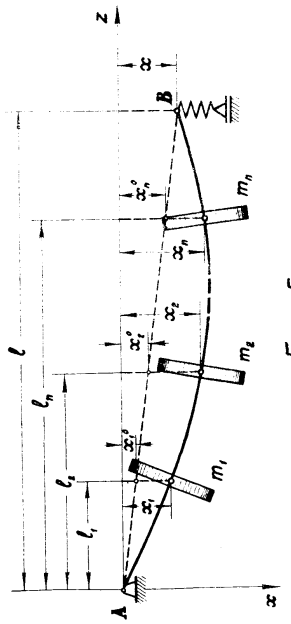

Fig. 5

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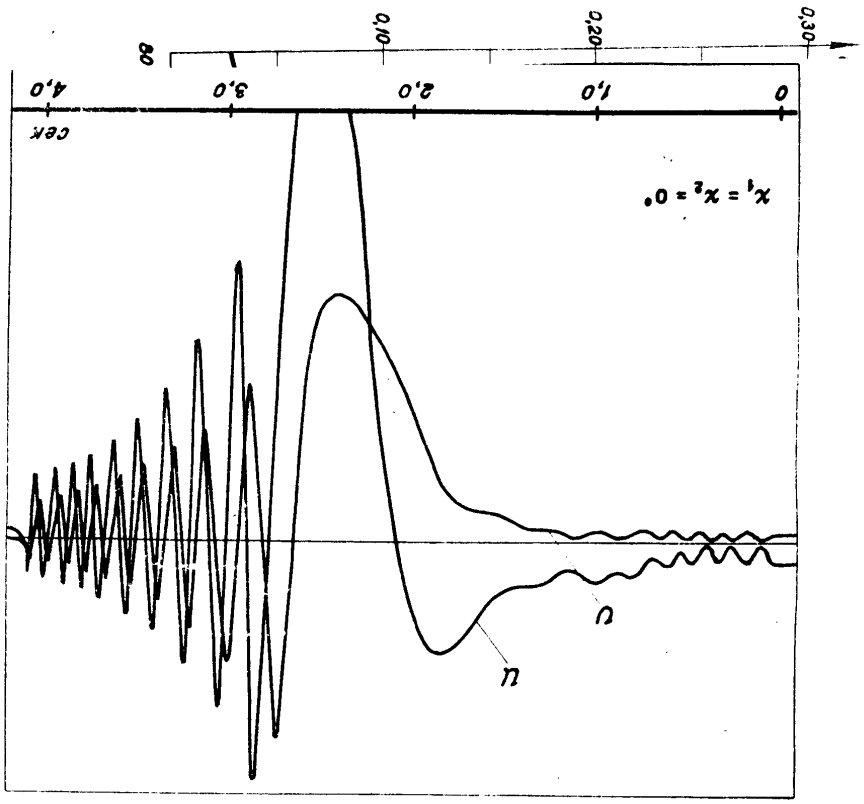

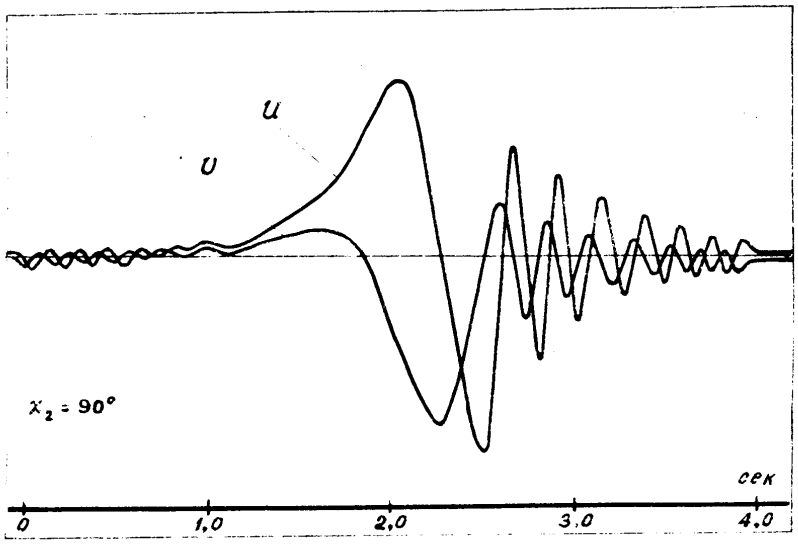

Fig. 7

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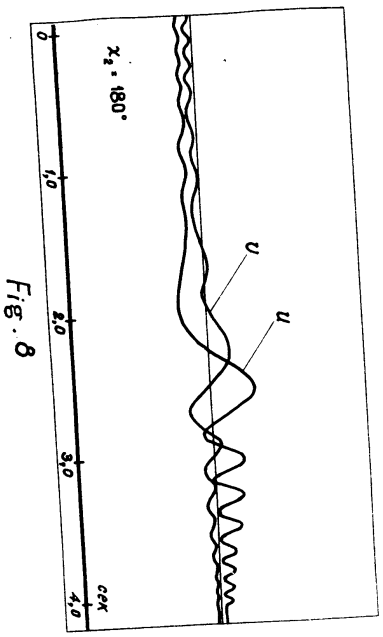

Fig. 8

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M. Z. KOLOVSKY — Leningrad, Polytechnical Institute

М. З. КОЛОВСКИЙ

О ПРИМЕНЕНИИ МЕТОДА МАЛОГО ПАРАМЕТРА ДЛЯ ОПРЕДЕЛЕНИЯ РАЗРЫВНЫХ ПЕРИОДИЧЕСКИХ РЕШЕНИЙ

ON THE APPLICATION OF THE SMALL PARAMETER METHOD FOR THE DETERMINATION OF DISCONTINUOUS PERIODIC SOLUTIONS.

M.Z.Kolovsky (Leningrad).

Poincare([1]) has used the small parameter method for the systems of differential equations, the right-hand sides of those being the analytical functions of the unknown quantities and the small parameter.

Further it was shown, that this method could be also used in cases, when the right-hand sides of the equations had the continuous derivatives of the second order([2]), or they were the piece-continuous functions([3]).

For the latter case the integral form of the necessary conditions of the existance of periodic solutions was received in([4]).

The application of the small parameter method for the determination of the periodic solutions, close to discontinuous solutions of the generating system is discussed in this paper.

The results received here can be used for the investigation of the systems, in which the impacts occur.

1. The formulation of problem. The conditions of gaps.

Consider an equation

$$\frac{dx}{dt} = X(x,t,\mu) \qquad (1.1)$$

(x and X are the n-dimensional column vectors; μ is the small parameter).

The right-hand side of this equation satisfies the following conditions:

1) function X is simply defined with arbitrary real t, for any values of μ from segment $0 \le \mu \le \mu_0$ and for all x from a certain n-dimensional domain G;

2) for all x from G and for $0 \le \mu \le \mu_0$ is continuous and periodic with respect to t with the period T

$$X(x,t+T,\mu) \equiv X(x,t,\mu)$$

3) domain G can be divided by the smooth surfaces

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$$\varphi_\kappa(x) = 0 \qquad (1.2)$$

referred further as "discontinuity surfaces" into domains $G_1, G_2, \ldots$, derivatives $\frac{\partial^2 X}{\partial x^2}$, $\frac{\partial^2 X}{\partial x \partial \mu}$ and $\frac{\partial^2 X}{\partial \mu^2}$ being continuous in each of them, the boundaries including.

It is necessary to find the periodic solution of the equation (1.1) with the period T satisfying the following "conditions of gaps" on the surfaces(1.2):

$$x^+ = \Psi_\kappa(x^-, \mu) \qquad (1.3)$$

where Ψ_κ has the continuous partial derivatives of the second order.

It is supposed that the generating system

$$\frac{dx_0}{dt} = X(x_0, t, 0) \qquad (1.4)$$

has the family of periodic solutions with the period T depending on ℓ independent parameters, all integral curves of this family

$$x_0 = x_0(t, a_1, \ldots, a_\ell) \qquad (1.5)$$

intersecting the same ensemble of discontinuity surfaces $\varphi_\kappa(x) = 0 \ (\kappa = 1, \ldots, m)$. In the points of intersections the following conditions are fulfilled.

$$x_0^+ = \Psi_\kappa(x_0^-, 0) \qquad (1.6)$$

$$\frac{\partial \varphi_\kappa}{\partial x} \frac{dx_0}{dt} \neq 0 \qquad (1.7)$$

(the line above means further a row vector).

Let G_s be the part of the domain G containing the parts of the trajectories (1.5), comprised between the surfaces $\varphi_s(x) = 0$ and $\varphi_{s+1}(x) = 0$.

As it has been shown in [4], the general solution of the equation(1.1)

$$x_s = x_s(t, C_s, \mu) \qquad (1.8)$$

having the continuous partial derivatives with respect to t, C_s and μ can be found in G_s with the accepted assumptions.

With $\mu \to 0$ this solution is continuously turned into one of the solutions of the equation (1.4). Therefore

[illegible] parts of the [illegible] some values [illegible].

For the determination of the periodic solution of the equation (1.1) we substitute (1.8) in (1.3) and (1.2).

Then we get the following expressions:

$$x_\kappa(\tau_\kappa, C_{\kappa-1}, \mu) = \Psi_\kappa[x_\kappa(\tau_\kappa, C_\kappa, \mu), \mu] \quad (\kappa = 2, \ldots, m) \qquad (1.10)$$

$$x_m(\tau_1 + T, C_m, \mu) = \Psi_1[x_1(\tau_1, C_1, \mu), \mu] \qquad (1.11)$$

$$\varphi_\kappa[x_\kappa(\tau_\kappa, C_\kappa, \mu)] = 0 \quad (\kappa = 1, \ldots, m) \qquad (1.12)$$

From these m vector and m scalar equations, the vectors $C_1, \ldots, C_m$ and the moments $\tau_1, \ldots, \tau_m$ of the intersections of the integral curves with the discontinuity surfaces, can be defined.

With $\mu = 0$ the system of equations (1.10)-(1.12) has the family of solutions depending on ℓ parameters.

Therefore the Jacobian of this system is equal to zero if $\mu = 0$, and its matrix defect is ℓ.

It is easy to prove that with sufficiently small μ the system (1.10)-(1.12) has solution close to the curves of the family (2.5), in which the values of parameters α_δ satisfy some equations

$$P_\nu(a_1, \ldots, a_\ell) = 0 \qquad (1.13)$$

If with these values of α_δ

$$\frac{\partial(P_1, \ldots, P_\ell)}{\partial(a_1, \ldots, a_\ell)} \neq 0 \qquad (1.14)$$

the solution of the system (1.10)-(1.12) will be simple and the separate solution of the equation (1.1) will correspond to it.

If the generating equation has the only periodic solution, satisfying the conditions (1.4), the equation (1.1) has also the only solution satisfying the conditions of gaps (1.3) and being close to the generating one.

The proof of these assertions is based upon the consideration of the system (1.10)-(1.12) as the transformation of points depending on the parameter μ. It follows immediately from the results, obtained in [3], and therefore will not be discussed in this paper.

If the generating solution satisfying the conditions (1.13) is determined, the solution of the system (1.10) - (1.12) with $\mu \neq 0$ can be determined by means of the iteration procedure.

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Now we are to find such a form of the conditions (1.13) that can be composed without determination of the general solutions (1.8).

2. The conditions of the existance of periodic solutions. For the simplicity of writing we shall further write down the expression (1.11) in the form of (1.10) with $k=1$, keeping in mind that τ_1 must be replaced by τ_1+T in its left-hand side.

Let the values of $\alpha_1^*, \dots, \alpha_\ell^*$ be defined satisfying (1.13). Thus one of the solutions (1.5) of the equation (1.4) and all C_k and τ_k become known.

For sufficiently small μ there is the solution of the system (1.10)–(1.12), in which the values of C_k and τ_k are close to those of C_k and τ_k in generating solution.

Then the system (1.10)–(1.12) is equivalent in the first approximation to the following:

$$\left\{\frac{\partial x_k}{\partial C_{k-1}}\right\}\delta C_{k-1}+\left[\frac{\partial \Phi_k}{\partial x^*}\right]\left[\frac{\partial x_k}{\partial C_k}\right]\delta C_k+\left\{\left(\frac{\partial x_{k+1}}{\partial \tau_k}\right)-\left[\frac{\partial \Phi_k}{\partial x^*}\right]\left(\frac{\partial x_k}{\partial \tau_k}\right)\right\}\delta\tau_k=$$
$$=\mu\left\{\left[\frac{\partial \Phi_k}{\partial x^*}\right]\left(\frac{\partial x_k}{\partial \mu}\right)-\left(\frac{\partial x_{k+1}}{\partial \mu}\right)+\left(\frac{\partial \Phi_k}{\partial \mu}\right)\right\} \quad (k=1,\dots,m) \qquad (2.1)$$

$$\left(\frac{\partial \varphi_k}{\partial x}\right)\left[\frac{\partial x_k}{\partial C_k}\right]\delta C_k+\left(\frac{\partial \varphi_k}{\partial x}\right)\left(\frac{\partial x_k}{\partial \tau_k}\right)\delta\tau_k=-\mu\left(\frac{\partial \varphi_k}{\partial \mu}\right)\left(\frac{\partial x_k}{\partial \mu}\right) \quad (k=1,\dots,m) \qquad (2.2)$$

In these equations $\mu=0$ is substituted in the elements of matrixes, enclosed in square brackets, and in vectors, enclosed in round brackets. $C_k+\delta C_k$ and $\tau_k+\delta\tau_k$ are the first approximations of the solution of the equations (1.10)–(1.12).

The determinant of the left-hand side of the linear inhomogeneous equations (2.1)–(2.2) coincides with the Jacobian of the system (1.10)–(1.12) with $\mu=0$ therefore its matrix defect is ℓ.

The system (2.1)–(2.2) has the solutions, if the vector, formed by the right-hand sides of these equations, is ortogonal to all ℓ independent solutions of the system conjugated with the homogeneous one. Its matrix is formed by the transposition of the matrix of the homogeneous system.

It is evident that the conditions of the ortogonality are the same as (1.13).

Let us transform the right-hand sides of the equations (2.1) ans (2.2).

Functions $\left(\frac{\partial x_k}{\partial \mu}\right)$ satisfy the equations

$$\frac{d}{dt}\left(\frac{\partial x_k}{\partial \mu}\right)=\left[\frac{\partial X}{\partial x_k}\right]\left(\frac{\partial x_k}{\partial \mu}\right)+\left(\frac{\partial X}{\partial \mu}\right) \quad (k=1,\dots,m) \qquad (2.3)$$

that can be received by differentiation of the equation (1.4). An appropriate homogeneous equation

$$\frac{dy_k}{dt}=\left[\frac{\partial X}{\partial x_k}\right]y_k \qquad (2.4)$$

which is a variational equation for the solution x_{0k}, has the fundamental system of solutions, forming the matrix $Y_k(t,\tau)$ and these solutions can always be chosen so that

$$Y_k(\tau,\tau)=E$$

where E is the unitary matrix.

In this case the function

$$\left(\frac{\partial x_k}{\partial \mu}\right)=\int_{\tau_k}^{t}Y_k(t,\tau)\left(\frac{\partial X}{\partial \mu}(\tau)\right)d\tau \qquad (2.5)$$

satisfies the equation (2.3) what can be easily proved by a substitution.

It is evident that

$$\frac{\partial x_k}{\partial \mu}(\tau_k)=0, \quad \frac{\partial x_{k-1}}{\partial \mu}(\tau_k)=\int_{\tau_{k-1}}^{\tau_k}\left[Y_{k-1}(\tau_k,\tau)\right]\left(\frac{\partial X}{\partial \mu}(\tau)\right)d\tau$$

Therefore

$$\left[\frac{\partial \Phi_k}{\partial x^*}\right]\left(\frac{\partial x_k}{\partial \mu}\right)-\left(\frac{\partial x_{k+1}}{\partial \mu}\right)+\left(\frac{\partial \Phi_k}{\partial \mu}\right)=-\int_{\tau_{k-1}}^{\tau_k}\left[Y_{k-1}(\tau_k,\tau)\right]\left(\frac{\partial X}{\partial \mu}(\tau)\right)d\tau+\left(\frac{\partial \Phi_k}{\partial \mu}\right) \qquad (2.6)$$

$$\left(\frac{\partial \varphi_k}{\partial x}\right)\left(\frac{\partial x_k}{\partial \mu}\right)=0 \qquad (2.7)$$

Transposing the matrix of the left-hand sides of the equations (2.1) and (2.2), we shall compose the conjugate homogeneous system, B_k and θ_k meaning the new unknown vectors and scalars:

$$-\left[\frac{\partial x_k}{\partial C_k}(\tau_k)\right]^*\left[\frac{\partial \Phi_k}{\partial x^*}(\tau_k)\right]^*B_k+\left[\frac{\partial x_k}{\partial C_k}(\tau_{k-1})\right]^*B_{k+1}+\left[\frac{\partial x_k}{\partial C_k}(\tau_k)\right]^*\left(\frac{\partial \varphi_k}{\partial x}(\tau_k)\right)\theta_k=0 \qquad (2.8)$$
$$(k=1,\dots,m)$$

$$\Delta_k B_k+\dot{\varphi}_k\theta_k=0 \quad (k=1,\dots,m) \qquad (2.9)$$

where asterisk denotes the transposition of matrix and

$$\Delta_k=\left(X^*(\tau_k)\right)-\left[\frac{\partial \Phi_k}{\partial x^*}(\tau_k)\right]^*\left(X^*(\tau_k)\right) \qquad (2.10)$$

$$\dot{\varphi}_k=\left(\frac{\partial \varphi_k}{\partial x}\right)\left(\frac{\partial x_k}{\partial \tau_k}\right) \qquad (2.11)$$

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In accordance with (1.7) $\dot{\varphi}_\kappa \neq 0$; therefore θ_κ can be found from (2.9) and substituted in (2.8).

$$-\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_\kappa)\right]^*\left[\frac{\partial \Phi_\kappa}{\partial x^*}(\tau_\kappa)\right]\theta_\kappa^*+\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_{\kappa+1})\right]^*\theta_{\kappa+1}^*-\frac{\bar{\Delta}_\kappa(\tau_\kappa)\theta_\kappa}{\dot{\varphi}_\kappa(\tau_\kappa)}\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_\kappa)\right]^*\left(\frac{\partial \varphi_\kappa}{\partial x}(\tau_\kappa)\right)\theta_\kappa=0 \quad (2.12)$$

The equation (2.12) has also ℓ independent systems of solutions $B_\kappa^{(1)},\ldots,B_\kappa^{(\ell)}$ $(\kappa=1,\ldots,m)$.

Consider now the conjugate variational equations

$$\frac{du_\kappa}{dt}+\left[\frac{\partial X}{\partial x_\kappa}\right]^* u_\kappa=0 \quad (2.13)$$

Suppose, that ℓ independent solutions of these equations have been found satisfying the conditions (2.12) with the substitution:

$$B_{\kappa+1}^{(i)}=u_\kappa^{(i)}(\tau_{\kappa+1}) \quad (2.14)$$

Then the conditions (2.12) aquire the following form:

$$-\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_\kappa)\right]^*\left[\frac{\partial \Phi_\kappa}{\partial x^*}(\tau_\kappa)\right]^* u_{\kappa-1}^{(i)}(\tau_\kappa)+\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_{\kappa+1})\right]^* u_\kappa^{(i)}(\tau_{\kappa+1})-$$
$$-\frac{\bar{\Delta}_\kappa(\tau_\kappa)\,u_{\kappa-1}^{(i)}(\tau_\kappa)}{\dot{\varphi}_\kappa(\tau_\kappa)}\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_\kappa)\right]^*\left(\frac{\partial \varphi_\kappa}{\partial x}(\tau_\kappa)\right)=0 \quad (2.15)$$

As (1.9) is the general solution of the equation (1.4) in the domain G_κ, the $\left[\frac{\partial x_\kappa}{\partial c_\kappa}(t)\right]$ is the solution matrix of the equation (2.4).

Therefore, in accordance with the well-known properties of the solutions of the conjugate systems, we have:

$$\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_{\kappa+1})\right]^* u_\kappa^{(i)}(\tau_{\kappa+1})=\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_\kappa)\right]^* u_\kappa^{(i)}(\tau_\kappa) \quad (2.16)$$

Substituting in (2.15) we receive

$$\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_\kappa)\right]^*\left\{u_\kappa^{(i)}(\tau_\kappa)-\left[\frac{\partial \Phi_\kappa}{\partial x^*}(\tau_\kappa)\right]^* u_{\kappa-1}^{(i)}(\tau_\kappa)-\frac{\bar{\Delta}_\kappa(\tau_\kappa)\,u_{\kappa-1}^{(i)}(\tau_\kappa)}{\dot{\varphi}_\kappa(\tau_\kappa)}\left(\frac{\partial \varphi_\kappa}{\partial x}(\tau_\kappa)\right)\right\}=0 \quad (2.17)$$

As the matrix $\left[\frac{\partial x_\kappa}{\partial c_\kappa}(\tau_\kappa)\right]$ is not singular,

$$u_\kappa^{(i)}(\tau_\kappa)-\left[\frac{\partial \Phi_\kappa}{\partial x^*}(\tau_\kappa)\right]^* u_{\kappa-1}^{(i)}(\tau_\kappa)-\frac{\bar{\Delta}_\kappa(\tau_\kappa)\,u_{\kappa-1}^{(i)}(\tau_\kappa)}{\dot{\varphi}_\kappa(\tau_\kappa)}\left(\frac{\partial \varphi_\kappa}{\partial x}(\tau_\kappa)\right)=0$$
$$(i=1,\ldots,\ell;\ \kappa=1,\ldots,m) \quad (2.18)$$

The conditions of ortogonality of the right-hand sides of the equations (2.1) and (2.2) to the characteristic vectors of the conjugate system can be written down in the following form:

$$\sum_{\kappa=1}^{m}\left\{\int_{\tau_{\kappa-1}}^{\tau_\kappa} u_{\kappa-1}^{(i)}(\tau_\kappa)\left[Y_{\kappa-1}(\tau_\kappa,\tau)\right]\left(\frac{\partial X}{\partial \mu}(\tau)\right)d\tau-\tilde{u}_{\kappa-1}^{(i)}(\tau_\kappa)\left(\frac{\partial \Phi_\kappa}{\partial \mu}(\tau_\kappa)\right)\right\}=0$$

Taking into account, that $\left[Y_{\kappa-1}(\tau_\kappa,\tau)\right]$ is the solution matrix of the equation (2.4), we obtain

$$\tilde{u}_{\kappa-1}^{(i)}(\tau_\kappa)\left[Y_{\kappa-1}(\tau_\kappa,\tau)\right]\left(\frac{\partial X}{\partial \mu}(\tau)\right)=\left(\frac{\partial X}{\partial \mu}(\tau)\right)\left[Y_{\kappa-1}(\tau_\kappa,\tau)\right]^* u_{\kappa-1}^{(i)}(\tau_\kappa)=$$
$$=\left(\frac{\partial X}{\partial \mu}(\tau)\right)u_{\kappa-1}^{(i)}(\tau)$$

and the necessary conditions of the existance of the periodic solution of the equation (1.1), close to one of the solutions (1.5) can be finally written down as following:

$$P(\alpha_1^*,\ldots,\alpha_\ell^*)=\int_0^T\left(\frac{\partial X}{\partial \mu}(\tau)\right)u^{(i)}(\tau)\,d\tau-\sum_{\kappa=1}^{m}\left(\frac{\partial \Phi_\kappa}{\partial \mu}(\tau_\kappa)\right)u_{\kappa-1}^{(i)}(\tau_\kappa)=0$$
$$(i=1,\ldots,\ell) \quad (2.19)$$

These equations can be used for the determination of the generating solution.

In (2.19) $u^{(i)}(\tau)$ are the solutions of the equations (2.13) depending on the parameters $\alpha_1,\ldots,\alpha_\ell$ and satisfying the conditions of gaps (2.18). If the solutions of the equations (1.1) and the functions $X(x,t,\mu)$ have no discontinuities on the surfaces (1.2), we obtain

$$\left[\frac{\partial \Phi}{\partial x^*}\right]=E\,,\qquad \Delta_\kappa=0$$

Then the conditions (2.18) become the conditions of continuity and periodicity of the functions $u^{(i)}(t)$, and the conditions (2.19) assume the usual form, obtained, for instance, in [2].

If the solutions must be continuous, but function $X(x,t,\mu)$ has the discontinuities of the first order on the surfaces (1.2), the conditions (2.18) and (2.19) coincide with those obtained in [4].

The conditions (2.18) can be expressed in another, more convenient form.

As the family of periodic solutions (1.5) satisfies the equations (1.4), the system of variational equations (2.4) has ℓ independent periodic solutions

$$\frac{\partial x_0}{\partial \alpha_1},\ldots,\frac{\partial x_0}{\partial \alpha_\ell} \quad (2.20)$$

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The values of these functions on the discontinuity surfaces and the moments of intersections of integral curves with these surfaces satisfy the homogeneous system of linear equations formed by the left-hand sides of the equations (2.1) and (2.2).

Substituting in them

$$\delta c_k = \frac{\partial c_k}{\partial \alpha_i}\delta\alpha_i \qquad (2.21)$$

we find

$$\left(\frac{\partial x_{k+1}}{\partial \alpha_i}\right)\delta\lambda_i - \left[\frac{\partial P_k}{\partial x'}\right]\left(\frac{\partial x_k}{\partial \alpha_i}\right)\delta\lambda_i + \left\{\left(\frac{\partial x_{k+1}}{\partial \tau_k}\right) - \left[\frac{\partial P_k}{\partial x'}\right]\left(\frac{\partial x_k}{\partial \tau_k}\right)\right\}\delta\tau_k = 0 \qquad (2.22)$$

$$\left(\frac{\partial q_k}{\partial x}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right)\delta\alpha_i + \left(\frac{\partial q_k}{\partial x}\right)\left(\frac{\partial x_k}{\partial \tau_k}\right)\delta\tau_k = 0 \qquad (2.23)$$

Determining $\delta\tau_k$ from (2.23), substituting it in (2.22) and cancelling $\delta\alpha_i$, we obtain

$$\left(\frac{\partial x_{k+1}}{\partial \alpha_i}\right) - \left[\frac{\partial P_k}{\partial x'}\right]\left(\frac{\partial x_k}{\partial \alpha_i}\right) - \Delta_k \frac{\left(\frac{\partial q_k}{\partial x}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right)}{\dot{q}_k} = 0 \qquad (k=1, 2, \ldots, n) \qquad (2.24)$$

It is evident that the transposition of the matrix of the system (2.24) gives the matrix of the system(2.18).

Let us multiply the equations (2.25) from the left by the vector $u_k^{(i)}(\tau_k)$

$$u_k^{(i)}\left(\frac{\partial x_{k+1}}{\partial \alpha_i}\right) - u_{k+1}^{(i)}\left[\frac{\partial x_k}{\partial x'}\right]\left(\frac{\partial x_k}{\partial \alpha_i}\right) - u_k^{(i)}\Delta_k\frac{\left(\frac{\partial q_k}{\partial x}\right)\left(\frac{\partial x_k}{\partial \alpha_i}\right)}{\dot{q}_k} = 0 \qquad (2.25)$$

On the other hand, multiplying (2.18) from the left by $\left(\frac{\partial x_k}{\partial \alpha_i}\right)$, we obtain:

$$\left(\frac{\partial x_{k+1}}{\partial \alpha_i}\right)u_{k+1}^{(i)} - \left(\frac{\partial x_k}{\partial \alpha_i}\right)\left[\frac{\partial P_k}{\partial x'}\right]^T u_{k+1}^{(i)} - \left(\frac{\partial x_k}{\partial \alpha_i}\right)\left(\frac{\partial q_k}{\partial x}\right)\frac{\Delta_k}{\dot{q}_k}u_{k+1}^{(i)} = 0 \qquad (2.26)$$

But

$$u_{k+1}^{(i)}\left[\frac{\partial x_k}{\partial x'}\right]\left(\frac{\partial x_k}{\partial \alpha_i}\right) = \left(\frac{\partial x_k}{\partial \alpha_i}\right)\left[\frac{\partial P_k}{\partial x'}\right]^T u_{k+1}^{(i)} + \left(\frac{\partial x_k}{\partial \alpha_i}\right)\left(\frac{\partial q_k}{\partial x}\right)\frac{\Delta_k}{\dot{q}_k}u_{k+1}^{(i)},$$

$$\left(\frac{\partial x_{k+1}}{\partial \alpha_i}\right)u_{k+1}^{(i)} - \left(\frac{\partial x_k}{\partial \alpha_i}\right)u_k^{(i)} = \Delta_k u_k^{(i)}.$$

Therefore it must be

[illegible] (2.27)

[illegible]

by $\left(\frac{\partial x_k}{\partial \alpha_i}\right)$ being constant.

On the other hand, if the condition (2.27) is fulfilled, the expression (2.26) is derived from (2.25). Therefore, in this case

$$\left(\frac{\partial x_k}{\partial \alpha_i}\right)\left\{u_k^{(i)} - \left[\frac{\partial P_k}{\partial x'}\right]^T u_{k+1}^{(i)} - \left(\frac{\partial q_k}{\partial x}\right)\frac{\Delta_k}{\dot{q}_k}u_{k+1}^{(i)}\right\} = 0 \qquad (2.28)$$

and the conditions (2.16) are fulfilled if the vector $\left(\frac{\partial x_k}{\partial \alpha_i}\right)$ is not ortogonal to the vector enclosed in braces.

3. Some generalizations.

a) The obtained results are easily generalized for the case of autonomous equation

$$\frac{dx}{dt} = X(x, \mu) \qquad (3.1)$$

where X satisfies the conditions mentioned in §1, and the solutions that have to be found satisfy the conditions of gaps (1.3) on the surfaces (1.2).

Let the generating system

$$\frac{dx_0}{dt} = X(x_0, 0) \qquad (3.2)$$

have the family of periodic solutions (1.5), their period being also the function of parameters, and solutions satisfying the conditions (1.6).

As the system (3.2) is autonomous, one of the parameters in (1.5) is the arbitrary phase. Therefore

$$x_0 = x_0(t+\alpha_1, \alpha_2, \ldots, \alpha_n) \qquad (3.3)$$

It is evident that the period of these solutions does not depend on α_1

$$T = T(\alpha_2, \ldots, \alpha_n) \qquad (3.4)$$

The system (3.1) is also autonomous; therefore if the system of equations (1.10)-(1.12) has the solution with [illegible], it has the family of solutions depending on α_1. This means that one of the values of [illegible] may be arbitrary. Let us suppose, for instance, that [illegible] at the same time the period [illegible] is the solution of the equation [illegible] are unknown. Thus the system [illegible] periodic [illegible] and [illegible] unknown number [illegible]

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the periodic solution remain correct.

Let us show that the form of the equations (1.13) also holds good.

Instead of the system (1.10)–(1.12) consider the equations:

$$x_{\kappa+1}(\tau_\kappa, C_{\kappa+1}, \mu) = \Phi_\kappa[x_\kappa(\tau_\kappa, C_\kappa, \mu), \mu] \quad (\kappa = 2, \dots, m) \tag{3.5}$$

$$x_m(T, C_m, \mu) = \Phi_1[x_1(0, C_1, \mu), \mu] \tag{3.6}$$

$$\varphi_\kappa[x_\kappa(\tau_\kappa, C_\kappa, \mu)] = 0 \quad (\kappa = 2, \dots, m) \tag{3.7}$$

$$\varphi_1[x_1(T, C_1, \mu)] = 0 \tag{3.8}$$

Or, in the first approximation:

$$\left[\frac{\partial x_{\kappa+1}}{\partial C_{\kappa+1}}\right]\delta C_{\kappa+1} - \left[\frac{\partial \Phi_\kappa}{\partial x_\kappa}\right]\left[\frac{\partial x_\kappa}{\partial C_\kappa}\right]\delta C_\kappa + \Delta_\kappa \delta\tau_\kappa = -\mu\left\{\int_{\tau_\kappa}^{\tau_{\kappa+1}}\left[Y_{\kappa+1}(\tau_{\kappa+1}, \tau)\right]\left(\frac{\partial X}{\partial \mu}(\tau)\right)d\tau - \left[\frac{\partial \Phi_\kappa}{\partial \mu}\right]\right\}$$
$$(\kappa = 2, \dots, m) \tag{3.9}$$

$$\left[\frac{\partial x_m}{\partial C_m}\right]\delta C_m - \left[\frac{\partial \Phi_1}{\partial x^1}\right]\left[\frac{\partial x_1}{\partial C_1}\right]\delta C_1 + \left[\frac{\partial x_m}{\partial T}\right]\delta T = -\mu\left\{\int_{\tau_m}^{T}\left[Y_m(\tau, \tau)\frac{\partial X}{\partial \mu}(\tau)\right]d\tau + \left[\frac{\partial \Phi_1}{\partial \mu}\right]\right\} \tag{3.10}$$

$$\left[\frac{\partial \varphi_\kappa}{\partial x}\right]\left[\frac{\partial x_\kappa}{\partial C_\kappa}\right]\delta C_\kappa + \left[\frac{\partial \varphi_\kappa}{\partial x}\right]\left[\frac{\partial x_\kappa}{\partial \tau_\kappa}\right]\delta\tau_\kappa = 0 \quad (\kappa = 2, \dots, m) \tag{3.11}$$

$$\left[\frac{\partial \varphi_1}{\partial x}\right]\left[\frac{\partial x_1}{\partial C_1}\right]\delta C_1 + \left[\frac{\partial \varphi_1}{\partial x}\right]\left[\frac{\partial x_1}{\partial T}\right]\delta T = 0 \tag{3.12}$$

where the right-hand sides are written down in accordance with (2.6) and (2.7).

It is easy to show that these equations can be obtained from (2.1)–(2.2) with the following substitution:

$$\tau_1 = 0, \quad \delta\tau_\kappa = \delta T, \quad \Lambda_{\kappa m} = \frac{\partial x_{\kappa m}}{\partial T}$$

Therefore it is not necessary to bring the further transformations, the following equations for the parameters of the generating solution being obtained as a result:

$$P_i(\alpha_2^*, \dots, \alpha_\ell^*) = \int_0^T\left[\frac{\partial X}{\partial \mu}(\tau)\right]U^{(i)}(\tau)d\tau - \sum_{\kappa=1}^{m}\left[\frac{\partial \Phi_\kappa}{\partial \mu}(\tau_\kappa)\right]U_{\kappa+1}^{(i)}(\tau_{\kappa+1}) = 0 \tag{3.13}$$

where

$$T = T(\alpha_2^*, \dots, \alpha_\ell^*)$$

and functions $U_\kappa^{(i)}$ satisfy the conditions (2.18).

b) The analogous conditions of the existence of periodic solutions can be obtained for more common suppositions, than in § 1. Instead of the equations (1.2) we can assume

$$\varphi_\kappa(x, t) = 0 \tag{3.14}$$

where φ_κ are the periodic functions of t with the period T.

All conclusions obtained before remain in this case, but the condition (1.7) must be replaced by

$$\dot{\varphi}_\kappa = \left[\frac{\partial \varphi_\kappa}{\partial x}\right]\frac{dx_\kappa}{dt} + \left[\frac{\partial \varphi_\kappa}{\partial t}\right] \neq 0 \tag{3.15}$$

It was supposed before that the representative point remains on the surface (1.2) when the integral curve intersects it. The exclusion of this supposition and replacement of the equations (1.2) by the equations

$$\varphi_\kappa(x^+) = 0 \tag{3.16}$$

or

$$\varphi_\kappa(x^-) = 0 \tag{3.17}$$

do not change the results, obtained before.

One may also suppose that the gap in the solution occurs for a finite time, and replace the conditions (1.3) by

$$x^+ = \Phi_\kappa(x^-, \tau^+, \mu), \quad \tau^+ = \psi_\kappa(x^-, \tau^+, \mu) \tag{3.18}$$

For this case the final results are given obtained after the transformations, analogous to those in § 2.

The equations for determination of the parameters are written down as following

$$\int_0^T\left[\frac{\partial X}{\partial \mu}(\tau)\right]U^{(i)}(\tau)d\tau + \sum_{\kappa=1}^{m}\left\{\left[\frac{\partial \Phi_\kappa}{\partial \mu}(\tau_\kappa)\right] - \frac{\partial \psi_\kappa}{\partial \mu}(\tau_\kappa)\dot{x}_\kappa(\tau_\kappa)\right\}U_{\kappa+1}^{(i)}(\tau_{\kappa+1}) = 0 \tag{3.19}$$
$$(i = 1, \dots, \ell)$$

The functions $U^{(i)}$ satisfy the conditions

$$U_\kappa^{(i)}(\tau_\kappa) - \left[\frac{\partial \Phi_\kappa}{\partial x^+}(\tau_\kappa)\right]U_{\kappa+1}^{(i)}(\tau_\kappa) + \frac{U_{\kappa+1}^{(i)}(\tau_\kappa)}{\varphi_\kappa(\tau_\kappa^+)}\left\{\left[\frac{\partial \Phi_\kappa}{\partial x^+}(\tau_\kappa)\right]\left[\frac{\partial x_\kappa}{\partial \tau_\kappa^+}(\tau_\kappa)\right] + \right.$$
$$\left. + \left[\frac{\partial \Phi_\kappa}{\partial \tau_\kappa^+}(\tau_\kappa)\right] - \dot{\psi}_\kappa(\tau_\kappa)\dot{x}_\kappa(\tau_\kappa)\right\}\left\{\left[\frac{\partial \varphi_\kappa}{\partial x}(\tau_\kappa)\right] + U_{\kappa+1}^{(i)}(\tau_\kappa)\left[\frac{\partial \varphi_\kappa}{\partial x^+}(\tau_\kappa^+)\right] = 0\right.$$
$$(i = 1, \dots, \ell; \ k = 1, \dots, m) \tag{3.20}$$

and the correlations

$$\overline{U}_{\kappa+1}^{(i)}\left(\frac{\partial x_{\kappa+1}}{\partial C_i}(\tau_\kappa)\right) = U_\kappa^{(i)}(\tau_\kappa)\left(\frac{\partial x_\kappa}{\partial C_i}(\tau_\kappa)\right) \tag{3.21}$$

By means of these correlations it is possible to find periodic solutions if the motion of the system in some of domains G_κ is determined not by the differential equations, but by the transformation of points (3.18).

4. The canonical generating equations.

Let the motion of the system be determined by the equations

$$\dot{q} = \frac{\partial H}{\partial p}(q, p) + \mu Q(q, p, t, \mu), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p) + \mu P(q, p, t, \mu) \tag{4.1}$$
$$\left(Q(q, p, t+T, \mu) = Q(q, p, t, \mu), \ P(q, p, t+T, \mu) = P(q, p, t, \mu)\right)$$

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where $q, p, \frac{\partial H}{\partial p}, \frac{\partial H}{\partial q}, Q, P$ are n-dimensional vectors; H is the Hamiltonian of the system, independent on time.

Suppose also that the periodic solution of the generating system with the period T

$$\dot{q}_0 = \frac{\partial H}{\partial p}(q_0, p_0); \quad \dot{p}_0 = -\frac{\partial H}{\partial q}(q_0, p_0) \tag{4.2}$$

exists satisfying the conditions of gaps

$$q^- = q^+; \quad p^- = \varphi_k(p^+, q) + \mu \psi_k(p^+, q, \mu) \tag{4.3}$$

on the surfaces

$$\varphi_k(q) = 0 \tag{4.4}$$

and with $\mu = 0$ in (4.3).

In other words, we have the canonical system, the motion of which is accompanied by the "impacts" on the surfaces (4.4). During these impacts the Hamiltonian, or the total energy in the conservative system, may be changed.

As the system (4.2) is the autonomous one in this case, it has the family of periodic solutions depending on the arbitrary parameter, the latter being the phase.

$$q_0 = q_0(t+\alpha); \quad p_0 = p_0(t+\alpha) \tag{4.5}$$

Then, as it is known, the system of variational equations for the generating systems has also the family of periodic solutions

$$z = \dot{q}_0(t+\alpha), \quad s = \dot{p}_0(t+\alpha) \tag{4.6}$$

and the conjugate system has the family of solutions

$$u = -\dot{p}_0(t+\alpha), \quad v = \dot{q}_0(t+\alpha) \tag{4.7}$$

The conditions (2.27) are fulfilled in this case, as

$$\bar{z}u + \bar{s}v = -\bar{p}_0\dot{q}_0 + \bar{p}_0\dot{q}_0 = 0$$

Consequently, generally speaking, the functions (4.7) satisfy the conditions (2.18) what becomes evident with the substitution.

Therefore the system (4.1) may have the periodic solution with the period T, close to one of the solutions of the generating system if the equation for the parameter α

$$\int_0^T \left\{ P(q_0, p_0, t, 0)\dot{q}_0 - Q(q_0, p_0, t, 0)\dot{p}_0 \right\} dt + \sum_k \Psi_k(p_k^+, q_{0k}, 0)\dot{q}_{0k}^- = 0 \tag{4.8}$$

has the real roots.

Let the system (4.2) be the conservative one and

$$Q(q, p, t, \mu) = 0$$

Then the integral in (4.8) determines in the first approximation the work of the perturbations P in the motion (4.5). The sum

$$\sum_k \Psi_k \dot{q}_k^-(\tau_k + \alpha) = \sum_k \Delta p_k \dot{q}_k^-(\tau_k + \alpha)$$

determines the variation of the kinetic energy during the impact. Therefore the condition (4.8) has in this case a simple physical meaning: the variation of the total energy of the system in the unknown periodic motion must be equal to zero in the first approximation.

5. The estimation of the convergence of iterations.

Having determined the parameters of the generating solution from the equations (1.13) and having examined them by means of the condition (1.14), it is possible to look for the periodic solution of the equation (1.1) using the iteration procedure to the system (1.10)-(1.12).

Let us estimate the influence of the number of the discontinuities m on the maximum value of the parameter μ, for which the convergence of the iterations can be guaranteed.

Mark the unknown scalar variables (τ_k and the components of the vectors c_k) by $z_1, \ldots, z_\rho$. It is evident that $\rho = m(n+1)$. Let $z_j = z_{j0}$ for the generating solution.

When $\mu \neq 0$, the system (1.10)-(1.12) has the only solution, as it is supposed. Consequently, it may be written down in the form:

$$z_j = z_{j0} + \mu \zeta_j(z_1, \ldots, z_\rho, \mu) \tag{5.1}$$

derivatives $\frac{\partial \zeta_j}{\partial z_i}$, $\frac{\partial \zeta_j}{\partial \mu}$ being continuous in accordance with the suppositions formulated in § 1.

Let us compose the iterations

$$z_{js} = z_{j0} + \mu \zeta_j(z_{1,s-1}, \ldots, z_{\rho,s-1}, \mu)$$
$$(j = 1, \ldots, \rho); \quad (s = 1, 2, \ldots)$$

The functions ζ_j and $\frac{\partial \zeta_j}{\partial z_i}$ are limited, being continuous. Let

$$\left|\frac{\partial \zeta_j}{\partial z_i}\right| \leq M \quad |\zeta_j| \leq N$$

where M and N are some positive numbers. Then

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$$|z_{j,s+1} - z_{j,s}| = \mu |\zeta_j(z_{1s}, \ldots, z_{ps}, \mu) - \zeta_j(z_{1,s+1}, \ldots, z_{p,s+1}, \mu)| \leq$$
$$\leq \mu M \sum_{i=1}^{p} |z_{is} - z_{i,s-1}|$$

Consequently

$$|z_{j1} - z_{j0}| \leq \mu N$$
$$|z_{j,n+1} - z_{jn}| \leq \mu^{n+1} M^n p^n N$$

Thus, the iterations converge if

$$\mu < \frac{1}{Mp} - \frac{1}{M(n+1)n} \qquad (5.2)$$

It is evident that maximum value of μ is diminished if the number of gaps increases.

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НЕКОТОРЫЕ АВТОНОМНЫЕ ЗАДАЧИ ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ

The considering report is dedicated to the research of the properties of autonomous oscillatory systems. Each of the considering system consists of the oscillatory system and the source of energy. The interaction of oscillatory system with the source of energy is an object of the research.

Oscillatory systems accomplish forced vibrations, parametrical and auto-oscillations owing to energy which they receive from the sources of energy.

It is possible to represent two types of the interaction of oscillatory system with the source of energy: 1) The first is characterized by the fact, that the source of energy receives a small influence from the oscillatory system and then it is possible to consider the motion of oscillatory system without influence to the source of energy.

Then it is possible to point out the following ideal property of the source of energy: the source of energy influence upon the oscillatory system but does not receive the influence from the oscillatory system. We shall call such source of energy as ideal energy. Most of the

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problems about oscillations of nonconservative system have been considered with the use of the ideal source of energy.

At the second type of the interaction of oscillatory system with the source of energy, the source of energy receives a considerable influence from the oscillatory system. The change of oscillatory parameters is accompanied by the change of working regime of the source energy. Such interaction is especially become active under the conditions, when the source of energy has a small reserve of the capacity.

The sources of energy which are receiving a considerable influence from the oscillatory system we shall call as nonideal. Almost all the technical real sources of energy may be considered as nonideal under certain conditions.

Oscillatory systems together with the sources of energy form autonomous systems. Oscillations of such systems depend upon the properties of non-ideal sources of energy. The solutions of the problems that are described below give an idea about the properties of such systems.

I. Statement of Problems

In the well known equation of the forced oscillations of the linear system

$$m\ddot{x} + \beta\dot{x} + cx = Q_0 \sin pt \qquad (1.1)$$

external force $Q_0 \sin pt$ is considered as the given one. This force is realized by the ideal source of energy, without influence from oscillatoty system. Here it is not necessary to consider the dynamical processes, proceeding in the source of energy. Since their influence upon the oscillatory system is perfectly determined by the known function $Q_0 \sin pt$.

But if we assume the possibilities of reciprocal action of the oscillatory system upon the source of energy, the statement of the problem must be changed. Then instead of the before known force $Q_0 \sin pt$ - perhaps it will be necessary to consider any force $Q(\varphi, \dot{\varphi})$, which we don't know yet, and which depends upon the variables φ , $\dot{\varphi}$ which define the working regime of the source of energy. This force must be determined from the equation of the motion, which describes inner processes of the energy source.

Let this movement be in the simpliest case as the rotation of the rotor of the engine with angular variable co-ordinate φ . The equation of the motion for such source of energy can be written in the following way:

$$J\ddot{\varphi} - H(\varphi, \dot{\varphi}) = L(\varphi, \dot{\varphi}) + R(\varphi, \dot{\varphi}, x, \dot{x}), \qquad (1.2)$$

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where J - is the moment of the rotor inertia,
$H(\dot{\varphi})$ - moment of the resistance forces to the rotation of the rotor,
$L(\varphi,\dot{\varphi})$ - is the moment of movement of the source of energy,
$R(\varphi,\dot{\varphi},x,\dot{x})$ - is the moment of forcesof resistance which depends upon the motion of the oscillation system. Then the interactions of the vibration system and the source of energy will be described as the systems of two differential equations. One of them is the equation of the source of energy, (1.2) and the other is the equation of vibration:

$$m\ddot{x} + \beta\dot{x} + cx = Q(\varphi,\dot{\varphi}) \qquad (1.3)$$

which becomes nonlinear.

In more general case, instead of (1.1) we may write the equation in which nonlinear and parametrical oscillations are described as the following form

$$m\ddot{x} + \beta(\dot{x}) + f_1(x,\rho t) = Q_1(\rho t).$$

Interaction of such oscillatory system with the source of energy will be described by the system of equations which will consist of the equation of the source of energy (1.2) and the equation of the oscillatory systems:

$$m\ddot{x} + \beta\dot{x} + f_2(x,\varphi) = Q_2(\varphi,\dot{\varphi}), \qquad (1.4)$$

where $f_2(x,\varphi)$ and $Q_2(\varphi,\dot{\varphi})$ represent the action of the source of energy to the vibration system.

It is necessary to introduce analogical changes into the equations which describe the motion

of more complicated oscillatory systems.

It is necessary to pay attention to the term $L(\varphi,\dot{\varphi})$ in the equation (1.2) which expresses the moment of engine, or in the general case, moving force of the source of the energy.

Here ~~in the first look~~ the including of this term into the equation of motion (1.2) may not give the exact meaning. Really the concrete shape of the function $L(\varphi,\dot{\varphi})$ is closely connected with the certain character of motion, more often $L(\varphi,\dot{\varphi})$ or $L(\dot{\varphi})$ is determined under the condition $\ddot{\varphi} = \text{const}$. Such dependence is called Statical characteristics of the source of energy.

But however, $L(\dot{\varphi})$ which is determined in such a way, may be used in the equations of motion which describe the processes of changing $\dot{\varphi}$, which are close to $\ddot{\varphi} = \text{const}$. i.e. the processes with slow change of the velocity $\dot{\varphi}$, or the processes with small acceleration $\ddot{\varphi}$. Just that very processes will be considered below. They represent the essential practical interest for many applications.

The possibility of using of statical characteristics of the source of energy for studying the regimes of motion which are close to $\ddot{\varphi} = \text{const}$, was confirmed by the experiments.

The demand for the small values of $\ddot{\varphi}$, which we shall introduce below at construction of the approximate solutions of concrete problems, is closely connected with the use of $L(\dot{\varphi})$, in the equations of motion.

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2. Forced Oscillations of the Simpliest Oscillator.

The problem about the motion of linear vibrated system together with the source of energy wes discussed in papers No.[1-4]. The problem about the interaction of the vibrated system with source of energy was discussed, according to the statement which was discribed in item 1 [5-8].

Let us consider the forced oscillations of the system with one degree of freedom. This system receives excitation from the ideal source of energy, with the known characteristic.

Let the dynamical scheme of this system represent fig.(1), where the source of mechanical energy is motor D. During the rotation of the motor, the crank with the radius r, deforms resilient coupling C_1. Owing to this arising force $C_1 r^2 \sin\varphi$ and moment $C_1 r^2 \cos\sin\varphi$ act on the motor. At the presence of the vibration the moment $C_1 r(x - r\sin\varphi)\cos\varphi$ will be acted upon motor.

The motion of the system is described by the following equations

$$m\ddot{x} + \beta\dot{x} + Cx = C_1 r\sin\varphi,$$
$$J\ddot{\varphi} + H(\dot{\varphi}) = L(\dot{\varphi}) + C_1 r(x - r\sin\varphi)\cos\varphi \qquad (2.1)$$

where m - is the mass of vibration system,
$C = C_0 + C_1$ - is stiffness,
J - is the moment of inertia of rotating masses.

It is taken into consideration in the equation: the resistance force to the oscillaroty motion $\beta\dot{x}$, the moment of the resistance forces

to the rotation of rotor, the function $H(\dot{\varphi})$ is given, the known moment of the source of energy $L(\dot{\varphi})$.

More active interaction of the vibration system and the source of energy is naturally expected in the resonance region. Thus we shall limit ourselves with the considerations of the forced oscillations close to the main resonance. According to $\omega - \Theta = \varepsilon\alpha_0$ where ω = frequency of the free oscillations of the system, $\Theta = \dot{\varphi}$ - the frequency of the exciting force, ε - a small positive parameter. Let us assume the exciting force $C_1 r\sin\varphi$ and the friction force $\beta\dot{x}$ be small and it concerns the value of $\ddot{\varphi}$ to be also small, which is mentioned above. Taking into account these limitings we shall introduce the small parameter into the equation (2.1) in the following way:

$$\ddot{x} + \omega^2 x = \varepsilon(q\sin\varphi - h\dot{x}),$$
$$\ddot{\varphi} = \varepsilon[M_1(\dot{\varphi}) + q_1(x - r\sin\varphi)\cos\varphi] \qquad (2.2)$$

where

$$\omega^2 = \frac{C}{m},\ \varepsilon q = \frac{C_1 r}{m},\ \varepsilon q_1 = \frac{C_1 r}{J},\ \varepsilon h = \frac{\beta}{m},\ \varepsilon M_1 = \frac{1}{J}[L(\dot{\varphi}) - H(\dot{\varphi})].$$

The particular solution of the system (2.2) is naturally expected in the form of the approximate harmonical oscillation, with the frequency close to constant. It is convenient to introduce the replacement of the variables, corresponding to these representations

$$x = A\cos(\varphi + \Xi),\ \frac{dx}{dt} = -A\omega\sin(\varphi + \Xi),\ \frac{d\varphi}{dt} = \Theta. \qquad (2.3)$$

The new variables, A, Ξ, Θ will be slowly changing

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functions of φ, which is quite clear from structure of the transformed equations:

$$\frac{d\Theta}{d\varphi} = \frac{\varepsilon}{\Theta}\{M_1(\Theta) + q_1[A\cos(\varphi+\Xi) - r\sin\varphi]\cos\varphi\},$$
$$\frac{dA}{d\varphi} = -\frac{\varepsilon}{\omega\Theta}[q\sin\varphi + A\omega h\sin(\varphi+\Xi)]\sin(\varphi+\Xi), \quad (2.4)$$
$$\frac{d\Xi}{d\varphi} = \frac{\varepsilon}{\Theta}\left\{\alpha - \frac{1}{\omega A}[q\sin\varphi + A\omega h\sin(\varphi+\Xi)]\right\}\cos(\varphi+\Xi).$$

These equations belong to the so called standard form for which N.N. Bogoljubov worked out the method of the theory of perturbations [9].

We shall use this method for the construction of the approximate solutions of the equations and others. This method allows to consider not only periodical but also nearly periodical regimes of motion, and regimes almost stationary which represent the essential interest for the examined problems. Besides, using this method the analysis of stability of the stationary regimes of motion is simplified. The particular approximate solution of the system (2.4) will be determinated in the following forms:

$$\Theta = \Omega + \varepsilon E_1(\varphi, \Omega, a, \xi), A = a + \varepsilon E_2(\varphi, \Omega, a, \xi), \Xi = \xi + \varepsilon E_3(\varphi, \Omega, a, \xi),$$

where $\varepsilon E_1(\varphi, \Omega, a, \xi), \varepsilon E_2(\varphi, \Omega, a, \xi), \varepsilon E_3(\varphi, \Omega, a, \xi)$ - are small periodical functions of φ.

The values Ω, a, ξ are determined in the first approximation from the equations

$$\frac{d\Omega}{d\varphi} = \frac{\varepsilon}{\Omega}[M_1(\Omega) + \tfrac{1}{2}q_1 a\cos\xi],$$
$$\frac{da}{d\varphi} = -\frac{\varepsilon}{2\omega\Omega}(q\cos\xi + h\omega a), \quad (2.5)$$
$$\frac{d\xi}{d\varphi} = \varepsilon\left(\frac{\alpha}{\Omega} + \frac{q}{2\omega\Omega a}\sin\xi\right), \alpha = \omega - \Omega,$$

which are received as the result of averaging

of the right parts (2.4) on the evidently contented independent variable φ.

The variables Ω, a, ξ which are determined as the integrals of the system (2.5) describe the slow processes of the frequency changing Ω, of the amplitude a and phase ξ of the forced oscillations. The equations (2.5) describe the same processes of passing of the oscillatory system tnrough resonance.

In the stationary regimes of motion

$$\frac{d\Omega}{d\varphi} = 0, \frac{da}{d\varphi} = 0, \frac{d\xi}{d\varphi} = 0,$$

thus the values Ω, a, ξ for the stationary regimes of motion are determined as the roots of the system of equations:

$$\frac{1}{\Omega}(M_1(\Omega) + \tfrac{1}{2}q_1 a\cos\xi) = 0,$$
$$q\cos\xi + h\omega a = 0,$$
$$\frac{1}{\Omega}\left(\alpha + \frac{q}{2\omega a}\sin\xi\right) = 0. \quad (2.6)$$

From here we shall find $a = \dfrac{q}{2\omega\sqrt{(\omega-\Omega)^2 + \frac{h^2}{4}}}$. (2.7)

The frequency Ω is determined as the root of the equations

$$L(\Omega) - H(\Omega) - \tfrac{1}{2}\beta\omega a^2 = 0. \quad (2.8)$$

The expression (2.7) coincides practically with the well known expression for the amplitude of forced oscillatory system under the influence of the force $q\sin\Omega t$ with the given frequency Ω. (The existing difference is characterized by the values of tne second order of diminition wnich

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are not accounted to the consideration by the first approximation).

It is impossible to choose arbitrarily the frequency of forced oscillation Ω as it was possible for the system with the ideal source of energy.

Here Ω - is a certain value and it is determined from the equation (2.8). It depends upon the amplitude of oscillations a and upon the characteristic of the source of energy.

The system of equations (2.7) and (2.8) in the region of resonance, has as a rule, one or three solutions. In the fig. (2) their corresponding points 1, 2, 3, - are the points of intersection of the graphs $M(\Omega)-L(\Omega)-H(\Omega)$ and $S(\Omega)=\frac{1}{2}\beta\omega a^2(\Omega)$. The necessity of the analysis of stability of these solutions appears.

It is possible to determine the stability of the stationary values Ω, a, ξ which satisfy the equation (2.6), by means of the analysis of the equations of variation worked out for the equations (2.5).

Such method is convenient because the equations of vibration will be linear equations with constant coefficients and then the problem about stability will reduce to the Routh-Gurwitz criteria for these equations.

These criteria composed by means of the well known method have the following form:

$$\frac{\beta}{m}-\frac{N}{J}>0$$

$$\beta\omega a^2(\omega-\Omega)-N\left[\frac{\beta^2}{4m^2}+(\omega-\Omega)^2\right]>0 \qquad (2.9)$$

$$\frac{\beta}{m}\left[\frac{\beta^2}{4m^2}+(\omega-\Omega)^2\right]-\frac{\beta}{Jm}N+\frac{\beta}{J^2m}N^2-\frac{m^2}{J^2}\omega a^2(\omega-\Omega)N>0.$$

The quantity $N=\frac{d}{d\Omega}[L(\Omega)-H(\Omega)]$ depends upon the steepness of the characteristic of the source of energy $\frac{d}{d\Omega}L(\Omega)$.

Further we shall account $N<0$, which is typical for the majority of the technical problems.

All three criteria (2.9) are satisfied when $\omega>\Omega$ i.e. on the left branch of resonance curve (fig.3), which is constructed by use (2.7). For the right branch of the same curve, where $\omega<\Omega$, the main role belongs to the second criterion (2.9). The more clear form of this criterion is geometrical form: [10]

$$-\frac{1}{\Omega^2}\left[(\omega-\Omega)^2+\frac{\beta^2}{4m^2}\right]\frac{d\Phi_1}{d\Omega}>0 \qquad (2.10)$$

or

$$\frac{d\Phi_1}{d\Omega}<0$$

where

$$\Phi_1=M(\Omega)-\frac{1}{2}\beta\omega a^2(\Omega).$$

It is easy to see the criterion (2.10) is satisfied in the points 1, 3 (fig.2.) and is not satisfied in the point 2, lying on the right branch of the curve $S(\Omega)$.

If we mind the family of characteristics which are similar to $M(\Omega)$, then in part TR the points will be arranged which correspond to unstable regimes of the motion. The points T and R which limit that part are the points of tangent "of border" characteristics M_1^r and M_2^r.

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On the resonance curve (fig.3) there are corresponding points T and R, which limit the unstable part of the resonance curve.

However except the part TR a little bit larger part of the resonance curve will not be realized. It is connected with the frustrations of stationary regimes of motion in the neighbouring points T and R. (Frustration is shown by the arrows in the fig.2).

Assuming the characteristic $M(\Omega)$ is moving translationaly (fig.2) because of the regulation of the source of energy so slowly that the regimes of motion remain practically stationary. Noticing that with the increasing Ω the point which representing stable stationary regime of motion of the system (for example point B), will transfer up along the ascending branch of the curve $S(\Omega)$ till the point T will be reached.

In the point T the stationary regime loses stability and the system transfers to the stable point H (which belongs to the same characteristic M_1^r), with unstationary motion (frustration) from point T to point H. Hence the part of the resonance curve arranging between the points T and H, will not be realized. With further increasing Ω the representating point will transfer along the curve $S(\Omega)$ to the right side from point H.

With the decreasing of the frequency Ω the representing point will transfer on the curve $S(\Omega)$ from right to left till it reaches the point R. In the point R the frustration

to stable point P will occur analogically. After that the representing point will transfer down on the graph $S(\Omega)$. The region of stationary regimes, between points R and P owing to this fact will not be realized. In the fig.4 the resonance curves are represented constructed with the use of (2.7) in which non-realized stationary regimes of motion are shown: a) with the increase of Ω, b) with the decrease of Ω.

The boundary points T,H,R,P correspond to the points of the same name are depicted on (fig.2).

The accounting interaction of the vibratory system with the source of energy permits us to define more exactly the representations about the form of the oscillatory motion and about the character of change of frequency $\frac{d\varphi}{dt}$ in the stationary regimes.

The presence in the second equation (2.1) of the member which depends upon x leads to the fact that periodical changes of the coordinate of the oscillatory motion x cause small periodical changes of frequency $\frac{d\varphi}{dt}$ and coordinate φ. It is easy to account these changes having determined the function $\varepsilon E_1(\varphi, \Omega, a, \xi)$. Then instead of $\frac{d\varphi}{dt} = \Omega = \cos t$, we shall receive more accurate expression for the frequency

$$\frac{d\varphi}{dt} = \Omega + \frac{C_1 r}{4J\Omega}\left[a \sin(2\Omega t + \xi) + r \cos 2\Omega t\right]. \qquad (2.11)$$

Naturally that such change of frequency will influence on the form of the oscillatory motions. This influence is represented by the functions $\varepsilon E_2(\varphi, a, \Omega, \xi)$ and $\varepsilon E_3(\varphi, a, \Omega, \xi)$.

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Having determined these functions we shall find that instead of

$$x = a\cos(\Omega t + \xi)$$

the oscillatory motion will have the following form

$$x = a\cos(\Omega t+\xi)+\varepsilon\ell_1\cos(\Omega t+\delta_1)+\varepsilon\ell_2\cos(3\Omega t+\delta_2) \quad (2.12)$$

where ℓ_1, ℓ_2, δ_1, δ_2 are the values depending upon a, Ω, ξ and upon the parameters of the system.

The properties of the source of energy influence not only on the established stationary processes of oscillations but also on the transitional processes. It is clearly displayed especially in the processes of passing through resonance.

Let us assume for example the stable stationary regime of motion of the system[15] characterized by the point B - point of intersection of graphs $M(\Omega)$ and $S(\Omega)$, laying outside of the resonance. We shall assume that at the start moment of time $\Omega = \Omega_*$, the amplitude of oscillations a and phase ξ correspond to the point D, the value of moment $M(\Omega_*)$ is characterized by the point C(fig.5).

The system which is left to itself from this state will pass through resonance, and after this the system will reach the point B. At the point B the stationary regime of oscillations will be settled.

The proceeding of transitional process can be calculated by means of intergration of the system of equations (2.5) under the initial conditions

which correspond to the points D and C. The results of such calculation are represented in fig.6 and reflect the specification of the transitional process, connected with the nonideal source of energy. The essential feature of the process is the close connection of the change of the frequency Ω with changes of the amplitude (a) owing to which the increase of frequency Ω slows down immediately when the amplitude reaches the maximal values, and as a consequence of this the nonlinear character of the change of Ω takes place at the moment of passing of the vibration system through resonance.

The more significant peculiarities mentioned above the less reserve of the capacity has the source of energy at the resonance (the relative characteristic of this reserve can be expressed by the piece ℓ in fig. 5.

This example makes us treat critically the calculations of the transitional process, which are susally based on the use of the prescribed law of changing Ω in the moment of passing of the resonance region. (More often linear law). For the systems with nonideal sources of energy such method may appear to be unacceptable.

The things discussed above in this section show how essential and manysided influence the nonideal source of energy gives to the beheavior of the vibration system such as elementary linear vibrator.

There are many difficulties when describing the questions of interaction of the vibration

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system with the source of energy while using the terminology: "linear vibrated system" and "nonlinear vibration system" in the generally accepted meaning because, all the systems with the nonideal source of energy become nonlinear. For deminition of these difficulties we called "linear" vibration system, that is those systems which are described by the linear differentional equations when they interact with the ideal source of energy. "Nonlinear" vibrant systems we call those vibrated systems which are described by the nonlinear differential equations, when they interact with the ideal source of energy.

Interaction of Nonlinear Vibrator with Source of Energy under Forced Oscillations

Let us examine for determination the nonlinear vibrating system with one degree of freedom, the oscillations of which are being excited by the inertia forces of an unbalanced mass m, rotating at a distance r [11].

Vibrating system (fig.7) has mass m_0 and resilence coupling with nonlinear characteristic $\Phi(x)=cx+f(x)$, where $f(x)$ is a small nonlinear function of x.

The motor D presents the source of energy of the examined system, the mass m is tied to the rotor of the motor, the moment of inertia of the rotor is I. Motor generates angular moment $L(\varphi)$. The moment of the forces of resistance to rotary motion $H(\varphi)$ is supposed to be known,

similarly the resistance force to oscillatory motion is supposed to be known. For the sake of simplicity, let us suppose the resistance force to be linear one $\beta\dot{x}$ and it is considered to be small.

Similarly the quantities $\frac{m}{m_0}$, $\frac{m'}{J}$ and $\frac{1}{J}L(\dot{\varphi})-H(\dot{\varphi})$ are small.

The equations of motion of such a system have the following form:

$$\ddot{x}+\omega^2 x=\varepsilon\left[-h\dot{x}-f_1(x)+q_2\dot{\varphi}^2\cos\varphi+q_2\ddot{\varphi}\sin\varphi\right],$$
$$\ddot{\varphi}=\varepsilon\left[M_1(\dot{\varphi})+q_3\dot{x}\sin\varphi+q_4\sin\varphi\right] \qquad (3.1)$$

where
$$\omega^2=\frac{c}{m_0},\ \frac{m}{m_0}r=\varepsilon q_2,\ \frac{\beta}{m_0}=\varepsilon h,\ \frac{m}{J}r=\varepsilon q_3,\ \frac{m}{J}gr=\varepsilon q_4,$$
$$\frac{1}{J}\left[L(\dot{\varphi})-H(\dot{\varphi})\right]=\varepsilon M_1(\dot{\varphi}),\ \frac{1}{m_0}f(x)=\varepsilon f_1(x).$$

In the right parts of the equations small quantities are grouped together, and the parameter ε points these small quantities.

With the help of the substitution of the variables

$$x=A\cos(\varphi+\Xi),\ \frac{dx}{dt}=-A\omega\sin(\varphi+\Xi),\ \frac{d\varphi}{dt}=\Theta$$

new unknown slowly changed functions Θ, A, Ξ are introduced. The equations (3.1) transform to standard form

$$\frac{d\Theta}{dt}=\varepsilon\left[M_1(\Theta)-q_3\omega A\Theta\sin\varphi\cos\varphi+q_4\sin\varphi\right],$$
$$\frac{dA}{dt}=-\frac{\varepsilon}{\omega}\left\{Ah\omega\sin^2\Psi-f_1(A\cos\Psi)+q_2\Theta^2\cos\Psi\right\}\sin\Psi+\varepsilon^2\ldots, \qquad (3.2)$$
$$\frac{d\Xi}{dt}=\varepsilon\left\{\alpha_0-\frac{1}{A\omega}\left[Ah\omega\sin\Psi-f_1(A\cos\Psi)+q_2\Theta^2\cos\varphi\right]\cos\Psi\right\}+\varepsilon^2\ldots,$$
$$\alpha_0=\omega-\Theta,\quad \Psi=\varphi+\Xi.$$

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The approximate solution of this system is found in the form of

$$\theta = \Omega + \varepsilon U_1(t,\Omega,a,\xi), A = a + \varepsilon U_2(t,\Omega,a,\xi), \Xi = \xi + \varepsilon U_3(t,\Omega,a,\xi),$$

where $\varepsilon U_1(t,\Omega,a,\xi)$, $\varepsilon U_2(t,\Omega,a,\xi)$, $\varepsilon U_3(t,\Omega,a,\xi)$ are small periodic functions of t.

After the substitution of these expressions in (3.2), and the following average till t, we get an equation for the determining of Ω, a, ξ :

$$\begin{aligned} \frac{d\Omega}{dt} &= \varepsilon\left[M_1(\Omega) + \frac{1}{2} q_3 \omega \Omega a \sin\xi\right], \\ \frac{da}{dt} &= -\frac{\varepsilon}{2}\left[ha + \frac{1}{\omega} q_2 \Omega^2 \sin\xi\right], \\ \frac{d\xi}{dt} &= \varepsilon\left[\alpha + \frac{G(a)}{\omega a} - \frac{q_2 \Omega^2}{2\omega a}\cos\xi\right] \end{aligned} \tag{3.3}$$

where

$$G(a) = \frac{1}{2\pi}\int_0^{2\pi} f_1(a\cos\psi)\cos\psi\, d\psi, \quad \alpha = \omega - \Omega.$$

Stationary conditions of motion are determined by the equations (3.3) with the conditions $\frac{d\Omega}{dt} = 0$, $\frac{da}{dt} = 0$, $\frac{d\xi}{dt} = 0$:

$$\begin{aligned} M_1(\Omega) + \frac{1}{2} q_3 \omega \Omega a \sin\xi &= 0, \\ h\omega a + q_2 \Omega^2 \sin\xi &= 0, \\ \alpha + \frac{G(a)}{\omega a} - \frac{q_2 \Omega^2}{2\omega a}\cos\xi &= 0 \end{aligned} \tag{3.4}$$

after excluding ξ , we get an equation for the determining the frequency Ω

$$L(\Omega) - H(\Omega) - \frac{\beta}{2\Omega}\omega^2 a^2 = 0 \tag{3.5}$$

and the expression for the amplitude of oscillations

$$a = \frac{m}{m_e}\frac{r\Omega^2}{\sqrt{4\omega^2(\omega_e - \Omega)^2 + \beta^2\frac{\omega^2}{m_e^2}}} \tag{3.6}$$

where

$$\omega_e = \omega + \frac{1}{\omega a} G(a).$$

The last expression practically coinsides with the well known in literature approximate expressions for the amplitude of oscillations of the system with nonlinear restoring force in the resonance region.
For example if $f(x) = \gamma x^3$

then $G(a) = \frac{3\gamma a^3}{8 m_e}$, and then (3.6) expresses that very amplitude of oscillations, which is obtained for the equation of Duffing type [12-14].

However, it does not necessarily mean, that the frequency Ω cannot be taken into the expression (3.6) arbitrarily; it must be defined as a root of the equation (3.5). The last expression contains the characteristics of the motor $L(\Omega)$. In this way, the stationary oscillations depend on the properties of the source of energy. It is not difficult to perceive, that the system of equations (3.5), (3,6) or the system of equations (3.4) can have few solutions. Each of these solutions Ω, a, ξ must be examined for stability.

Stability of stationary Ω, a, ξ - roots of systems of equations (3.4) - is explained by the analysis of equations in variations for the systems (3.3), which will be linear equations with constant coefficients. Then the problem

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comes to the criterions of Routh-Gurwitz, for these equations. It is observed that we come to the same conclusion in solutions of analogous problems by other methods /14-16/.

Following to the well known rules of criterion for the stability of stationary conditions Ω, a, ζ we get the following:

$$K_1>0,\ K_3>0,\ K_1K_2-K_3>0 \qquad (3.7)$$

where

$$K_1=\frac{1}{J}\left(\frac{\beta\omega^2a^2}{2\Omega^2}-N\right)+\frac{\beta}{m_e},\quad \omega_e'=\frac{d}{da}\omega_e,$$

$$K_2=\frac{\beta^2}{4m_e^2}+(\omega_e-\Omega)^2+a(\omega_e-\Omega)\omega_e'+\left(\frac{\beta\omega^2a^2}{2\Omega^2}-N\right)\frac{\beta}{Jm_e}+$$
$$+\frac{m_e\omega^2a^2}{J\Omega^2}\left[(2\omega_e-\Omega)(\omega_e-\Omega)+\frac{\beta^2}{8m_e^2}\right],$$

$$K_3=-\left[(\omega_e-\Omega)^2+a(\omega_e-\Omega)\omega_e'\right]\left(\frac{\beta\omega^2a^2}{2\Omega^2}+N\right)\frac{1}{J}+$$
$$+\frac{\beta\omega^2a^2}{J\Omega^2}\left[(2\omega_e-\Omega)(\omega_e-\Omega)+\frac{3\beta^2}{8m_e^2}\right]-\frac{\beta^2}{4Jm_e^2}N.$$

The expression for K_3 can be put in more compact form suitable for geometrical interpretation

$$K_3=-\left[\frac{\beta^2}{4m_e^2}+(\omega_e-\Omega)^2+a(\omega_e-\Omega)\omega_e'\right]\frac{dF}{d\Omega}$$

where

$$F=\frac{1}{J}\left[L(\Omega)+H(\Omega)-\frac{\beta}{2\Omega}\omega^2a^2(\Omega)\right].$$

The analysis of the obtained criterions of stability elucidates the role of the characteristic of motor, and its influence on the stability of stationary oscillations of nonlinear system.

On the basis of such analysis, we can hold that disposition of staole and unstable points on the resonant curve of the nonlinear oscillating

system essentially depends on the properties of the source of energy. An important role is played by the derivative $N=\frac{d}{d\Omega}M(\Omega)$ where $M(\Omega)=L(\Omega)-H(\Omega)$ is in the conditions of stability. It is not very difficult to convince that changing the magnitude $\frac{d}{d\Omega}L(\Omega)$ we can reach that the stable condition of the standard oscillations changes into unstable one and vice versa. Let us illustrate the conclusion on the example of oscillating system with characteristic resistance coupling $\varphi(x)=cx+\gamma x^3$. As it is well known, the movement of this system has been studied in great detail in the supposition that frequency of exciting force Ω is given and can be changed arbitrary, independent from the oscillating system [12-14].

The studying interaction of this system with the source energy, gives more wide notion about the conditions of oscillations and their stability.

The detailed calculations, which satisfy this system on the equations (3.5), (3.6) and the analysis of criterion of stability (3.7) allow to obtain the following expositions of possible conditions of oscillations.

I. Oscillating system with "rigid" characteristic of resilient coupling($\gamma>0$). Resonance curve $a(\Omega)$ of such system is well known in literature, and the expression (3.6) represents the same resonance curve. For further detailed exposition resonance curves are not used, rather the graph $S(\Omega)=\frac{\beta}{2\Omega}\omega^2a^2$ is used. The graphs $S(\Omega)$

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and $a(\Omega)$ have similar forms and owing to this fact, it is easy to use the notion of resonance curve.

Graphic representation of the possible standard conditions of oscillations gives the construction as it is shown in fig.8. Here the graph $S(\Omega)$ is plotted for the system with $\gamma > 0$ and the graph $M(\Omega)$, the reflecting property of the motor. The intersection of these graphs corresponds to the equation (3.5) and defines the points C_1, C_2, C_3, characterizing the possible standard conditions of oscillations. Therefore it is possible to examine the construction on fig.8 as the graphical solution of equation (3.5) and the abscissas C_1, C_2, C_3, as its roots.

For example, with the help of conditions of stability (3.7), it is seen that the points C_1 and C_3 are the points of stability and C_2 is the point of unstability. From the same criterions, (7) it follows, as it is shown in fig.8, that the character of the mutual dispositions of stable and unstable states one preserved and for any other deminishing characteristics $M(\Omega)$. Thus, for example, the point C_1 will be stable in all cases when the tangent of the characteristic $M(\Omega)$ at this point passes in the interior of shaded quadrant. If we take into account, that the characteristics $M(\Omega)$ in this process of regulation of the motion remain parallel to itself, then on the branches BT and RD the points of stability are placed and on the branch TR the points unstability (dotted line) are placed.

Disarrangements of oscillations under the changed frequency Ω are characteristic of nonlinear system, and in our case, they will depend on the characteristics of the motor. Having in view, that the control of velocity of rotation corresponds to the shift characteristics $M(\Omega)$, it is not difficult to notice that when Ω increases, changes in amplitude take place along the continuous line, and when Ω decreases, then changes of amplitude take place along the dotted line, and shift takes place from the point R to point P. (For the sake of simplicity, it is again supposed here that the characteristics $M(\Omega)$ shifts parallelly to itself).

2. The oscillating system with "soft" characteristics of resilient forces $(\gamma < 0)$.

For this system that very representation with the help of graphs $S(\Omega)$ and $M(\Omega)$ is useful. The characteristic peculiarity of the resonant curve $a(\Omega)$, when $\gamma < 0$ and its inclination to the side of deminishing frequency is repeated again as in graph $S(\Omega)$ in fig.9.

The conditions of stability (3.7) allow us to establish that with "soft" nonlinear system, the disposition is unstable with the change of hardness of the characteristic of the motor. It is interesting to note the following two possibilities:

If the characteristics of the motion appear to be "steep", i.e. N is of great modulus, then points of stability are disposed on the branch TD and on the part BR. On the part TR (dotted

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line) points of unstable positions are placed. For every point of stable part, it is possible to determine from (3.7), the region of significance N, under which the stability is maintained. This region is represented in fig.9a by shaded sector. For the point of stability C_1 it is necessary that the tangent to the characteristics $M(\Omega)$ passing through C, should be contained within the interior of the shaded sector (fig.9a).

If, on the other hand, the characteristics of the motion appear to be "gradual" i.e. N is of small modulus, then the points of stability are placed on the branch BT and in the part RD. In this case, points of unstability are placed on the part TR. The shaded sector represents the region of significance N, under which, the point E can remain as stable position.

In this way, due to the changing of N, unstable states shift from the growny branch of resonance curve to its deminishing branch.

The disposition of the standard conditions of motion for the case, when the tangent to the characteristic $M(\Omega)$ passes through the inside of the shaded area of fig.9b, it is shown in fig.10. It is clear that E_1 and E_3 - are the points of stability and E_2 - is unstable. Here it is shown also the scheme of change of amplitude and the discurangement of oscillations under the growth of Ω (continuous arrows) and under deminishing Ω (dotted arrows).

It is not difficult to see, that the schemes, described in literature, on the changes in amplitude under quasistandard growth, and under deminishing Ω and the schemes of discurangement of oscillations for nonlinear systems with hard and soft characteristic of resilient forces [12,13], can be interpreted in exposition of higher representation as events oscillations, excited by source of energy with infinitely big hardness of characteristic. Then these events will correspond to those on the graph fig.8, 9, 10 to characteristics $M(\Omega)$, having vertical straight line position, i.e. the characteristic of ideal sources of energy.

Angular velocity $\frac{d\varphi}{dt}$, except the basic magnitude Ω, acquires a small harmonic components

$$\frac{d\varphi}{dt}=\Omega+\frac{mr}{4J}\omega a\cos(2\Omega t+\xi)-\frac{mgr}{J\Omega}\cos\Omega t.$$

Coordinate of the oscillating motion, excepting its own chief value

$$x=a\cos(\Omega t+\xi)$$

acquires small supplementing periodic additive terms however, now their concrete aspect depends not on only nonlinear members $f(x)$, but also on the manner of connection with the source of energy. The equation (3.3) is suitable for calculation of the processes of passing of nonlinear system through resonance. Calculations show [18], that nonlinear systems as well have nonlinear character of change of frequency when the system passes through resonance.

The fundamental conclusions about mutual influence of oscillatory system and its source of energy under forced vibrations mentioned above, are confined by the experiments [17,18], which

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confirm the possibility of using of statistical characteristics of the sources of energy in the examined problems.

4. Interaction of Parametrical Oscillatory System with the Source of Energy

It is convenient to consider the interaction of parametrical oscillatory system with the source of energy as an example of simple mechanical model represented in fig.11. The resilient bar AB is under the action of periodical force in the direction of axis of the bar and therefore the bending stiffness of the bar experiences the periodical changes.

Under certain conditions these changes may cause the appearance of oscillations of the bar in the direction of axis y.

The peculiarity of considering system is that the change of parameter of oscillatory system is realized by the source of energy (motor) with the definite (generally speaking-limited) capacity. Therefore we suppose that parametrical oscillations may influence on the running motor and on the behavior of the oscillatory system [19].

In such statement the dependence between the rotating motion of motor's rotor and oscillatory motion of resilient bar is essential.

We shall represent the equations of motion in simple form for the 1st form of transversal oscillation of the bar which supposed $y(x,t)=y(t)\sin\frac{\pi x}{\ell}$:

$$\ddot{y}+(\omega^2+\varepsilon\lambda\sin\varphi)y=-\varepsilon(h\dot{y}+\gamma_0 y^3),$$
$$\ddot{\varphi}=\varepsilon[M_1(\dot{\varphi})-\sigma_2 y^2\cos\varphi-\sigma_3\sin 2\varphi-\sigma_4\cos\varphi]. \quad (4.1)$$

Here: $\varepsilon M_1(\dot{\varphi})=\frac{1}{J}[L(\dot{\varphi})-H(\dot{\varphi})]$, $\varepsilon h=\frac{\beta}{m}$, $\varepsilon\lambda=\frac{C_2}{m}$, $\varepsilon\sigma_2=\frac{C_2}{2J}$, $C_2=-\frac{\pi^2 r}{2\ell}C_1$, $\varepsilon\sigma_3=\frac{r^2C_1}{2J}$, $\varepsilon\sigma_4=f_0\frac{rC_1}{J}$, $m=\frac{1}{2}m_1\ell$, $\gamma=\varepsilon\gamma_0$, $\gamma=\frac{\pi^4C_1}{8\ell^2m}$, $\omega^2=\frac{\pi^4}{\ell^4m}(1-\frac{P_0}{P_1})EJ_x$, $P_0=f_0C_1$, $P_1=\frac{\pi^2}{\ell^2}EJ_x$.

The inertia forces of mass in the direction of axis (x) are not taken into account. It is considered that magnitude w (displacement of the end of the bar $x=\ell$ in the direction of axis x) is connected with y by approximate relation:

$$w=\frac{1}{4\ell}\pi^2y^2$$

In the equations:

m_1 - is the mass of unity of length of the bar,
I - is the moment of inertia of motor's rotor and another rotating parts,
r - is radius of crank deformated of the spring
f_0 - is static spring deformation,
I_x - is the moment of inertia of crossed section of the bar,
E - is elasticity modulus of the bar material,
βy - is the resistance force of the linear relation taken for the simplicity of the problem.

The moment of motor $L(\dot{\varphi})$ (characteristic of motor) and the moment of resistace to the rotating of motor $H(\dot{\varphi})$ is represented as polinomials with $\dot{\varphi}$. It is assuming that modulstion of stiffness of the system is small.

Generalizing the problem we shall assume that

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coefficient γ may be not only positive but also negative. Than it will be possible to consider the behaviour of system not only with "stiff" but also with "soft" characteristic of nonlinear restoring force. In the latter case it is necessary, of course, to consider additional factors that are acted in the system.

Members written in the right parts of the equations and items $\lambda y \sin\varphi$ are assumed as small and a small parameter ε shows to this.

We shall limit ourselves by the consideration of the oscillations which are nearly harmonical in the region of main parametrical resonance with assuming that φ is changed slowly.

We shall introduce the replacement of variables

$$y = A\cos\left(\frac{\varphi}{2}+\Xi\right),\ \frac{dy}{dt} = -A\omega\sin\left(\frac{\varphi}{2}+\Xi\right),\ \frac{d\varphi}{dt} = \Theta,$$

with the help of which the equations (4.1) will be transformed as following:

$$\frac{d\Theta}{d\varphi} = \frac{\varepsilon}{\Theta}\left[M_1(\Theta) - \sigma_2 A^2\cos^2\psi\cos\varphi - \sigma_3\sin 2\varphi - \sigma_4\cos\varphi\right],\ \psi = \frac{\varphi}{2}+\Xi,$$

$$\frac{dA}{d\varphi} = \frac{\varepsilon}{\omega\Theta}\left[\lambda A\cos\psi\sin\varphi + \gamma_0 A^3\cos^3\psi - h\omega A\sin\psi\right]\sin\psi,$$

$$\frac{d\Xi}{d\varphi} = \left(\omega - \frac{\Theta}{2}\right)\frac{1}{\Theta} + \frac{\varepsilon}{\omega\Theta A}\left[\lambda A\cos\psi\sin\varphi + \gamma_0 A^3\cos^3\psi - h\omega A\sin\psi\right]\cos\psi.$$

The solution of this system we shall find in the following form

$$\Theta = \Omega + \varepsilon U_1(t,\Omega,a,\xi),\ A = a + \varepsilon U_2(t,\Omega,a,\xi),\ \Xi = \xi + \varepsilon U_3(t,\Omega,a,\xi).$$

For the determination of values of Ω, a, ξ in the first approximation we shall receive the following system of equations

$$\frac{d\Omega}{d\varphi} = \frac{\varepsilon}{\Omega}\left\{M_1(\Omega) - \frac{1}{4}\sigma_2 a^2\cos 2\xi\right\},$$
$$\frac{da}{d\varphi} = \frac{\varepsilon}{2\Omega}\left(\frac{\lambda}{2\omega}\cos 2\xi - h\right)a, \qquad (4.2)$$
$$\frac{d\xi}{d\varphi} = \frac{\varepsilon}{\Omega}\left(\omega - \frac{\Omega}{2} + \frac{3\gamma_0 a^2}{8\omega} - \frac{\lambda}{4\omega}\sin 2\xi\right).$$

Values of Ω, a, ξ at the stationary regimes of motion are determined as the roots of following systems of equations:

$$L(2\nu) - H(2\nu) - \frac{1}{8}\sigma_2 a^2\cos 2\xi = 0,$$
$$\left(\frac{\sigma_2}{2\omega}\cos 2\xi - \beta\right)a = 0,$$
$$\omega - \nu + \frac{3\gamma_0 a^2}{8\omega} - \frac{\sigma_2}{4m\omega}\sin 2\xi = 0,\ \nu = \frac{\Omega}{2}, \qquad (4.3)$$

which are received from (4.2) under conditions

$$\frac{d\Omega}{d\varphi} = 0,\ \frac{da}{d\varphi} = 0,\ \frac{d\xi}{d\varphi} = 0.$$

As result of elimination of ξ from these equations we shall receive the equation which is connected with the characteristic of motor $L(2\nu)$ with the amplitude of parametrical oscillations a

$$L(2\nu) - H(2\nu) - \frac{1}{4}\beta\omega a^2 = 0 \qquad (4.4)$$

and the expression for amplitudes of parametrical oscillations

$$a^2 = \frac{4}{3\gamma}\left[(\nu-\omega)2\omega \pm \frac{1}{2m}\sqrt{\sigma_2^2 - 4\beta^2\omega^2}\right]. \qquad (4.5)$$

The last expression has such form as it is known for the amplitude of parametrical oscillations in nonlinear system without the account of properties of the source of energy [20, 12, 13].

But the equation (4.5) has such peculiarity that it is impossible to take the frequency of

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modulation Ω (or $\nu=\frac{1}{2}\Omega$) arbitrary as it takes place usually in the problems about parametrical oscillations, without taking into consideration the properties of the source of energy.

In our case the value of ν must be determined from the equation (4.4) and therefore it will depend on the properties of motor. It is interesting to note that the equation (4.4) has such structure as for the case of forced oscillations (3.5).

It is possible to have some solutions of nonlinear equations (4.5) at points 1, 2, 3 (Fig.12) correspondingly where $M(\nu)=L(2\nu)-H(2\nu)$ and $S=\frac{1}{4}\beta\omega a^2$, thus mentioned points correspond to the roots of equation (4.4). Having in mind real relations between $L(\dot{\varphi})$ and $H(\dot{\varphi})$ it is possible to consider typical cases of crossing of graphs M(ν) and S(ν) at the points 1 and 2 and point 3, which appear at the crossing M(ν) with axis $\psi\omega$ and correspond to non-oscillation regime of motion.

Analysis of stability has an important interest; because the results of such analysis sometimes bring us to the conclusion which changes the traditional representation of stability of parametrical oscillations without the properties of the source of energy.

The stability of stationary solutions (roots of system 4.3) may be received in the first approximation with use of Routh-Gurwitz criterions for the system of differential equations (4.2).

Making use of well known rules we have criterions of stability in the following form:

$$K_1>0,\ K_3>0,\ K_1K_2-K_3>0. \tag{4.6}$$

Here

$$K_1=\frac{\beta}{m}-\frac{N_2}{2J},$$

$$K_2=\frac{\omega a^2}{2J}(\omega_e-\nu)(m+3J\frac{\gamma}{\omega^2})-\frac{N_2\beta}{2Jm},$$

$$K_3=\frac{\omega a^2}{2J}(\omega_e-\nu)(\beta-\frac{3\gamma}{2\omega^2}N_2),$$

$$N_2=\frac{d}{d\nu}[L(2\nu)-H(2\nu)],\quad \omega_e=\omega+\frac{3\gamma a^2}{8\omega}.$$

The values ν, a, ξ must be substituted to these criterions for each of finding stationary solutions correspondingly to points 1, 2, 3 in Fig.12.

Thus the values ν, a, ξ found from equations (4.3) together with the results of analysis of their stability and with the use criterions (4.6) give us the solution of the problem in first approximation.

The stability of parametrical oscillations depends on: the properties of the source of energy; on the value of parameter γ; on the amplitude of oscillations; on the moment of inertia I and on others parameters of system.

These dependences are in the criterions (4.6) but their detailed analysis is rather complicated and it is not considered here.

We shall discuss only such results which determine the role of characteristics of the source of energy and the nonlinear parameter γ.

Characteristics of the source of energy is represented in the criterions of stability (4.6) as derivative N_2 i.e. angle of bank of tangent

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to the graph M ($\dot{\nu}$) in the considering point is important.

For real conditions M ($\dot{\nu}$) more often is positive decreasing function and its derivative N_2 is negative value. The limit graphs for M ($\dot{\nu}$) will be vertical straight line ($N_2 = -\infty$) and horizontal straight line ($N_2 = 0$). The tangents to all others characteristic and passing through E will be arranged in the limits of forth and second quadrants (shading sectors in Fig.13).

Let us consider two typical cases for nonlinear parametrical systems.

1. The case when oscillatory system has a "stiff" characteristics of nonlinear restoring force i.e. $\gamma > 0$. The graph S for this case shown in Fig.13 is similar to graph of resonance curve a^2 ($\dot{\nu}$). We shall use this similarity for the shortening of describing. Analysis of criterions (4.6) for this case gives the following conclusion points at the branch BT (so as correspondingly points at the resonance curve correspond to stable regimes of motion when a tangent to characteristic M passes through point E, in the limit shading sectors (Fig.13) i.e. when characteristic M is decreasing. The point at the branch TR of graph S (and correspondingly points at resonance curve) will correspond to the unstable regimes of motion for the same characteristic M .

As it is shown in Fig.12 there are three points of typical "equilibrium". The point 3 in Fig.12, so as the point of crossing characteris-

tics M with axis (in Fig.13) correspond to stable non-oscillatory regimes of motion, i.e. rotation with the frequency Ω .

The reciprocal disposition of stable and unstable points of "equilibrium" determines the character of possible "frustrations" of oscillations with moving characteristics M at the process of motor control.

Let for the simplicity of characteristics M be rectilinear (Fig.13) and at the motor control moves parallelly to itself. It is evident that with increasing the point E will move up along the branch BT. At the environment of point T the stability of stationary regime will be lost and the system will transfer to the new state, determined by point H. At the point T parametrical oscillations will "frustrate" and disappear as the system will transfer from state T to state H. If than will decrease to a point, which represents the state of the system, it will move along the axis $\dot{\nu}$ from H to R. At the point R parametrical oscillation will appear. They will be settled as the system will transfer from point R to point P with subsequent decreasing of the stable stationary oscillatory regime at the point P may be changed so as the point representing a process will move from P to B.

2. The case, when a system has "soft" characteristic of restoring force, i.e. $\gamma < 0$. The graph S for this case is shown in Fig.14 and Fig.15, Graph S is similar to resonance curve

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(ν), known from literature.

For $\gamma < 0$ the disposition stable and unstable points in the graph S may be changed considerably with the change of incline of characteristic M. It is more interesting to observe the following peculiarities:

a) for the fixed γ there is a region of values N_2 at which stable points are at the right branch of graph S. The tangent to the characteristic M for such points should pass at the limit of shaded sector in Fig.14. Other points on the left branch of curve S correspond to unstable regimes of motion.

b) for the fixed γ there is other region of values N_2 at which stable points are on the left branch of graph S. The tangent to the characteristic M for points of this part should pass at the limit of shaded sector in Fig.15. The central angle of sector represents the value of the above mentioned region of values N_2. It is not difficult to find this value from inequalities (4.6).

As we can see the states of system which were stable may be unstable if the incline of characteristic of source of energy change in the proper way.

The origin of parametrical oscillations and their frustrations at the change of frequency ν (by means of motor control) is not difficult to observe by the above mentioned method for the system with the stiff nonlinear characteristics.

Schemes of frustrations of oscillations are shown by pointers in Fig.14 and Fig.15. It is assuming in these schemes that point T and R are near the boundaries of stability region.

It is easy to see that the well known in literature [20, 12, 13] the cases of nonlinear parametrical oscillations will be interpreted, in the above discribed scheme, as the cases of parametrical excitation from the source of energy with unlimited large rigidity of characteristic M.

It is interesting to notice that solutions (4.4), (4.5) and conditions of stability (4.5) are disseminated to the case of parametrical system with the linear resilient force ($\gamma = 0$). As it is known that the solution of the problem about parametrical oscillations of linear system without properties of energy source allows only to find the conditions of arising of oscillations and to determine the boundaries of region of the parametrical resonance.

But from the solutions (4.4), (4.5) it is possible to determine two amplitudes corresponding to points B_1 and B_2 (if characteristic M is decreasing function (Fig.16)). The point B_1 corresponds to stable regime of motion, point B_2 - to the unstable.

We shall notice that for every considering case the frequency ν changes slowly (quasi-stationary) since the stationary regimes are considered here.

The main part of the solution of the differential equations has been considered here: harmonic $c\cos(\nu t + \xi)$ and average value of an-

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gular velocity Ω . Besides that the solution has small harmonics of combination frequencies of oscillatory motion and small periodical component of angular velocity.

5. Interaction of the Autooscillatory System with the Source of Energy

It is seen that traditional problems arise while discussing the system of autooscillations with the ideal source of energy which predetermines the independence of the oscillatory system from the source of energy. On the contrary the system with the real source of energy with the known properties indicates the dependence of the oscillations on the properties of the source of energy [2].

Let us consider the interaction of an autooscillatory system with the source of energy which confirms to the well known mechanical model of the autooscillating system, with the type of an infinite length, and also that it gets energy from the real source of energy - motor with the known characteristic $L(\dot{\varphi})$.

Oscillations appear in consequence of the action of the force of friction T , at the point of contact of the mass "m" with the tape "1" which moves with the velocity $v = r\dot{\varphi}$, where (φ) is the angle of rotation of the rotor of engine.

Equations of motion can be expressed in the following way:

$$J\ddot{\varphi} = L(\dot{\varphi}) - H_1(\dot{\varphi}) - rT\{(v - r\dot{x})\},$$
$$m\ddot{x} + cx = -H_2(\dot{x}) + T\{(v - \dot{x})\} \qquad (5.1)$$

where J - is the sum of the moments of inertia of the rotational parts,

m - is the mass of the oscillatory system,

C - is the rigidness of the oscillatory system,

$H_1(\dot{\varphi})$ - is the moment of the forces which resists to the rotation, the sum of of which is equal to ( ).

$H_2(\dot{x})$ are the forces of resistance to oscillatory movement. Except the forces $H_1(\dot{\varphi})$ and $H_2(\dot{\varphi})$ which act in the system the force of friction T which appears to be an odd function of the velocity related to the slide $(v - \dot{x})$. Functions $L(\dot{\varphi}), H_1(\dot{\varphi})$ are continuous, slowly changed. (Tape "1" is considered to be inextensible and the connection of the motor with the system is rigid). In the system (5.1) the first equation describes the behavior of the source of energy, the 2nd equation describes the motion of the oscillatory system.

The 2nd equation is studied in greater detail assuming that v is given. Considering the most simple form of the 2nd equation we will take for granted that $H_2(\dot{x}) = \beta\dot{x}$ and we suppose that $v > 0$, $v > |\dot{x}|$. Under these conditions $T\{(v - \dot{x})\} = K_0 + K_1(v - \dot{x}) + K_2(v - \dot{x})^2 + K_3(v - \dot{x})^3$, K_0, K_1, K_2, K_3 are constants. The oscillations tend to be simple

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harmonic, forces $\beta\dot{x}$ and T are small. It is clear that autooscillations would be possible only at decreasing part of the functions $T[(r\dot{\varphi}-\dot{x})]+\beta\dot{x}$.

In the first equation the right part expresses the algebraic sum of the moving moment $L(\dot{\varphi})$ and the moments of the resisting forces. Taking for granted that this sum is small, we are able to say that the resulting acceleration $\ddot{\varphi}$ also will be small.

Considering the observations in the equation (5.1) we can introduce the small parameter ε in the following manner:

$$\ddot{x}+\omega^2 x=-\varepsilon[h\dot{x}-T_1\{(r\dot{\varphi}-\dot{x})\}],$$
$$\ddot{\varphi}=\varepsilon[M_1(\dot{\varphi})-T_2\{(r\dot{\varphi}-\dot{x})\}] \qquad (5.1)$$

here it is denoted

$$\varepsilon M_1(\dot{\varphi})=[L(\dot{\varphi})-H(\dot{\varphi})]\frac{1}{J},\ \omega^2=\frac{c}{m},\ \varepsilon h=\frac{\beta}{m},$$
$$\varepsilon T\{(r\dot{\varphi}-\dot{x})\}=\frac{1}{m}T\{(r\dot{\varphi}-\dot{x})\},\ \varepsilon T_2\{(r\dot{\varphi}-\dot{x})\}=\frac{r}{J}T\{(r\dot{\varphi}-\dot{x})\}$$

Substitutions for the variables

$$\frac{d\varphi}{dt}=\Theta,\ x=A\cos(\rho t+\Xi),\ \frac{dx}{dt}=-A\omega\sin(\rho t+\Xi) \qquad (5.3)$$

these equations can be expressed in standard form:

$$\frac{d\Theta}{dt}=\varepsilon[M_1(\Theta)-T_2\{(r\Theta+A\omega\sin\Psi)\}],\ \Psi=\rho t+\Xi,$$
$$\frac{dA}{dt}=-\frac{\varepsilon}{\omega}[h\omega A\sin\Psi+T_1\{(r\Theta+A\omega\sin\Psi)\}]\sin\Psi,$$
$$\frac{d\Xi}{dt}=\omega-\rho-\frac{\varepsilon}{\omega A}[h\omega A\sin\Psi+T_1\{(r\Theta+A\omega\sin\Psi)\}]\cos\Psi. \qquad (5.4)$$

here ρ is the frequency of autooscillations.

By means of the theory of perturbation [9] it is possible to determine Θ, A, Ξ in the first approximation consisting of two items:
$\Theta=\Omega+\varepsilon Z_1(t,\Omega,a,\xi), A=a+\varepsilon Z_2(t,\Omega,a,\xi), \Xi=\xi+\varepsilon Z_3(t,\Omega,a,\xi),$
where Ω, a, ξ are constants at slowly changed values of the unknown variables which represent the main part of the solution.

εZ_1, εZ_2, εZ_3 - small periodical functions of t.

$$\frac{dU}{dt}=\frac{r}{J}[M(\frac{U}{r})-rT(U)-\frac{r}{2}(\kappa_2+3\kappa_3 U)A_1^2],$$
$$\frac{dA_1}{dt}=-\frac{1}{2\omega}(\beta+\kappa_1+2\kappa_2 U+3\kappa_3 U^2+\frac{3}{4}\kappa_3 A_1^2)A_1,$$
$$\frac{d\xi}{dt}=\omega-\rho$$

where: $U=\Omega r$, $A_1=\omega a$, $M(\frac{U}{r})=L(\frac{U}{r})-H(\frac{U}{r})$.

Stationary regimes of motion are determined from the following conditions $\frac{dU}{dt}=0, \frac{dA_1}{dt}=0, \frac{d\xi}{dt}=0$, which make it possible to determine U and A_1 from the equations

$$M(\frac{U}{r})-rT(U)-\frac{r}{2}(\kappa_2+3\kappa_3 U)A_1^2=0, \qquad (5.6)$$
$$A_1(\beta+\kappa_1+2\kappa_2 U+3\kappa_3 U^2+\frac{3}{4}\kappa_3 A_1^2)=0 \qquad (5.7)$$

and similarly the equation for the frequency ρ of the autooscillations $\omega-\rho=0$.

From the equation (5.7) it follows an expression for the amplitude of the oscillations.

$$A_1^2=-\frac{4}{3\kappa_3}(\beta+\kappa_1+2\kappa_2 U+3\kappa_3 U^2) \qquad (5.8)$$

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which agrees with the solution of the equation (5.2) when it is given that $v = r\Omega$ [13].
Equation (5.6) after excluding A_1

$$M\left(\frac{v}{r}\right) - rT(v) + 2r\left(v + \frac{\kappa_2}{3\kappa_3}\right)\left[\frac{d}{dv}T(v) + \beta\right] = 0 \qquad (5.9)$$

serves to determine the stationary values of the velocity v at which stationary automatic oscillations are established.

As it was expected the autooscillations can be produced only under certain values of "u" satisfying eq.(5.9) (or at few values of "u", if the eqation (5.9) has few roots). This value of "u" depends on the character of the source of energy and on the amplitude of the oscillations "a", which is obtained from the equation (5.6). Here one of the reverse influences of the autooscillations on the source of energy appear.

From the equation (5.8) it follows that autooscillations are possible under the conditions when

$$\left(\beta + \kappa_1 + 2\kappa_2 v + 3\kappa_3 v^2\right)\frac{1}{\kappa_3} < 0. \qquad (5.10)$$

This condition A_1 corresponds to the known demand that "u" should be a decreasing portion of the graph $T(v) + \beta v$. One can be easily convinced that it coincides with the condition of stability of autooscillations of the same oscillating system but also with the ideal source of energy, i.e. with the condition of stability of periodical solution of the first equation of the system (5.2) when v = const.

Steadiness of the autooscillations system

with the real source of energy has a special interest because it seems to depend on the properties of the source of energy.

Analysis of stability of the approximate periodical solutions with the consideration of the equation (5.2) can be brought to the consideration of stability of the statistical solutions of the first and the second equations of the system (5.5). For the stability of the stationary values u and A_1, and determined as the roots of (5.7), it is necessary in order that the roots of the characteristic equation of equations in variations, which are formed for the first two equations of the system (5.5) so that they had negative real parts.

These demands bring us to the following criterion of stability.

$$N + \beta r - \frac{3}{4}r\kappa_3\left(1 + \frac{J}{mr^2}\right)A_1^2 < 0, \qquad (5.11)$$

$$\left[N + \beta r - \frac{3}{4}r\kappa_3 A_1^2 + \frac{4r}{3\kappa_3}\left(\kappa_2 + 3\kappa_3 v\right)^2\right]\kappa_3 < 0 \qquad (5.12)$$

where $N = \frac{d}{dv}M\left(\frac{v}{r}\right) = \frac{d}{dv}\left[L\left(\frac{v}{r}\right) - H_1\left(\frac{v}{r}\right)\right]$.

Almost in all real sources of energy

$$\frac{d}{dv}L\left(\frac{v}{r}\right) < 0.$$

We shall consider that $L\left(\frac{v}{r}\right) > H_1\left(\frac{v}{r}\right)$, $N < 0$.

Magnitude N reflecting the steepness of characteristic of the source of energy essentially influences on the stability.

The condition (5.11) can be done when β is small, which we had stated earlier. The condition (5.12) is decisive.

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In this manner autooscillations can show unsleadiness if the characteristic of the source of energy is not sufficiently steep.

Using the example we shall illustrate this calculation. From the condition (5.10) it follows that for the existance $A_1 \neq 0$ and for the stability of autooscillations with the ideal source of energy the following condition should be satisfied

$$u < \left[\left(\frac{\kappa_2}{3\kappa_3}\right)^2 - \frac{\kappa_1+\beta}{3\kappa_3}\right]^{\frac{1}{2}} - \frac{\kappa_2}{3\kappa_3} = u_i$$

(it is assumed that for example $K_1<0$, $K_1+\beta<0$, $K_2>0$, $K_3>0$). For the stability of the autooscillatory system with the real source of energy the following condition should be satisfied:

$$u < \left[\frac{N_0 - r\beta}{12 r\kappa_3} + \frac{A_1^2}{16}\right]^{\frac{1}{2}} - \frac{\kappa_2}{3\kappa_3} = u_r .$$

The last inequality is obtained from (5.12) in the supposition that $N = -N_0$, $N_1 > 0$. Dependings on the value of N_0 the 2nd bonder can be less than the first.

For the selection of the characteristic source of energy which will provide the stability of the autooscillations it is convenient to use the graphical method which is shown in Fig.18. After calculating the stationary values of $A_1 = A_c$, $v = v_c$ from the equations (5.8) and (5.9) the graph of the function $S(u) = rT(u) + \frac{r}{2}(K_2 + 3K_3 u)A_1^2$ is being built around the point v_c . Instead of A^2 (5.8) is used. At the point B /point - $S(u)$ of the graph with the abscissa v_c the tangent "bb" is being drawn to the graph. Autooscillations will be stable if the tangent to the characteristic source of energy drive from the point B will be kept in limit to the shaded sector. This condition is equivalent to the criterion (5.12) and it is based on the work [10].

Let us pay attention to one more side of interaction of the autooscillatory system with the source of energy. The origin of autooscillations leads us to the appearance of periodicals which constitute the angular velocity $\frac{d\varphi}{dt}$.

It was found above as the root of the equation (5.9), stationary value $\Omega = \frac{v}{r}$ can be added with small periodicals consistants of εZ, . Such improved solution in the first approximation has the following form:

$$\frac{d\varphi}{dt} = \Omega - \frac{ra\omega}{Jp}\left[\beta\cos(pt+\xi) - \frac{a\omega}{4}(K_2+3K_3 r\Omega)\sin 2(pt+\xi) + \frac{1}{12}K_3\omega^2 a^2 \cos 3(pt+\xi)\right].$$

Here the angular velocity contains odd and even harmonics as long as the function $T\{(r\varphi - x)\}$ reflects connections between coordinates of the initial equation of motion, contain even and even odd degrees of coordinates.

6. Interaction of Complicated Oscillating Systems with the Source of Energy

At interaction with the source of energy of more complicated oscillating systems show similar specific peculiarities of motion, which were displayed on simple oscillatory systems.

The study of the equation of interaction of complicated oscillatory systems with the source

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of energy is essentially simplified if the consideration of the motion is limited in the spheres of resonances. This kind of approach introduce an acceptable, as far as namely in the sphere of resonance naturally expect the most active manifestation of interaction.

As an example the oscillatory system with n degrees of freedom performs forced oscillations, excited by the deformation elastic connection C on the scheme presented in fig.19. Oscillatory system contains concentrations of masses $m_1, m_2, \dots m_n$ and stiffness $C_1, C_2, \dots, C_n$. Elastic connection between masses m_1 and m_2 has nonlinear characteristic $\Phi(z) = C_2 z + \varepsilon f(z), z = y_1 - y_2$. Forces resisting to the motion are accepted to be proportional to the velocity of motion in each coordinates $\dot{\varphi}, \dot{y}_1, \dot{y}_2, \dots, \dot{y}_n$ with coefficients of proportionality $\beta_1, \beta_2, \dots, \beta_n$.

Equations of motion of the system have the form of

$$\begin{aligned}
&J\ddot{\varphi} = L(\varphi) - \beta\dot{\varphi} + Cr(y_1 - r\sin\varphi)\cos\varphi,\\
&m_1\ddot{y}_1 + C_1(y_1 - y_2) + Cy_1 = -\beta_1\dot{y}_1 - \varepsilon f(y_1 - y_2) + Cr\sin\varphi,\\
&m_2\ddot{y}_2 + C_1(y_2 - y_1) + C_2(y_2 - y_3) = -\beta_2\dot{y}_2 + \varepsilon f(y_1 - y_2),\\
&m_3\ddot{y}_3 + C_2(y_3 - y_2) + C_3(y_3 - y_4) = -\beta_3\dot{y}_3,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
&m_n\ddot{y}_n + C_{n-1}(y_n - y_{n-1}) + C_n y_n = -\beta_n\dot{y}_n. \qquad (6.1)
\end{aligned}$$

The terms on the right part of the equation are considered to be small.

Special attention is paid to the study of private solutions of the system, corresponding

to solo-frequent regimes of oscillations in the sphere of resonance. In connection with this we shall assume that peculiarity of oscillations in the sphere of resonance have a harmonic form, angular velocity "u" (frequency) changes slightly to the extent of the period of oscillations, frequency ω_s of the particular oscillations appears to be simple roots of the characteristic of the equations of the system

$$\begin{aligned}
&m_1\ddot{y}_1 + C_1(y_1 - y_2) + Cy_1 = 0,\\
&m_2\ddot{y}_2 + C_1(y_2 - y_1) + C_2(y_2 - y_3) = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
&m_n\ddot{y}_n + C_{n-1}(y_n - y_{n-1}) + C_n y_n = 0
\end{aligned}$$

magnitudes $\pm k\omega_s (k = 2, 3, \dots)$ do not satisfy the characteristic equation.

Linear replacement of variables with constant coefficients

$$y_j = \sum_{\sigma=1}^{n} \kappa_{j\sigma} x_\sigma \quad (\sigma = 1, 2, \dots, n)$$

allows us to represent the system (6.1) in the form of total elementary vibrations, interacting with the source of energy by means of small members of the right part of the equations

$$\begin{aligned}
&\ddot{x}_s + \omega_s^2 x_s = \varepsilon F_s\left\{\varphi, \sum_\sigma \kappa_{1\sigma}\dot{x}_\sigma, \dots, \sum_\sigma \kappa_{n\sigma}\dot{x}_\sigma, \sum_\sigma (\kappa_{1\sigma} - \kappa_{2\sigma})x_\sigma\right\},\\
&\ddot{\varphi} = \varepsilon F(\varphi, \dot{\varphi}, \sum_\sigma \kappa_{1\sigma} x_\sigma). \qquad (6.2)
\end{aligned}$$

The study of the solo-frequent resonance regimes of forced oscillations in such a system can be carried in the first approximation to the consideration each time only two equations of the system (6.2): of the first equation, which appears

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to be the equation of the source of energy, and one of the n equations of the oscillating motion. The selection of the necessary equation of oscillating motion is fixed by the value, of the frequency of the natural oscillations at the environment of that it is necessary to find the regimes of forced oscillations. If such frequency is ω_s then the system of two equations will be considered

$$\ddot{x}_s + \omega_s^2 x_s = \varepsilon F_s\{\varphi, x_{1s}\dot{x}_s, \ldots, x_{ns}\dot{x}_s, (x_{1s}-x_{2s})x_s\},$$

$$\ddot{\varphi} = \varepsilon F(\varphi, \dot{\varphi}, x_{1s}x_s). \qquad (6.3)$$

These equations are of the same type as the equations (2.2), (3.1), (4.1). Their subsequent analysis is not difficult to carry out in the analogical manner.

Thus, for each resonance region the problem is reduced to the construction of the particular solution of the system of two equations of such type as (6.3). Therefore each resonance regime of complicated system will have all those specific peculiarities of motion at interaction with source of energy, which are typical for such systems but with one degree of freedom in oscillatory motion.

The results received when studying definite systems with some degrees of freedom [22-24] show that qualitative peculiarities of motion are displayed by analogical manner for each resonance region. Experiments [23, 24] proved that particular solution of equations of the form (6.3) which really corresponds to the observed real oscilla-

tory processes. At nonstationary regimes of motion, which are not limited by small acceleration $\ddot{\varphi}$ specific peculiarities of change of frequency $\dot{\varphi}$ in the region of resonance may be more abrupt till to some decreasing of Ω at large amplitudes of oscillation of the system [25].

In some systems with automatical control the properties of the source of energy may influence and in the nonresonance conditions, as it is shown, for example, in paper [26].

The discussed above permits to have a representation of essential influence of the nonideal source of energy to the motion of oscillatory system.

The choice of mechanical systems for the sake of examples made on clearness.

It goes without saying that all specific peculiarities of oscillatory systems with the source of energy were considered above and they are applied equally to electric, acoustics and other systems.

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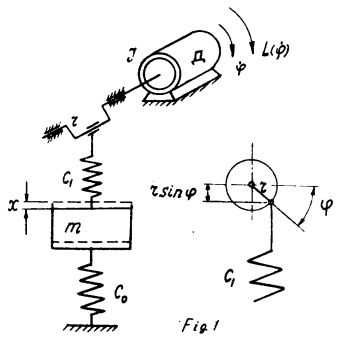

Fig 1

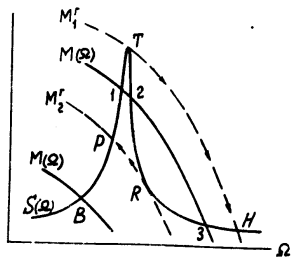

Fig 2

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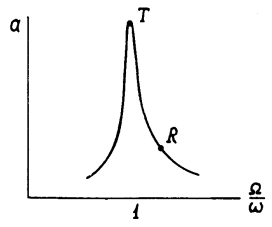

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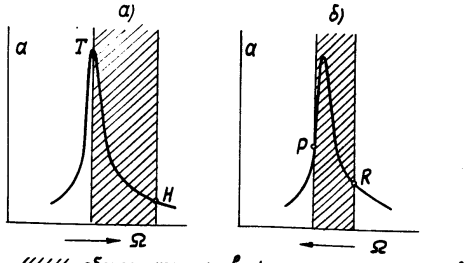

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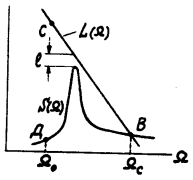

Fig. 5

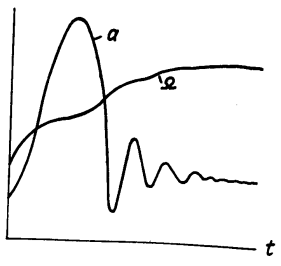

Fig. 6

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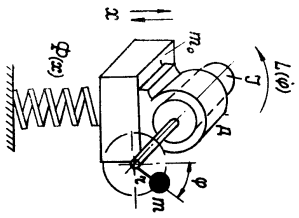

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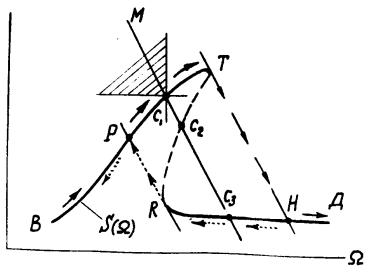

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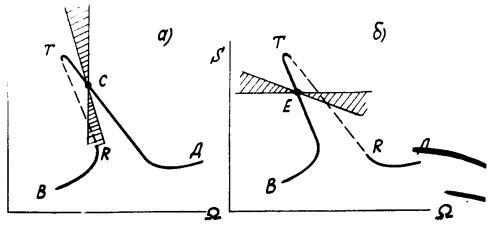

Fig. 9

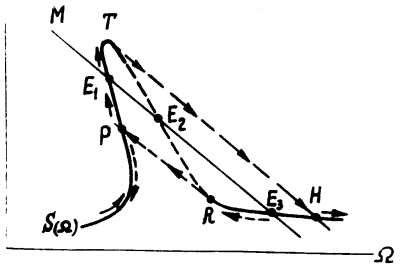

Fig. 10

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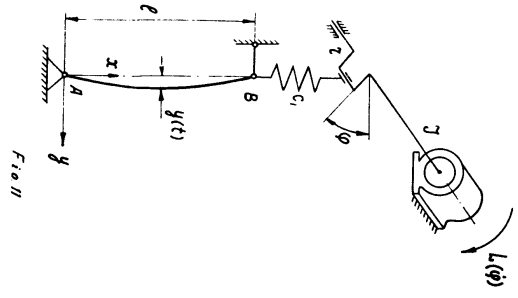

Fig 11

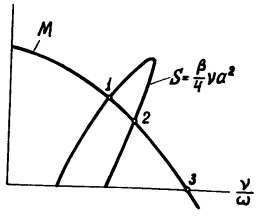

Fig 12

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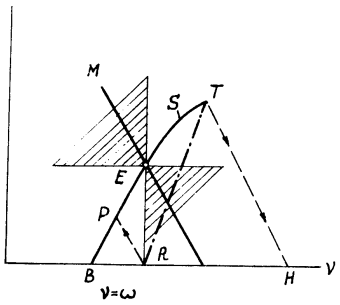

Fig. 13

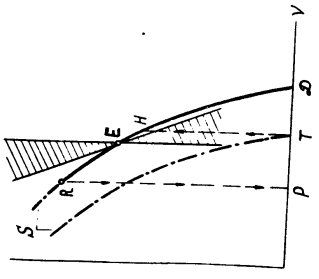

Fig. 14

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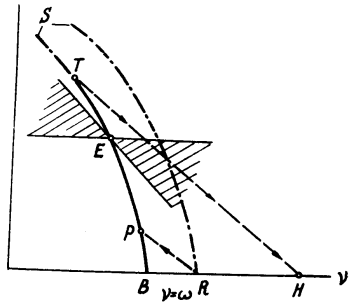

Fig. 15

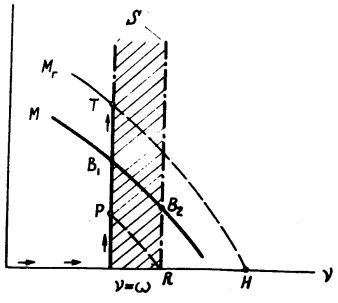

Fig. 16

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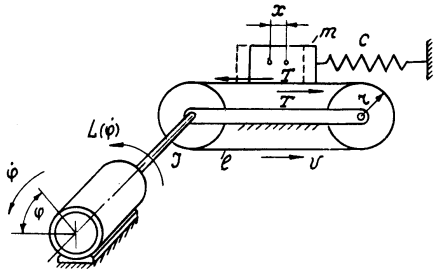
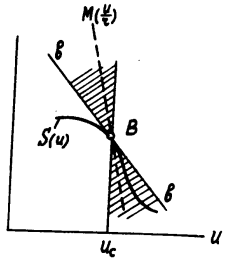


Fig. 17



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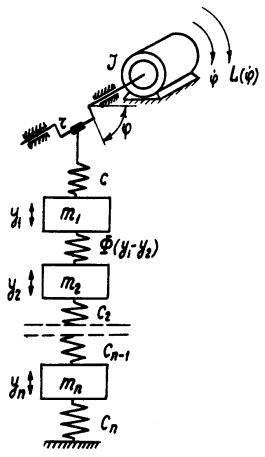

Fig. 19

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О НЕКОТОРЫХ ПРИЗНАКАХ СУЩЕСТВОВАНИЯ ПЕРИОДИЧЕСКИХ РЕШЕНИЙ У СИСТЕМ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

In the present paper we want to consider some general methods of establishing of the existence - theorems of periodic solutions for the systems of the ordinary differential equations of the I-st order

$$\dots \qquad /1/$$

and for the systems of the second order

$$\dots \qquad /2/$$

It is convenient to write the systems in question in the vector form

$$\frac{dx}{dt} = f(t, x) \qquad /1'/^*$$

and

$$\frac{d^2x}{dt^2} = \varphi\left(t, x, \frac{dx}{dt}\right). \qquad /2'/^*$$

It will be considered further that the right-hand sides of the equations are everywhere periodic with respect to t, ω - being the period, and sufficiently smooth in corresponding domains /many of the formulated proposals given bellow holding even if the right hand sides of the equations are not smooth/.

I. The method of two functionals. Poincare's method of the point transformations is one of the principal methods to prove the existence of periodic solutions for the system /I/ /see[1,2]/. The method is the following: the integral curve $x(t) = \rho(t, X)$ of the system /1/ is let out from each point $X = (X_1, \dots, X_n)$ of some domain G of the n dimensional space E^n, and the point $U(X) = \rho(\omega, X)$, determined by shifting on the integral curve, corresponding to the point X, when $0 \le t \le \omega$. Periodic solutions are determined by the stationary points of the transformations U. It should be noted that the construction of the operator U may turn out to be impossible, unless all the

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solutions of the system may be continued to [illegible] /see [illegible]/. In that case one has to make additional constructions, but we don't dwell upon them in the paper. Thus, it is necessary to show the operator $\mathcal{U}$ to have stationary points, in order to prove the existence of the periodic solutions. In most cases the principle of compressed mappings or that of Braker is used. One may obtain a number of some new features of the existence of periodic solutions, when stronger topological theorems are used

It is known that for the operator $\mathcal{U}$ to have a stationary point, it is sufficient that the rotation /see[6,7]/ of the vector field $U(X)-X$ schould not be zero an the boundary of some domain. Then one has the following statement.

Lemma 1. [illegible]. Let the vector field $f(0,X)$ have non-zero rotation on the boundary S of some domain $\mathcal{G}$ and let the condition of irrevocability be satisfied

$$\partial(t,X)\neq X \quad (X\in S,\ 0<t\leq\omega). \qquad /3/$$

Then the system /I/ has at least one ω- periodic solution.

The application of this general principle requires that the rotation of the vector field schould be calculated, this being possible in many cases /see[6,7,10-12]/. Sometimes the condition of irrevocability is verified by using some functions /the signs being not necessarily constant/ analogous to Lyapunov's functions in the theory of stability; the functions admitting simultanious estimation of the rotation of the field $f(0,X)$.

We shall give a simple statement resulting from a more general theorem [8].

Theorem 1. Let $\lambda(x)$ and $u(x)$ be even differentiable functions in n variables and let the inequalities be satisfied

$$(f(t,x),\ \mathrm{grad}\,\lambda(x)) = \sum_{i=1}^{n} f_i \frac{\partial\lambda}{\partial x_i} > 0$$

$$(f(t,x),\ \mathrm{grad}\,u(x)) = \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} > 0$$

and

$$\lim_{\|x\|\to\infty} \{|\lambda(x)|, |u(x)|\} = \infty$$

where $\|x\| = \sum_{i=1}^{n} |x_i| \geq R$.

Then the system /I/ has at least one periodic solution. This theorem seems to be convenient since its conditions are of analitic character only. One can deduce a [illegible] statements close to those ones. The proof [illegible] principle is usially required, and also theorems close to Birjuk's results. One can formulate various other theorems taking functions $\lambda(x)$ and $u(x)$ to be different.

More thin discussions associated in particular with the usage of differential inequalities, result in theorems of some other character. Let one of them be given [illegible].

Theorem 2. Let $\lambda(x)$ and $u(x)$ be even differentiable functions of n variables $u(x)>0$ and let the inequalities

$$(f(t,x),\ \mathrm{grad}\,\lambda(x)) = \sum_{i=1}^{n} f_i \frac{\partial\lambda}{\partial x_i} > \varepsilon\,\lambda(x),$$

$$(f(t,x),\ \mathrm{grad}\,u(x)) = \sum_{i=1}^{n} f_i \frac{\partial u}{\partial x_i} > L\,u(x), \qquad /8/$$

and

$$\lim_{\|x\|\to\infty} \{|\lambda(x)| + |u(x)|\} = \infty \qquad /9/$$

be satisfied when $\|x\| = \sum_{i=1}^{n} |x_i| \geq R$, ε and L being positive.

Then the system /I/ has at least one periodic solution.

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2. The usage of invariant cones in the phase space.

A number of features of the existence of periodic solutions may be obtained if one can choose some cone K [7,11], the points of which do not overstep it in moving on integral curves (t increases). For example, the cone K is invariant to the operator $U(x)$, if K - is some vector cone with non-negative components and inequalities

$$f_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \geq 0 \quad (x_j \geq 0,\ i, j = 1, \dots, n) \qquad /10/$$

are satisfied.

Various theorems on the existence of the stationary points, on the uniqueness and non-uniqueness of stationary points are known for nonlinear operators, leaving the cone invariant. Moreover, the low and upper values of a number of stationary points are also known. Extending the statement of these theorems to the operator $U(x)$ one can obtain

the corresponding statements on the periodic solutions.

For example, suppose the conditions /10/ to be satisfied and the system /I/ to have some zero solution

$$f_i(t, 0, \dots, 0) \equiv 0 \quad (i = 1, 2, \dots, n). \qquad /11/$$

Then suppose the right hand sides of the system (1) to be differentiable in the neighbourhood of zero

$$a_{ij}(t) = \frac{\partial f_i(t, 0, \dots, 0)}{\partial x_j}, \quad (ij = 1, \dots, n) \qquad /12/$$

and to have assimptotic in the infinity ($x_1, \dots, x_n$ - are not negative) as linear functions, i.e. let some continues periodic functions $b_{ij}(t)$ exist such, that

$$\lim_{\|x\| \to \infty} \frac{f_i(t, x_1, \dots, x_n) - \sum_{j=1}^{n} b_{ij}(t) x_j}{|x_1| + \dots + |x_n|} = 0. \qquad /13/$$

Consider linear systems

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) y_j, \quad \frac{dz_i}{dt} = \sum_{j=1}^{n} b_{ij}(t) z_j \qquad /14/$$

and assume $Y(\omega)$ and $Z(\omega)$ to be matrixes of monodromy for the systems. Then it follows from /10/ that elements, which are not on the main diagonal of the matrixes, are nonnegative. It follows from Perron's theorem that each of these matrixes has non-negative eigenvalue, an eigenvector with non-negative components corresponding to it. Suppose these eigenvalues to be simple and denote them λ_0 and λ_∞ respectively.

Theorem 3. Let one of the conditions $\lambda_0 < 1 < \lambda_\infty$ or $\lambda_0 > 1 > \lambda_\infty$ be satisfied. /15/

Then the system /I/ but zero one has at least one more periodic solution with non-negative components.

3. On differential equations in Banach spaces. Infinite system of ordinary differential equations, integral - differential equations and some boundary problems may be considered /see [4] and [21]/ to be ordinary differential equations in some Banach space. The method of point transformations makes it possible to prove here some theorems on periodic solutions /see [8], [22]/. It is quite natural to use the principle of stationary points for operators acting in the infinite -dimensional spaces. J.G.Borisovich's works, where the theory of weak - continues vector fields is used, should be mentioned here [22-24].

4. Odd periodic solutions. Consider the system /2/, the right hand sides of which satisfy some additional condition:

$$\varphi_i(-t, -x_1, \dots, -x_n, y_1, \dots, y_n) = \qquad /16/$$
$$= -\varphi_i(t, x_1, \dots x_n, y_1, \dots, y_n), \ (i = 1, \dots, n).$$

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Suppose we succeded in constructing of the solution $x(t)$ of the two-point problem

$$\frac{d^2x}{dt^2} = \varphi\left(t, x, \frac{dx}{dt}\right), \quad x(0) = x\left(\frac{\omega}{2}\right) = 0. \qquad /17/$$

Continue the function $x(t)$ in the interval $\left[-\frac{\omega}{2}, 0\right]$ as an odd function and then continue it as ω- periodic function to all values of t, the notation $x(t)$ being the same for the new function. It follows from /16/ that the function $x(t)$ is a periodic solution of the system /2/. Thus, in a number of impotant cases the construction of the periodic solution for the system /2/ may be reduced to the two-point problem.

Fundamental results for scalar case $(n=1)$ in the problem /17/ have been establisned in a well known Bershtein work [2]. Some other condition for the solution of the two-point problem were obtained by many authors. There is a new statement for the scalar problem /17/.

Theorem 4. Let $\varphi(t, x, y)$ satisfy the conditions

$$\varphi(t, x, 0) > 0 \text{ for } x > M, \quad \varphi(t, x, 0) < 0 \text{ for } x < -M \qquad /18/$$

and

$$|\varphi(t, x, y)| \leq \mathcal{L}(|y|), \qquad /19/$$

where $\mathcal{L}(u)$ is a positive function such that

$$\int_0^\infty \frac{u\,du}{\mathcal{L}(u)} = \infty.$$

Then the boundary problem /17/ has a solution.

The boundary problem /17/ may have several solutions. In some cases the number of these solutions can be estimated from bellow. Consider one theorem of [16]. Assume the problem /17/ to have zero solution, i.e. $\varphi(t, 0, 0) \equiv 0$.

Theorem 5. Let the conditions of the theorem 4 be satisfied and let the solution of the initial problem

$$\frac{d^2u}{dt^2} - \frac{\partial\varphi(t,0,0)}{\partial x}u + \frac{\partial\varphi(t,0,0)}{\partial y}\frac{du}{dt}, \quad u(0)=0, \quad \frac{du(0)}{dt} = 1, \qquad /20/$$

have ℓ zeroes in the interval $(0, \frac{\omega}{2})$ and $u(\frac{\omega}{2}) \neq 0$. Then the number of non-zero solutions of the problem /17/ is not less than 2ℓ.

The analysis of the two-point problem for systems with n degrees of freedom is more complicated. Apparently, the simplest way of investigation of the problem /17/ is a transition to the system of nonlinear integral-differential equations

$$x_i(t) = \int_0^\omega K(t,s)\varphi_i\left(s, x_1(s), \dots, x_n(s), \frac{dx_1(s)}{ds}, \dots, \frac{dx_n(s)}{ds}\right)ds, \qquad /22/$$

where $K(t,s)$ is Green's function of the operator $\frac{d^2}{dt^2}$ with zero boundary conditions. Replace the system /13/ by the equivalent system of the integral equations

$$v_i(t) = \varphi_i\left(t, Av_1(t), \dots, Av_n(t); Bv_1(t), \dots, Bv_n(t)\right), \qquad /23/$$

where

$$Av(t) = \int_0^\omega K(t,s)v(s)ds, \qquad /24/$$

$$Bv(t) = \int_0^\omega \frac{\partial K(t,s)}{\partial t}v(s)ds. \qquad /25/$$

The results on periodic solutions obtained in this way are given in [13]. Some conditions of solubility of the two-point problem, based on the two-sided values, are given in [34].

5. One - sided values provided the existence of the periodic solutions for the system /2/. Now take the system /2/ neglecting /16/. Then the following statement may be obtained from the theorem 2.

Theorem 6. If inequalities

$$\sum_{i=1}^n \lambda_i \varphi_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \geq \varepsilon \sum_{i=1}^n x_i^2 - m\sum_{i=1}^n y_i^2 - \alpha, \qquad /26/$$

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$$\sum_{i=1}^{n} y_i \varphi_i(t, x_1, \dots, x_n, y_1, \dots, y_n) \geq -L\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2\right)^{\frac{m+1}{2}} + \varepsilon, \qquad /27/$$

where ε, L – are positive, $m<1$ are satisfied, the system /2/ has a periodic solution.

6. Features arising in the investigation of autonomous systems. Here we consider a system

$$\frac{d^2 x_i}{dt^2} = \varphi_i\left(x_1, \dots, x_n, \frac{dx_1}{dt}, \dots, \frac{dx_n}{dt}\right) \quad (i = 1, \dots, n) \qquad /28/$$

or in the vector form

$$\frac{d^2 x}{dt^2} = \varphi\left(x, \frac{dx}{dt}\right). \qquad /28^*/$$

It is difficult to obtain periodic solutions of such a system as the period is a priori unknown. In other words, the problem of obtaining of periodic solutions of autonomous system is a problem with an arbitrary parameter, the value of which is to be chosen.

Let $$t = \omega\tau \qquad (0 \leq \tau \leq 1). \qquad /29/$$

Then the system /28/ is in the form

$$\frac{d^2 x}{dt^2} = \omega\varphi\left(x, \frac{1}{\omega}\frac{dx}{d\tau}\right). \qquad /30/$$

To obtain periodic solutions of the system /28/ is to obtain periodic solutions with some known period equal to I for the system /30/. The values of the parameter ω, when the system /30/ has one periodic solution, are the periods of solutions for the system /28/.

Greem functions being used, the problem of I- periodic solutions for the system /30/ may be rewritten as a system of non-linear integral equations in the vector form

$$x(t) = A\{x(t), \omega\}. \qquad /31/$$

Thus, the problem of periodic solutions for autonomous system proves to be the problem of obtaining stationary points of a completely continuous operator $A\{x(t), \omega\}$

acting in some space of the vector functions and depending on the scalar parameter ω. To investigate the equation /31/ one should use all the methods of non-linear functional analysis. Some results obtained in this way are given in [33]*).

There are some of the considerations associated with two problems only.

Assume the system /28/ to have a zero-solution. Investigate non-zero periodic solutions with small amplitudes. For simplicity consider the case, when

$$\varphi_i(-x_1, \dots, -x_n, y_1, \dots, y_n) = -\varphi_i(x_1, \dots, x_n, y_1, \dots, y_n) \quad (i = 1, \dots, n). \qquad /32/$$

As it was shown in 4. the periodic solutions of the system /28/ may be constructed using solutions of the two-point problem, if conditions /32/ are satisfied. In this case operator equation /31/ can be written in the form

$$v_i(t) = \omega\varphi_i\left(Av_1(t), \dots, Av_n(t), \frac{1}{\omega}Bv_1(t), \dots, \frac{1}{\omega}Bv_n(t)\right), \qquad /33/$$

where A and B are linear operators determined by the inequalities /24/ and /25/.

In order to investigate the system /31/ it is natural to use general theory of bifurcation points. It is easy to describe the fundamental idea of topologic methods in the theory of bifurcation points.

Let the equation /31/, linearized in zero, have non-zero solution, when $\omega = \omega_0$ /that is, let ω_0 be a period of non-zero periodic solution of the linearized solution /28/. Find the topological index of some zero point of a completely continuous vector field, the critical value being ω_0, the precritical value $\omega < \omega_0$, postcritical value $\omega > \omega_0$.

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*) Unfortunately, there are many misprints and inaccuracies in [33].

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In main cases, when pre-critical values are close to ω_0, the index $\gamma(\omega_0 - 0)$ of the zero point is known not to depend on ω; analogously, when post-critical values of ω are close to ω_0, the index $\gamma(\omega_0 + 0)$ is the same. Denote the index of zero point of the field $x - A(x, \omega_0)$ by $\gamma(\omega_0)$. If out of three numbers $\gamma(\omega_0 - 0), \gamma(\omega_0), \gamma(\omega_0 + 0)$ at least two are different, then with respect to every $\varepsilon > 0$ one can obtain such ω, that $|\omega - \omega_0| < \varepsilon$ and the equation /3I/ has non-zero solutions with the norm less than ε. For the application of the above mentioned principle one should calculate the index of a zero point of a completely continuous vector field. The index was calculated for different particular cases in the reports [7,35-37]. Recently P.I.Zabreiko and one of the authors obtained algorithm, which makes it possible to calculate an index for a sufficiently smooth operator practicaly in all cases.

It should be noted that general theorems make it possible to describe the character of the periodic solutions in terms of some of the characteristics of the right-hand sides of the equations, to determine the form of amplitude curves, without obtaining exact or approximate solution beforehand. Some theorems for simple cases are repreesented in [33,36].

When the conditions of the type /2/ are fullfiled the operator eq. /33/ has, as a rule, isolated solutions for fixed values of the parameter ω. The topological index of these solutions differs from zero. Such solutions are stable with respect to weak perturbations of the right - hand side of the equation. It explains the simplicity of the investigation of the operator equation and posibility to get sufficiently complete results.

In general case the autonomous system together with every periodic solution has continuum of other periodic solutions, obtained from the first by shifting the argument. Considering the operator equation /3I/ we have continuum of stationary points forming closed curve in the functional space. Let us enclose that curve Γ by some domain $\mathcal{G}$ of torus type and denote the boundary of the domain by S. The rotation of vector field $\Phi(x, \omega) = x - A(x, \omega)$ on S is a zero, because however small the change of ω be in domain $\mathcal{G}$, the field $x - A(x, \omega_0)$ will not have stationary points. In other words, one has to find curve of stationary points with zero summary topological index in general problem on periodic solutions of autonomous system. It explains a comparative complexity of obtaining periodic solutions of autonomous system with greater number of degrees of freedom.

We shall give here general scheme which allows to reduce the problem on periodic solutions of autonomous systems to the problem of stationary points of non-zero index. Now we can obtain not only some of the qualitative results but also substatiate the applicability of some approximative methods.

At first consider in detail the character of the vector field $x - A(x, \omega)$ in the neighbourhood of curve Γ (ω_0 - is a period of the periodic solution for the autonomous system). The point of the curve Γ are zeroes of the vector field. It turns out to be possible to construct a family of planes $\Pi(x)$ $(x \in \Gamma)$ each of them intersect the curve Γ in the point x with non-zero angle. The vector of the field are in corresponding planes $\Pi(x)$ up to the terms of high orders.

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If one considers $\Phi(x,\omega_0)$ in the plane $\Pi(x_0)$ in the neighbourhood of the point x_0, then X_0 will be an isolated stationary point and its index will not be zero: the index is the same for all points $x_0 \in \Gamma$.

Let's now consider the field $\Phi(x,\omega_0+\delta\omega)$. It turns out that all the vectors of the field $\Phi(x,\omega_0)$ have / up to small value of high orders/ non-zero supplement, directed along the curve Γ. If $\delta\omega$ are of opposite signs, the supplement will be opposite in the directions.

The described geometrical method makes transition from the operator equation $x = A(x,\omega)$ to the equation

$$x = A(x,\omega(x)) \qquad /34/$$

possible, where $\omega(x)$ is snitably chosen /for example, the linear/ functional, the values of which may change with the transition from one plane $\Pi(x)$ to anotner. It is obvious, that the points $x_0 \in \Gamma$, where the functional $\omega(x)$ has the value ω_0, and only they are the solutions of the equation /34/.

These solutions are also the solutions for /31/, i.e. they will determine the periodic solution of the autonomous system. The value of functional $\omega(x)$ for the solution determines the unknown period of the solution /34/ are already isolated and they have non-zero topological index 1.

The described scheme is applicable to the analysis of equations of the type known as Van - der - Polle's equations and for his multy -dimensional analoges. This scheme requires further development and detalisation.

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О СИНТЕЗЕ КОРРЕКТИРУЮЩИХ ЦЕПЕЙ В НЕЛИНЕЙНЫХ КОЛЕБАТЕЛЬНЫХ И РЕГУЛИРУЕМЫХ СИСТЕМАХ

The oscillations in oscillating and regulated systems are often required to be limited in a definite way around a stable equilibrium state when the initial disturbances are bounded in a prescribed region. To realize the required limitation of the oscillations, active and passive correcting devices are introduced in the system. Here we study some methods of the construction of the correcting devices characteristics in non linear oscillating and regulated systems.

I. A Non-Linear System of the Second Order Acted upon by an External Force.

Let us consider an oscillating system, which is described by an ordinary differential equation

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$$x'' + \Phi(x, x')x' + h(x) = p(\tau), \qquad \left(' = \frac{d}{d\tau}\right), \tag{I}$$

with respect to a dimentionless value x, where τ is a dimentionless time, Φ - the characteristic of the damping force, h - that of the restauring force and p - that of the disturbing force. The functions $\Phi(x, x')$, $h(x)$ and $p(\tau)$ are such that in a certain region $\mathcal{D}$

$$|x| \leq X, \quad |y| \leq Y$$

of the variables x and y, there exists a unique solution $x(\tau)$ of the equation (I), passing through the given point (in the region $\mathcal{D}$)

$$x(0) = \xi, \quad y(0) = \eta \tag{2}$$

and which is defined upon the whole positive semiaxis of the independent variable τ:

$$0 \leq \tau \leq \infty. \tag{3}$$

In the mentioned class of functions, the problem is stated to synthesize the damping characteristic Φ, which would ensure the prescribed limitation of the oscillation of the system around the equilibrium point $x = y = 0$ ($h(0) = 0$); this point corresponds to the case $p \equiv 0$ and the oscillations in question are induced by the action of the disturbing force and the initial conditions (2). In this problem it is reasonable to utilize a criterium of the ultimate

boundedness of the differential equation solutions, the importance of which was pointed out by V.V.Niemytzky [1].

H.A.Antosiewicz [2] delivered the following criterium of the ultimate boundedness of the solution of the equation (I) upon the semiaxis (3).

If I. $\Phi(x, x') \geq 0$ for all x, x';

II. $H(x) = \int_0^x h(\sigma)d\sigma > 0$ for all $x \neq 0$;

III. $H(x) \to \infty$ as $|x| \to \infty$;

IV. $\int_0^\infty |p(\tau)|d\tau < \infty$,

then any solution of the equation (I) satisfies $|x(\tau)| < C_1 = const$, $|x'(\tau)| < C_2 = const$ as $\tau \to \infty$.

The direct application of this criterion to many physical systems is obstructed by the conditions II, III and IV. Really, even the characteristic $\sin x$ of the restauring force of the physical pendulum in a homogenous gravity field does not satisfy the conditions III. A lot of physical systems cease to be described by the equation (I) when $|x| > a$ ($a = const$, $0 < \alpha \leq X$) due to the liaisons, imposed upon the coordinate x by limiting rods, boundary regulators etc. The condition IV excludes out of the consideration the vibration, which is on of important forcing actions. Therefore we shall

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try to weaken the conditions II, III and IV of the boudedness of the solution of the equation (I) in such a way as to render the given criterion useful for the investiga - tion of applied problems. Namely, let us substitute those conditions by the follo - wing ones :

II'. $H(x) > 0$ as $0 < |x| \leq a$;

III'. the function $H(x)$ increases as $0 < |x| \leq a$;

IV'. $|\rho(\tau)| \leq \rho = const$ in the semiaxis (3).

Besides, we shall consider the characteristic Φ as depending only of the dimentionless velocity $y = x'$; $\Phi \equiv \Phi(y)$ as in real systems it is comparatively easy to realize the damping by means of passive networks whereas the construction of a passive damping action according to the law

$$\Phi(x, y)y \quad (\partial\Phi/\partial x \neq 0)$$

presents a considerable difficulty. As to the introduction of the coordinate x in the characteristic through active devices does so - metimes considerably complicate the system.

Following the paper [2], let us consider the function $R = y^2 + 2H(x)$. From the condition II' we learn that $R \geq 0$. Suppose the inequality $R \leq R_0$ (4) is valid, where $R_0 = R(\xi, \eta)$. Then at any case

$$H(x) \leq \frac{1}{2} z^2, \quad |y| \leq z \quad (z = +\sqrt{R_0}),$$

in the semiaxis (3) and in virtue of the condition III',

$$|x(\tau)| \leq C; \quad |y(\tau)| \leq z,$$

where C is the absolute value of the greatest in absolute value root of the equation $2H(x) = z^2$. The graphical solution to this equation is given in the fig. Ia-Id, $C < a$.

Let us examine the conditions under which the inequality (4) does hold. In virtue of the equation (I)

$$R' = \frac{dR}{d\tau} = 2(\rho y - \Phi y^2) \leq 2|\rho(\tau)||y| - 2\Phi(y)y^2$$

From the condition IV' it follows that $R' \leq -2q$; $q = \Phi(y)y^2 - \rho|y|$. For real values $y^2 = |y|^2$ and therefore $q = |y|[\Phi(y)|y| - \rho]$. Let us compose the characteristic $\Phi(y)$ such that $\Phi(\eta)|\eta| - \rho > 0 \quad (\eta \neq 0)$; this is always possible due to the condition 1. Then $q > 0$ for $z = 0$ and the value R will decrease at least until it reaches the value $R = S < R_0$, corresponding to the first vanishing of q. Besides, $S = R(\lambda, \rho)$, where λ - the root of the equation $q = 0$ which is the nearest to the value η and $|\eta| > |\rho|$; λ is the corresponding value of the coordinate x. As for all $|x| < |\lambda|$ $R' \leq 0$, the oscillations of the system under the initial values (2) are ultimately subject to the limitation $R \leq S$: $|y(\tau)| \leq |\eta|$, $|x(\tau)| \leq |\lambda|$. To valuate the number λ [illegible]

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us make use of the inequality $S < R_*$:

$$H(\lambda) < \frac{1}{2}(\eta^2 - \mu^2) + H(\xi) = H^*.$$

Thus $|\lambda| < \lambda_0$, where λ_0 is the absolute value of the greatest in absolute value root of the equation

$$H(\lambda) = H^*. \qquad (5)$$

As an example let us consider the characteristic of the damper for a physical pendulum - the plomb line detector. The support of the pendulum performs an angular vibration in the plan of its swinging, which acts upon the pendulum through friction in the hinge. The primary equation is one of the form:

$$\ddot{\varphi} + g(\dot{\varphi})\dot{\varphi} + \Omega^2 \sin\varphi = Q(t) \quad (\Omega^2 = const > 0), \quad \left(\dot{} = \frac{d}{dt}\right)$$

where φ is the deviation of the pendulum, g - the characteristic to be found, Q - the forcing action, $|Q| \le \rho_0$. Through the substition $\tau = t\Omega$, $x = \varphi/(\pi/2)$ this equation is transformed to the general form (I), where

$$\Phi(x') = \frac{1}{\Omega} g\left(\frac{\pi\Omega}{2} x'\right), \; h(x) = \frac{2}{\pi} \sin\frac{\pi x}{2}, \; \rho(\tau) = \frac{2}{\pi\Omega^2} Q\left(\frac{\tau}{\Omega}\right).$$

Suppose that according to the scheme of the pendulum, its maximum swing is limited by the interval $|\varphi| \le \varphi_0 < \frac{\pi}{2}$ e.g. $a = \frac{2\varphi_0}{\pi} < 1$ and $|f| < a$. Let us choose the function Φ in the form $\Phi = \kappa/y/$, $\kappa = const$. From the condition I it follows that $\kappa > 0$. In this case the problem of the synthesis comes

to the appointment of the coefficient κ to ensure the demand of the prescribed ultimate boundedness of the pendulum oscillations : $|x| \le \varepsilon \quad (\varepsilon < a)$. The condition II' is fulfilled as

$$H = \frac{8}{\pi^2} \sin^2\frac{\pi x}{4}.$$

The condition III' is fulfilled by the definition of the constant $a \quad (|x| < a < 1)$. The condition IV' keeps in virtue of properties of the function Q and $\rho = 2\rho_0/\pi\Omega^2$. Further, $\eta = |y|/(\kappa y^2 - \rho)$, $|\mu| = \sqrt{\rho}/|\eta|$. Let us choose the coefficient κ such that $\kappa > |\mu|$. Then $S = \frac{\rho}{\kappa + 2H(\lambda)}$ and taking consideration of the equation (5) we see that $|x| < \lambda_0$. It follows that we have to satisfy the inequality $\lambda_0 \le \varepsilon$. To this, it is enough to fulfill the inequality $H^* \le \min(H_1, H_2)$, where $H_1 = H(-\varepsilon)$, $H_2 = H(\varepsilon)$. In the fig.2, to fix the ideas, $H(-x) \ge H(x)$ and H_1 . From the above we conclude:

If : (1) $\eta \ne 0$, (2) $\kappa > |\mu| = \frac{\sqrt{\rho}}{|\eta|}$, (3) the initial values (z) are limited in the interval belonging to the region D and such that the greatest in absolute value root of the equation

$$\sin^2\frac{\pi\lambda}{4} = \frac{\pi^2(\eta^4 - \rho)}{16\eta^2} + \sin^2\frac{\pi\varepsilon}{4} \qquad (6)$$

respecting the unknown λ , considered in the segment $|\lambda| \le a$, lies in the segment $[-\varepsilon, \varepsilon]$, then the deviation of the pendulum is

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ultimately bouned by the inequality $|x| \leq \varepsilon$. Besides, the unequality $|y| \leq |\eta|$ will hold too.

The equation (6) may have real roots in the segment $|\lambda| \leq a$; that follows from a simple example : let $\eta = \pm(1+\delta)\sqrt[n]{\rho}$ and $\xi = \pm\delta$. If we choose for the constant δ a sufficiently small number, the right side member in the equation (6) becomes a sufficiently small positive value and the equation (6) will be satisfied by a small value of the unknown λ.

2. Stationary Non-Linear Systems of the n-th Order.

Let us consider a system, described by the nonlinear equations

$$\left.\begin{aligned} \dot{z}_i &= Z_i(z_1, z_2, \dots, z_n), \\ \dot{z}_n &= Z_n(z_1, z_2, \dots, z_n, \delta) \end{aligned}\right\} \qquad (7)$$

$$(i = 1, 2, \dots, n-1)$$

where δ is the characteristic of the correcting device. The problem is to synthesize the characteristic δ for a certain class Z of functions Z_j ($j = 1, 2, \dots, n$) so that as long as the initial values $\zeta_j = z_j(0)$ of the depending variables z_j lie in the intervals

$$|\zeta_j| < \theta_j \quad (\theta_j = const_j > 0)$$

the variables z_j never leave the corresponding segments in the whole time semiaxis

$$0 \leq t \leq \infty \qquad (8)$$

and tend to zero asymptotically :

$$|z_j(t)| \leq \theta_j, \quad z_j(\infty) = 0.$$

Transforming the variables : $x_j = \frac{z_j}{\theta_j}$ we change the system (7) and the latter inequalities into :

$$\left.\begin{aligned} \dot{x}_i &= f_i(x_1, x_2, \dots, x_n), \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n, \delta) \end{aligned}\right\} \qquad (9)$$

$$|\xi_j| < 1,$$

$$|x_j(t)| \leq 1, \qquad (10)$$

$$x_j(\infty) = 0, \qquad (11)$$

where

$$f_i = \frac{1}{\theta_i} Z_i(\theta_1 x_1, \theta_2 x_2, \dots, \theta_n x_n), \dots, \xi_j = \zeta_j / \theta_j .$$

To synthesize the characteristic in question we shall generalize the method based on the divergence of a certain time integral /3/. Let the class Z satisfy the following conditions :

I. the functions f_j are continuous in all the arguments in the segments $|x_j| \leq 1$;

II. $f_i(0, 0, \dots, 0) = 0$, $f_n(0, 0, \dots, 0, 0) = 0$;

III. there exists a function F of some of

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n - I variables $x_1, x_2, \ldots, x_n$ - and for that variable we shall assume the value x_1 without loss of generality - continuous and derivable in the segment /-I,I/ and posessing the properties :

a) $F(x_1) \neq 0$ in the intervals (-I,0) and (0,-I),

b) $F(-1) = F(0) = F(1) = 0,$

c) $|F'(x_1)| < \infty$ in the segment $|x_1| \leq 1$

and

$F'(-1) > 0,\ F'(0) < 0,\ F'(1) > 0,$

d) there exist real functions $x_2 = X_2(x_1), \ldots, x_n = X_n(x_1), X_{i+1}(0) = 0, |X_{i+1}(x_1)| \leq 1$ for $|x_1| \leq 1$, continuous and derivable in the segment $|x_1| \leq 1$ and satisfying in the system of differential equations

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= F(x_1) \\ f_2(x_1, x_2, \ldots, x_n) &= X_2'(x_1) F(x_1) \\ &\ldots \\ f_{n-1}(x_1, x_2, \ldots, x_n) &= X_{n-1}'(x_1) F(x_1) \end{aligned}\right\}, \quad (I2)$$

IV the equation

$$X_n'(x_1) F(x_1) = f_n[x_1, X_2(x_1), \ldots, X_n(x_1), \delta]$$

has the solution $\delta \equiv \delta(x_1)$ which is real and continuous in the segment /-I,I/.

Then the solution $x_1 \equiv x_1(t)$ of the equation $\dot{x}_1 = F(x_1)$ under the initial condition $|\xi_1| < 1$ will satisfy to the requirements (I0) and (II) in the whole semiaxis (8); all other dependent variables will be subject to the similar limitations. In truth, the integral

$$t \equiv T(A) = \int_{\xi_1}^{A} \frac{d\sigma}{F(\sigma)}$$

determining the time T in which the variable x_1 reaches a certain value A, holds in this case the following properties:

$$T(-1) = -\infty,\ T(1) = -\infty,\ T(0) = +\infty.$$

That proves the above assertion. As to the relations $X_{i+1}(x_1) = x_{i+1}$ they represent first integrals of the system (9).

The system (9) containes only one correction function, which ensure all the required properties of the system. When there are more correction functions, the class Z of the functions f_i, in which the required limitation of oscillations is feasible, becomes larger. As an example of function which does not satisfy to the condition "d" let us mention the function $f_i = a_1 x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2\ (a_j > 0)$.

As an application let us consider a control system for two parameters : the angular velocity ω and the pressure p in an air impeller with a single control device. The controlled variables oscillations are described by the following system of two nonlinear equations [4] :

$$\dot{p} = \bar{G}(p, \omega) - \bar{H}(p) \equiv Z_1(p, \omega),$$

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$$\dot{\omega} = \bar{M}(\omega,\delta) - \bar{N}(\omega,\rho) \equiv Z_2(\rho,\omega).$$

Here $Z_1(0,0) = Z_2(0,0) = 0$ and the variables ρ and ω are measured from their respective operation values to be stabilized, which serve as reference points. Upon the introducing the new variables $x_1 = x = \frac{\rho}{\beta_1}$, $x_2 = y = \frac{\omega}{\beta_2}$ the latter equations take the form

$$\dot{x} = G(x,y) - H(x), \qquad (I3)$$

$$\dot{y} = M(y,\delta) - N(y,x). \qquad (I4)$$

It is supposed thet the segments $/-\beta_1, \beta_1/$ and $/-\beta_2, \beta_2/$ belong to the working region of the plant.

For a large set of air impellers

$G \equiv \mathcal{L}(x)(1+\mu y)$, $\mu = const > 0$.

The graphs of the functions $\mathcal{L}(x)$ and $H(x)$ are plotted in the fig.3. $\mathcal{L} > 0$ and $H > 0$ for $|x| \leq 1$.

The conditions I and II of this section are fulfilled in virtue of properties of the functions Z_1 and Z_2. We shall seek the region of values of the parameter μ, corresponding to the condition III. This condition requires the property $|y(x)| \leq 1$ in the segment $|x| \leq 1$. It follows from the equation (I3) that

$$y = Y(x) = \frac{[F(x) + H(x) - \mathcal{L}(x)]}{\mu\,\mathcal{L}(x)}, \quad Y(0) = 0. \qquad (I5)$$

According to the inequality $-1 \leq y \leq 1$

$$\underline{F} \equiv (1-\mu)\mathcal{L} - H \leq -F_0 < F < F_0 \leq (1+\mu)\mathcal{L} - H \equiv \bar{F}, \qquad (I6)$$

where $F_0 = \max |F(x)|$ in the segment I-I, IV. Obviously $\bar{F} - \underline{F} = 2\mu\mathcal{L} > 0$. The function $\underline{F}$ ought to be negative, the function $\bar{F}$ positive. Therefore

$$\mu \geq \mu_1 = \max \frac{|\mathcal{L} - H|}{\mathcal{L}}$$

in the segment $|x| \leq 1$. To hold the inequalities (I6) it is sufficient that

$\max \underline{F} \leq -F_0 < F < F_0 \leq \min \bar{F}$; $|x| \leq 1$.

It is clear that (fig.3)

$\max \underline{F} \leq \underline{f} = \mathcal{L}_2 - \mu\mathcal{L}_1 - H_1$, $\min \bar{F} \geq \bar{f} = (1+\mu)\mathcal{L}_1 - H_2$

Composing the difference $\Delta = \bar{f} - \underline{f}$ we shall find that $\Delta > 0$ when

$$\mu > \mu_2 = \frac{(\mathcal{L}_2 - \mathcal{L}_1 + H_2 - H_1)}{2\mathcal{L}_1} \qquad (\mu_2 > 0).$$

Let us choose the value F_0 as follows :

$$F_0 = \min(\bar{f}, |\underline{f}|), \quad \mu > \mu_2.$$

The function

$$F(x) = -\frac{3\sqrt{3}}{2} F_0 x (1 - x^2)$$

satisfies to the condition III a),b),c) and to the condition III d) with respect to the function $Y(x)$ (I5).

Thus the limitation $|x(\tau)| \leq 1$ is ensured and the relation (I5) becomes a first integral of the system.

Through an examination of the equation (I4) it is possible to find likewise the sufficient conditions to answer to the requirement IV. Then the correction characteris-

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tic $\delta = \delta(x)$ will be determined. Hence it follows, that for the plants to be stabilized of the considered type which posess the characteristics that are qualitatively plotted in the fig.3 and for which is valid the property

$$(\mu_1 > 0),\ \mu_2 > 0,\ \mu > \max(\mu_1, \mu_2)$$

it is possible to achieve the required limitation of oscillation by means of a single correcting action ; the latter is applied in the chain of the control of the plant angular velocity.

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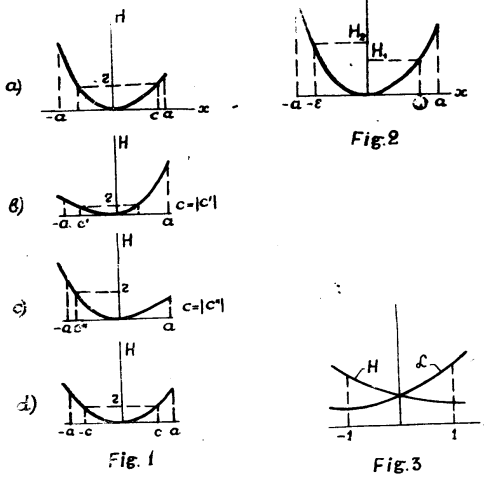

Fig. 1

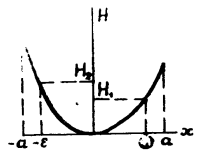

Fig. 2

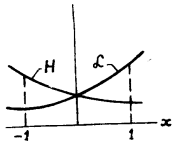

Fig. 3

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W. W. LUGOWSKY — Leningrad, USSR

В. В. ЛУГОВСКИЙ

ПРИЛОЖЕНИЕ МЕТОДОВ ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ К ИССЛЕДОВАНИЮ КАЧКИ СУДОВ НА ВОЛНЕНИИ

THE APPLICATION OF THE METHODS OF THE NON-LINEAR OSCILLATION THEORY TO THE ANALYSIS OF SHIP OSCILLATIONS IN A SEAWAY.

Lugovsky V.V., (Leningrad, USSR.)

The linear theory of ship oscillations widely used in ship research and well-founded with respect to hydrodynamics [1]-[2] cannot be employed for the analysis of ship motion conditions connected with the safety of floating, for the basic linear theory assumption about small amplitudes of oscillations is found in full contradiction with the physical meaning of the problem. The transition to a pure non-linear theory is, however, connected with some important difficulties, and modern methods of analysis do not always hold in overcoming these difficulties. The existence of velocity potential being assumed, the general non-linear problem of ship oscillations under the influence of the infinite perturbate fluid in wave motion may be formulated as follows: to find the velocity potential of the motion of a fluid $\Phi(\xi, \eta, \zeta, t)$, satisfying Laplace's equation

$$\Delta \Phi = 0 \qquad (1)$$

the boundary condition on the free fluid surface:

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial \zeta} + 2\left(\frac{\partial \Phi}{\partial \xi} \frac{\partial^2 \Phi}{\partial t \partial \xi} + \frac{\partial \Phi}{\partial \eta} \frac{\partial^2 \Phi}{\partial t \partial \eta} + \frac{\partial \Phi}{\partial \zeta} \frac{\partial^2 \Phi}{\partial t \partial \zeta} \right) +$$

$$+ \left(\frac{\partial \Phi}{\partial \xi} \right)^2 \frac{\partial^2 \Phi}{\partial \xi^2} + \left(\frac{\partial \Phi}{\partial \eta} \right)^2 \frac{\partial^2 \Phi}{\partial \eta^2} + \left(\frac{\partial \Phi}{\partial \zeta} \right)^2 \frac{\partial^2 \Phi}{\partial \zeta^2} +$$

$$+ 2\left(\frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi}{\partial \eta} \frac{\partial^2 \Phi}{\partial \xi \partial \eta} + \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi}{\partial \zeta} \frac{\partial^2 \Phi}{\partial \xi \partial \zeta} + \frac{\partial \Phi}{\partial \eta} \frac{\partial \Phi}{\partial \zeta} \frac{\partial^2 \Phi}{\partial \eta \partial \zeta} \right) = 0 \qquad (2)$$

the boundary condition on the surface of an oscillating ship

$$v_n(t) - \frac{\partial \Phi}{\partial n} = 0 \qquad (3)$$

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the infinite conditions, and the initial conditions with t = o; to determine fluid pressure, making use of Lagrange-Cauchy's integral; to compose differential equations of ship motions and to find their solution.

A system of coordinates $O\xi\zeta\zeta$, fixed in space, the $O\xi\zeta$ - plane being assumed to coinoide with the undisturbed fluid surface, the $O\zeta$ -axis being directed vertically upward, and the outer normal to the hull's surface n are introduced here. An exceptional difficulty of such a problem is quite evident: the solution is to be found in a sphere the boundaries of which vary in time. The change is dependent on ship oscillations, which, in their turn, are determined by the solution of the boundary problem; all types of ship oscillations interfere with each other, besides, all of them depend on the advance velocity of ship motion; the boundary problem of the wave theory as well as the differential equations of ship motion are essentially non-linear.

The immediate solution of the problem seems to be impossible at present.

Perhaps the method of small parameter, allowing the transition from the non-linear problem to the sequence of linear ones, may be considered the only effective method for its solution.

Evidently, F.John [3] was the first to make use of this method for the investigation of body oscillations on the free fluid surface at zero advance velocity. F.John strictly linearised the problem by systematically developing in a formal series the expressions for ship's deviations from the state of equilibrium, the velocity potential and the elevation of the free fluid surface; then he limited himself by the investigation of the linear approximation.

This method was further developed by A.S. Peters and J.J. Stoker [4] , who constructed the general linearized theory of ship motion in a seaway. To connect the theory of ship oscillations with the classic theory of wave

resistance to the ahead motion of a ship, Peters and Stoker make use of a thickness-length ratio of the ship, called a "slenderness" parameter β , as a small parameter of the problem. They believe that amplitudes of fluid oscillations as well as of all types of ship oscillations are small of the same order as the slenderness parameter β . Systematic series developments of motion equations and boundary conditions with respect to β give a system of integral equations solved by Peters and Stoker for a ship of Michell's type only in a linear approximation. It seems impossible to construct the effective theory of ship oscillations with a finite amplitude, formally keeping the terms of higher order with respect to β in the developments. At any rate the authors make no attempt to do it, they only note exceptional difficulties of the transition to the non-linear theory.

The problem is essentially simplified if the small parameter is not introduced formally but in compliance with physical properties of various types of ship oscillations. A non-linear theory is necessary for the analysis of ship oscillations with great amplitudes arising, as usual, in the conditions of resonance. But natural frequencies of rolling are lower than frequencies of pitching and heaving, and resonant rolling is accompanied, as a rule, by pitching, swaying and yawing, amplitudes of which being small. The construction of the non-linear theory of rolling therefore, may be limited only by taking into account the higher degrees of rolling, assuming the terms proportional to the degrees, beyond the first one, of all other types of oscillations, neglectingly small. On the contrary, the non-linear terms, characteristic of rolling, can be left out of consideration in constructing the non-linear theory of pitching.

Such an approach seems to be rather effective in a very important, from the point of view of stability in a seaway, problem of rolling with great amplitudes of a ship in beam seas and at small advance speed. In this case the

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developments of translatory oscillations of the center of gravity (Si, i = 1, 2, 3), angles of heel θ , of the trim ψ and yawing may be represented in the form:

$$\left.\begin{aligned} S_i &= \varepsilon_1 S_{i1} + \varepsilon_1^2 S_{i2} + \dots \quad (i = 1, 2, 3), \\ \Psi &= \varepsilon_1 \Psi_1 + \varepsilon_1^2 \Psi_2 + \dots, \\ \Psi &= \varepsilon_1 \Psi_1 + \varepsilon_1^2 \Psi_2 + \dots, \\ \Theta &= \varepsilon_2 \Theta_1 + \varepsilon_2^2 \Theta_2 + \dots \end{aligned}\right\} \quad (4)$$

where ε_1 and ε_2 - small parameters of the problem, characteristic of the slenderness order of various types of oscillations. Thus, from the one-parameter developments, the necessity of which was doubted by N.N. Moiseev [5] we proceed to the two-parameter developments, allowing to take into special account physical properties of the subject. Remembering that resonant rolling amplitudes reach 30 - 40° , whereas pitching and yawing amplitudes do not exceed 3 - 5°, and relative oscillations of the center of gravity $-\frac{1}{30} - \frac{1}{40}$, the quantities proportional to ε_2^3 may be said to possess the slenderness order ε_1 Then the developments to within ε_1 will take the form:

$$\left.\begin{aligned} S_i &= \varepsilon_1 S_{i1} \quad , \quad (i = 1, 2, 3), \\ \Psi &= \varepsilon_1 \Psi_1 \quad , \\ \Psi &= \varepsilon_1 \Psi_1 \quad , \end{aligned}\right\} \quad (5)$$

$$\Theta = \varepsilon_2 \Theta_1 + \varepsilon_2^2 \Theta_2 + \varepsilon_2^3 \Theta_3 = \varepsilon_1 \Theta_1^I \quad (6)$$

Attention should also be paid to the results of the approximate solution of the non-linear problem of wave induced on the surface of a heavy incompressible fluid by the pulsating source, the intensity of which varies in time in accordance with the biharmonic law [7] . Non linearity of a boundary condition on the free surface at frequencies, typical of the zone of rolling resonance, do not practically influence hydrodynamic forces, wave resistance of the source, in particular, even at great amplitudes of oscillations. The said above gives the possibility of linearizing, differently than in [7], [8], the boundary problem of hydrodynamic rolling forces assuming velocity potential $\Phi(\xi, \eta, \zeta, t)$, the free surface elevation $\zeta(\xi, \eta, t)$ and pressure p to be the terms of the first order:

$$\Phi = \varepsilon_1 \Phi_1 \quad , \quad \zeta = \varepsilon_1 \zeta_1 \quad , \quad p = \varepsilon_1 p_1 \quad (7)$$

but taking into account, with the aid of [6], the change of form and quantity of the wetted area of an oscillating ship. Substituting (5) - (7) into (1) - (3) , choosing the terms of the first order with respect to ε_1 , making use of Lagrange-Cauchy's integral and of the equation for hydrodynamic moment relative to the longitudinal axis, we receive the possibility to compose the differential equation of non-linear rolling with taking into account small pitching and swaying oscillations and yawing.

2. For the analysis of non-linear rolling characteristics an equation of motion in a somewhat simplified form should be employed; assuming that pitching and yawing do not much influence the resonant rolling of a ship in beam seas, and swaying of the center of gravity varies in accordance with the harmonic law, we have:

$$\ddot{\theta} + \bar{F}_1(\dot{\theta}) + \omega^2 \left[\bar{F}_0(\theta) + \bar{z}_2 \theta \right] = \varkappa_0 \left[\varkappa_1(\theta) \sin \sigma t + \varkappa_2(\theta) \cos \sigma t \right] \quad (8),$$

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where $\tilde{\mathcal{F}}_1(\dot{\theta})$ represents the non-linear function characteristic of roll damping, $\mathcal{F}_0(\dot{\theta})$ – the non-linear odd function, determining the shape of static stability diagramm,

ω – the undapmed natural frequency of rolling,

$\tilde{z}_2$ – heaving characteristic,

α_0 – the maximum angle of the wave slope,

σ – wave frequencies

$\chi_1(\theta)$ and $\chi_2(\theta)$ – non-linear functions, which may be written as follows:

$$\chi_1(\theta) = \kappa_1 + \kappa_2\theta^2$$

$$\chi_2(\theta) = \kappa_3 + \kappa_4\theta + \kappa_5\theta^2 + \kappa_6\theta^3$$

Coefficients Ki (i = 1,2,...6) take into account the influence of a changing hull's shape in a seaway, of a thickness – wave-length ratio, swaying of the center of gravity etc, on the hydrodynamic moment.

It is of interest that in earlier investigations the equations for rolling of a finite amplitude, constructed on the ground of intuitive considerations (for instance, [7] , [8]) are easily receivedvas particular cases.

The non-linear differential equation (8) is easily solved with the aid of Bogolubov's asymptotic method [9]. Such an approach has a certain physical basis as it is experimentally known that even at large amplitudes rolling shape is nearly harmonic. In order to consider possible resonances the solution is to be written as follows:

$$\theta = a\sin\left(\frac{p}{q}\sigma t + \delta\right) + \varepsilon U_1(a, \sigma t, \delta) + \qquad (9),$$

determining the amplitude a and the initial phase δ with the aid of a system:

$$\left.\begin{aligned}\frac{da}{dt} &= \varepsilon A_1(a,\delta) + \varepsilon^2 A_2(a,\delta) + \\ \frac{d\delta}{dt} &= \omega - \frac{p}{q}\sigma + \varepsilon B_1(a,\delta) + \varepsilon^2 B_2(a,\delta) +\end{aligned}\right\} \qquad (10)$$

where p and q – integer, prime numbers, and ε – a small parameter of the problem. The application of the expressions of the monograph [9] for the 1 -st and 2-nd approximations leads to the following, rather interesting conclusions. The solution for the 1-st approximation, to within ε , gives possibility of revealing two resonant regimes of rolling: the basic –

$$\theta = a\sin(\sigma t + \delta) \qquad (11)$$

and the main subharmonic:

$$\theta = a\sin\left(\frac{\sigma t}{2} + \delta\right) \qquad (12)$$

Equations for establishing of oscillations in basic resonance regime may be written as [10] :

$$\left.\begin{aligned}\frac{da}{dt} &= -f_1(a) + \frac{\left(\kappa_1 + \frac{3}{4}\kappa_2 a^2\right)\sin\delta}{\omega+\sigma} - \frac{\left(\kappa_3 + \frac{3}{4}\kappa_5 a^2\right)\cos\delta}{\omega+\sigma}, \\ \frac{d\delta}{dt} &= \omega - \sigma + \frac{\omega}{a}\left[f_0(a) - \frac{a}{2}\right] + \frac{\left(\kappa_1 + \frac{3}{4}\kappa_2 a^2\right)\cos\delta}{a(\omega+\sigma)} + \frac{\left(\kappa_3 + \frac{3}{4}\kappa_5 a^2\right)\sin\delta}{a(\omega+\sigma)}\end{aligned}\right\} \qquad (13)$$

and a stationary ratio of an oscillation amplitude to the wave frequency as:

$$\sigma^2 = \frac{\omega_1^2(a) - 2\frac{f_1^2(a)}{a^2} - \alpha_0^2\frac{\mathcal{P}_1\mathcal{P}_2}{a^2}}{1 - \frac{\alpha_0^2}{a^2}\left(\mathcal{P}_2^2 + \mathcal{P}_3^2\right)} \pm$$

$$\pm \frac{\sqrt{\left[\omega_1^2(a) - 2\frac{f_1^2(a)}{a^2} - \frac{\alpha_0^2}{a^2}\mathcal{P}_1\mathcal{P}_2\right]^2 - \left[\omega_1^2(a) - \frac{\alpha_0^2}{a^2}\mathcal{P}_1^2\right]\left[1 - \frac{\alpha_0^2}{a^2}\left(\mathcal{P}_2^2 + \mathcal{P}_3^2\right)\right]}}{1 - \frac{\alpha_0^2}{a^2}\left(\mathcal{P}_2^2 + \mathcal{P}_3^2\right)} \qquad (14)$$

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Here, for the same of abbreviation, the functions, dependent on the coefficients ki and on the square rolling only, are denoted through $\mathcal{P}_j$, it is easy to determine these functions preliminarily,

and $\omega_1(a)$ - the undamped natural frequency of rolling with a finite amplitude, defermined in the first approximation;

$$\left.\begin{aligned} f_0(a) &= \frac{1}{2\pi}\int_0^{2\pi} \overline{F_0}(a\sin\Psi)\sin\Psi\, d\Psi = \frac{\omega_1^2(a)}{\omega^2}\frac{a}{2} \\ f_1(a) &= \frac{1}{2\pi\omega}\int_0^{2\pi} \overline{F_1}(a\omega\cos\Psi)\cos\Psi\, d\Psi \end{aligned}\right\} \quad (15)$$

Experiments show that at great rolling amplitudes the resistance law is nearly square: $\overline{F_1}(\dot{\theta}) = w\,\dot{\theta}\,|\dot{\theta}|$, where w — a relative damping coefficient;

in this case $$f_1(a) = \frac{4\omega}{3\pi} w\, a^2 \quad (16)$$

Formulae, characteristic of the main subharmonic resonance, may be represented in the same way. The stationary amplitude, in partioular, is connected with frequency by the equation:

$$\left(\frac{\sigma}{2}\right)^2 = \omega_1^2(a) + 2\left(\frac{\alpha_2}{4}\right)^2\left(\frac{2_{2m}}{2\alpha_0} - \mathcal{P}_4\right)^2 - 2\frac{f_1^2(a)}{a^2}\mathcal{P}_5^2 \pm$$

$$\pm\sqrt{\left[\omega_1^2(a) + 2\left(\frac{\alpha_2}{4}\right)^2\left(\frac{2_{2m}}{2\alpha_0} - \mathcal{P}_4\right)^2 - 2\frac{f_1^2(a)}{a^2}\mathcal{P}_5^2\right]^2 - \omega_1^4(a)} \quad (17)$$

Resonance curves, drawn in compliance with (14) - (17), illustrate well-known properties of the basic and main subharmonic resonances. Resonant curves deviate from the corresponding linear ones, and parts of these curves which do not fulfil the stability conditions at small disturbances:

a) in case of the basic resonance

$$\omega(a) > \sigma \quad , \quad \frac{d\sigma}{da} > 0 \quad (18)$$

b) in case of the main subharmonic resonance:

$$2\omega(a) > \sigma \quad , \quad \frac{d\sigma}{da} > 0 \quad (19)$$

correspond to the unsteady, practically non-existing motion conditions.

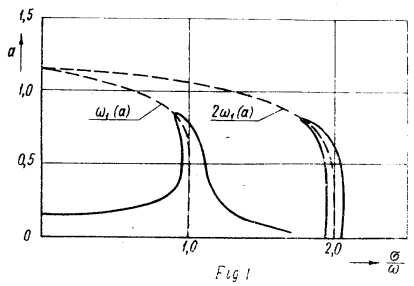

Fig 1

The physical nature of the main subharmonic resonance is very interesting. The latter arises owing to the periodic change of restoring moment in time and may, therefore, be called the parameter resonance. This change is conditioned, first, by the orbital ship motion like a motion of a wave particle, and, second, by heaving. If transverse ship dimensions are commensurable with wave length or waves are irregular, the second factor is to be decisive. Model tests and ship navigation confirm the existence of the parameter rolling resonance.

3. Resonance curves of the basic and main subharmonic resonances, drawn in accordance with the formulae of the first approximation, do not much differ from the corresponding curves of the second approximation, and it seems inexpedient to introduce such a specification into practical

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calculations. The transition to the theory of the second approximation gives, however, a qualitatively new result: the possibility of existing of the third resonant regime of rolling, which may be called a subharmonic resonance of the third order.

$$\vartheta = a \sin\left(\frac{\sigma t}{3} + \delta\right) + \frac{1}{\pi} \sum_{m=2} \frac{1}{\omega_1^2 (m^2-1)} \left\{ \cos m\left(\frac{\sigma t}{3} + \delta\right) \int_0^{2\pi} f_1(a\omega\cos\Psi)\cos m\Psi\, d\Psi + \sin m\left(\frac{\sigma t}{3} + \delta\right) \int_0^{2\pi} \omega^2 f_0(a\sin\Psi)\sin m\Psi\, d\Psi \right\} + \frac{\alpha_0 \mathcal{P}_6 \sin 6t}{2\omega(\omega+\sigma)} \qquad (20)$$

The shape of oscillations in this regime is of complicated, polyharmonic character and contains overtones of natural frequency, dependent on non-linear properties of a damping and restoring moments, as well as a harmonic component with wave frequency. It is clearly seen from an experimental rolling curve, shown in Fig.2: against the background of the main harmonic component, the frequency of which is a one third of an exciting moment frequency, a harmonic component of greater frequency is clearly seen.

Fig.2

Subharmonic resonance amplitudes of the third order as functions of the wave frequency may be calculated with the aid of the following formula:

$$\left(\frac{\sigma}{3}\right)^2 = \frac{\omega_2^2(a) - \omega_1^2 \delta_1^2(a) - \frac{81}{64}\frac{\alpha_0^2}{a^2}\mathcal{P}_7\mathcal{P}_9}{1 - \frac{81}{64}\frac{\alpha_0^2}{a^2}\left(\mathcal{P}_7^2 + \mathcal{P}_8^2\right)} \pm \frac{\sqrt{\left[\omega_2^2(a) - \omega_1^2\delta_1^2(a) - \frac{81}{64}\frac{\alpha_0^2}{a^2}\mathcal{P}_7\mathcal{P}_9\right]^2 - \left[\omega_2^4(a) - \frac{81}{64}\frac{\alpha_0^2}{a^2}\mathcal{P}_9^2\right]\left[1 - \frac{81}{64}\frac{\alpha_0^2}{a^2}\left(\mathcal{P}_7^2 + \mathcal{P}_8^2\right)\right]}}{1 - \frac{81}{64}\frac{\alpha_0^2}{a^2}\left(\mathcal{P}_7^2 + \mathcal{P}_8^2\right)} \qquad (21)$$

Here $\omega_2(a)$ - the undamped natural frequency of rolling with a finite amplitude in the second approximation:

$$\omega_2^2(a) = \omega_1^2(a) + \frac{\omega^2}{\alpha}\sum_{m=2}^{N}\frac{f_{om}(a)}{m^2-1}\frac{\partial f_{om}(a)}{\partial a} \qquad (22),$$

$$\left.\begin{aligned} \delta_1(a) &= \frac{4}{3\pi}\left(3 - \frac{f_0(a)}{a} - \frac{\partial f_0(a)}{\partial a}\right) \\ f_{om}(a) &= \frac{\omega^2}{\pi}\int_0^{2\pi} f_0(a\sin\Psi)\sin m\Psi\, d\Psi \end{aligned}\right\} \qquad (23)$$

In Fig. 3 resonance curves of rolling for a ship with a small metacentric height and a stability diagramm of S-like shape have been given. The resonance of the third order is very likely to appear in ships of such a type.

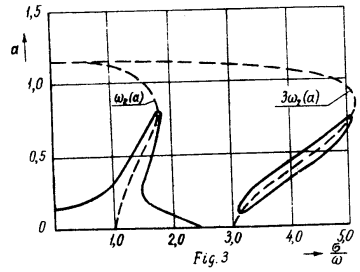

Fig. 3

The subharmonic resonance of the third order is wholly conditioned by non-linearity of a restoring moment, that is why the increase of non-linearity of a stability diagramm is attended by the expansion of the zone of wave frequencies, inducing this regime, and the probability of its rise is increased.

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It should be noted, that resonance curves are limited by rolling damping not only from above, as at the basic and main resonances, but also from below, becoming closed. Consequently, rolling in the resonance regime of the third order cannot arise at smooth, very slow changing of frequency of running waves; here an external push (blow) of a given force is necessary. A gust or single irregularity of waves in motion, a steeper wave, for instance, may serve as such a push. As a rule, resonance curves of the third order are disposed in the zone of wave frequencies three times higher than the ship's natural frequencies. Resonant rolling, therefore, can arise under the influence of rather short waves, considerably removed from the basic resonance zone and characteristic of inner seas.

4. It is important to know, from the viewpoint of safety of a ships's navigation, the possible maximum amplitude of rolling in real conditions of floating. Under the influence of regular waves upon a ship the maximum amplitude ϑ_m is near to the ordinate of the intersection point of the curve $a\left(\frac{g}{\omega^2}\right)$ with the frequency curve of natural oscillations $\omega(a)$ (Fig. 1,3,4). Practically such regimes of rolling, however, rarely exist; there are, usually, external disturbances, the influence of which breaks a stationary regime and may considerably change the quantitive characteristics of rolling.

The inequalities (18) show that conditions, corresponding to the parts BC of resonance curves are unsteady at rather small disturbances and, hence, are impracticable in reality. Conditions, corresponding to parts AB and CD are steady "in small"; the phase plane method may be employed for estimating the degree of their steadiness, i.e. for defining the value of disturbances, at which the steadiness is lost.

For the dynamic system, considered, the phase plane is conveniently built up in polar coordinates a (an amplitude) and δ (an initial phase) [12]. In order to define phase trajectories it is necessary to solve the equations (13)

In finite form, however, these equations cannot be integrated, as their right-hand sides are non-linear and depend on a as well as upon δ; that is why their numerical integration ought to be resorted to. Introducing into (13) the expression for $f_0(a)$ and $f_1(a)$, in compliance with the formulae (15) - (16), and assuming time to be a parameter, we may write equation for establishing of rolling as:

$$\frac{da}{d\delta} = \frac{1-\frac{\omega^2(a)}{\delta^2}-\frac{\alpha_0}{a}\left[\left(\kappa_1+\frac{3}{4}\kappa_2 a^2\right)\cos\delta+\left(\kappa_3+\frac{3}{4}\kappa_5 a^2\right)\sin\delta\right]}{\frac{d}{3\pi}w a^2-\alpha_0\left[\left(\kappa_1+\frac{3}{4}\kappa_2 a^2\right)\sin\delta-\left(\kappa_3+\frac{3}{4}\kappa_5 a^2\right)\cos\delta\right]} \qquad (24)$$

Curves $\delta(a)$, being the solutions of the equation (24) at various values of a, give the totality of phase trajectories, corresponding to various initial values of rolling amplitudes.

Of special interest are phase trajectories going through points of parts BC as they may be considered, to some extent, boundaries, "separatresses", separating "attraction" spheres of nonstationary rolling conditions of the upper AB and lower CD by steady parts. The totality of separatresses, corresponding to exciting force frequencies from σ_1 to σ_2 (Fig. 4), allows to define the degree of steadiness of rolling conditions AB and CD.

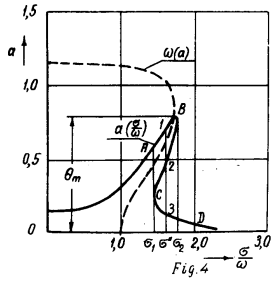

Fig. 4

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Phase planes for these curves are drawn in Fig. 5, a, b, c, where separatresses and spheres of "attraction" by AB and CD are shown. Fig. 5a corresponds to the frequency $\sigma' = \sigma_1 + \frac{1}{4}\overline{\sigma_1\sigma_2}$ Fig. 5b - to the frequency $\sigma'' = \sigma_1 + \frac{1}{2}\overline{\sigma_1\sigma_2}$, Fig. 5c - to the frequency $\sigma''' = \sigma_1 + \frac{3}{4}\overline{\sigma_1\sigma_2}$. It is clear that frequency displacement from σ_1 to σ_2 leads to the decrease of the sphere of "attraction" by the conditions AB and to the increase of the sphere of "attraction" by the conditions CD.

The beginning of the phase plane coordinates is located in the sphere of "attraction" by CD, therefore, the influence of wave motion at frequency $\left(\sigma_1 + \frac{1}{4}\overline{\sigma_1\sigma_2}\right) < \sigma < \sigma_2$ on a ship disturbs the state of equilibrium, and after some time, necessary for the establishing of oscillations, rolling becomes stationary with an amplitude and phase, corresponding to lower steady parts CD of resonance curves.

An additional impulse is necessary for setting a ship in rolling condition AB, i.e. a gust, for instance, or an impact of waves. The value of such an impulse depends, in the main, upon the difference $\sigma_1 - \sigma$, where σ is the frequency of swell, running on a ship. At frequencies $\sigma > \sigma_1 + \frac{1}{4}\overline{\sigma_1\sigma_2}$ the ship may pass from the lower steady condition CD to the upper AB at the only condition, that the rolling amplitudes instantly magnify in 2 - 3 times. If the phase of oscillations, will be such, that the point of the phase plane, showing non-stationary condition of the ship, remain in the "attraction" sphere of lower steady regime CD, under the impulse action, even such considerable increase of rolling amplitudes will not lead the ship to the upper stationary condition AB. For such a transition, therefore, the external disturbance, acting upon the ship in a given moment, is necessary.

In real conditions of floating this moment cannot be predicted or calculated. It means that even knowing the

exact value of the impulse disturbance we may speak only about the probability of passing to the upper stationary condition AB. This probability is increased with approaching of wave frequency to σ_1, and decreased with approaching to σ_2. The more is the value of the external impulse, acting upon a ship, the larger is the probability for given wave parameters (length, height). It is clear from Fig. 5 that for the replacement of the lower stationary condition CD by the upper AB at the frequency $\sigma > \sigma_1 + \frac{1}{4}\overline{\sigma_1\sigma_2}$ the value of the external disturbance must be rather considerable, and for $\sigma > \sigma_1 + \frac{1}{2}\overline{\sigma_1\sigma_2}$ the probability of replacing regimes still will be very small. On the contrary, at such frequencies a small external disturbance is enough to make the probability of passing from the upper AB to the lower CD very large. This conclusion may be verified by the numerical integration [13] of the rolling differential equation (8).

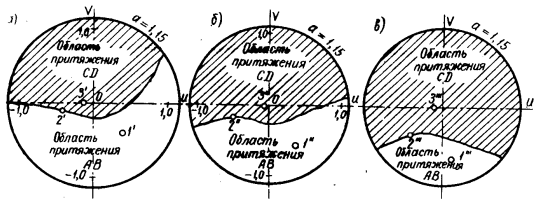

Fig. 5

Consequently, in real conditions of floating maximum possible stationary rolling amplitude is, as a rule, considerably smaller than the resonance amplitude ϑ_m,

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under the influence of swell and external disturbances, such as gusts and wave irregularities. The value of this decrease depends on the shape of the static stability diagramm and may be defined by the phase plane method in every specific case.

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ИССЛЕДОВАНИЕ РЕШЕНИЙ НЕЛИНЕЙНЫХ СИСТЕМ, БЛИЗКИХ К ИНТЕГРИРУЮЩИМСЯ, С ПОМОЩЬЮ МЕТОДА ИНТЕГРАЛЬНЫХ МНОГООБРАЗИЙ*

1. By an integral manifold for a system of differential equations we mean a hypersurface with the property that if some value of a solution of the system belongs to the manifold, the solution will wholly belong to that manifold.

The method of integral manifolds consists in that for a system of nonlinear differential equations the existence of an integral surface is proved, which is completely covered by the integral curves of some family of particular solutions. Some properties of this surface, and the ensuing properties of the family of solutions, are established.

The importance of the method of integral manifolds lies in the fact that it enables to reduce the investigation of the individual solutions of the above system of nonlinear differential equations in an n phase space to the investigation on a manifold whose dimensionality is less than that of the starting phase space. This is especially important if the manifold is of dimension 1 or 2.

The significance of the method of integral manifolds consists also in the following. In the investigation of oscillatory systems with many degrees of freedom, the single-frequency method is

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known to be widely applied, involving the construction of an approximate family of solutions, which depends on two arbitrary constants and corresponds to the single-frequency oscillatory process.

From the physical point of view, it is very important that this approximate family of solutions possess the property of attracting any solutions, i.e. that the single-frequency process under consideration be stable, which can be ascertained by using the method of integral manifolds.

For the first time, the method of integral manifolds was developed by N.N.Bogoliubov in 1945 in his well known monograph "On Some Statistical Methods in Mathematical Physics", as applied to systems of nonlinear differential equations containing a small parameter, in the standard form

$$\frac{dx}{dt} = \varepsilon X(t, x) \qquad (1)$$

where x, X - are n- vectors.

The method of integral manifolds has been furthner developed in the works of prof.Yu.A.Mitropolsky, et al., and a number of American scientists.

We introduce the following notations:

E_n - Eucledian real n-space;
R - infinite interval of variation t;
Ω_{ε_0} - real range of variation t;
D - some bounded range of variation of x in E_n;
U_{δ_0} - δ_0-neighbourhood of point $x_0 = 0$;

U_ρ - range of variation of h, ξ, τ^*.

The function $F(t, x, \varepsilon)$ belongs to class $O^k(x, \varepsilon)$, $(\overline{O}^k(x, \varepsilon)$ if it is periodic in t of period 2π, and bounded and uniformly continuous in the range of variation of its arguments together with its partial derivatives of the first order /any order/ with respect to x and ε.

The function $F(t, x, y, \varepsilon)$ belongs to class $M(\varepsilon, \rho^2)$ if

$$|F(t, x, 0, \varepsilon)| \leq M(\varepsilon, \rho^2),$$

where $M(\varepsilon, \rho^2) \to 0$ as $\varepsilon \to 0$, $\rho^2 \to 0$.

For the function $F(t, x, \varepsilon)$ in the range of variation of its arguments $F(t, x, \varepsilon) \in Lip(x, \lambda(\varepsilon, \rho^2))$ will hold, if F is a continuous function of its arguments and is a Lipschitzian with respect to x with Lipschitz constant $\lambda(\varepsilon, \rho^2)$ where λ is continuous with respect to ε, ρ^2 and tends to zero as $\varepsilon \to 0$, $\rho^2 \to 0$.

2. We shall consider the following system of nonlinear differential equations which is close to the exactly integrable system

$$\frac{dx}{dt} = X(x) + \varepsilon X^*(t, x, \varepsilon) \qquad (2)$$

where ε is a small positive parameter, X, X^* are functions whose values are the points of an Euclidean n-space E_n, defined for all real t varying in region R, $\varepsilon \in \Omega_{\varepsilon_0}$ and for $x \in D$. $\varepsilon X^*(t, x, \varepsilon)$ characterize the small perturbations acting permanently on the system

$$\frac{dx}{dt} = X(x). \qquad (3)$$

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The following theorem holds:

Theorem I

Let the following conditions be satisfied for system (2):

1. Equations (3) have the zero solution

$$x_0 = 0, \quad X(0) = 0.$$

2. The characteristic equation

$$|\gamma_n z - X_x'(0)| = 0 \tag{4}$$

has two purely imaginary roots $z_1 = i\omega$, $z_2 = -i\omega$, while all the other roots have negative real parts.

3. The functions $X(x) + \varepsilon X^*(t, x, \varepsilon)$ belong to class $C^r(x, \varepsilon)$ in region $R \times U_{\sigma_0} \times \Omega_{\varepsilon_0}$.

Then positive numbers σ_1, ε^*, σ_1, ρ_1 can be given such that for any positive $\varepsilon < \varepsilon^*$ the following statements will hold:

1. The system of equations (2) has a unique two-dimensional integral manifold S lying in the neighbourhood of $U_{\sigma_1}(0)$ $(\sigma_1 < \sigma_0)$.

2. On manifold S the system (2) is equivalent to two equations

$$\frac{d\xi}{dt} = i\omega\xi + \varepsilon P_1(t, \xi, \xi^*, \varepsilon),$$
$$\frac{d\xi^*}{dt} = -i\omega\xi^* + \varepsilon Q_1(t, \xi, \xi^*, \varepsilon), \tag{5}$$

where P_1, Q_1 possess properties similar to those of $X + \varepsilon X^*$.

3. Manifold S is parametrically representable as

$$x^* = A\xi + B\xi^* + Df(t, \xi, \xi^*, \varepsilon) = \Phi(t, \xi, \xi^*, \varepsilon) \tag{6}$$

where $\Phi(t, \xi, \xi^*, \varepsilon)$ is defined in region $R \times U_{\rho_1} \times \Omega_{\varepsilon^*}$, $\varepsilon^* < \varepsilon_0$ and belongs to class $C^r(t, \xi, \xi^*, \varepsilon)$ x/.

x/ Note that if function $X + \varepsilon X^*$ were assumed

4. Any solutions of system (2) which do not belong to the manifold and whose initial values lie in the vicinity of the integral manifold, are attracted to it according to the law

$$|x(t) - \Phi(t, \xi_t, \xi_t^*, \varepsilon)| < C(\varepsilon, \rho^*) e^{-\beta(t - t_0)} \tag{7}$$

/as long as ξ_t, $\xi_t^* \in U_{\rho_1}$/.

Note 1. If in equation (6) ξ, ξ^* are considered not as constant parameters, but as functions of time determined from system (5), then the expression

$$x^*(t) = A\xi_t + B\xi_t^* + Df(t, \xi_t, \xi_t^*, \varepsilon) = \Phi(t, \xi_t, \xi_t^*, \varepsilon) \tag{8}$$

will represent a two-parametric family of solutions covering the same integral surface S.

Herein the parameters are arbitrary constants appearing in the expression for ξ, ξ^* as determined from system (5) as functions of time. The concrete integral curves of the family correspond to the particular solutions of system (5).

Note 2. From statement 4 establishing the tendency of the trajectories of any solutions towards the manifold, does not follow their tendency to concrete integral curves of the manifold. This question requires additional investigation.

The procedure of proof of the theorem is as follows.

to possess bounded and uniformly continuous derivatives of any order with respect to x, ε, then the manifold function $\Phi(t, \xi, \xi^*, \varepsilon)$ would also possess bounded and uniformly continuous derivatives of any order with respect to ξ, ξ^*, ε.

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First of all, by introducing a change of variables

$$x = A\xi + B\xi^* + Dh \qquad (9)$$

where A, B are constant vectors and D is a constant matrix, the original system of equations is reduced to a form more convenient for consideration

$$\begin{aligned} \frac{d\xi}{dt} &= i\omega\xi + P(t, \xi, \xi^*, h, \varepsilon), \\ \frac{d\xi^*}{dt} &= -i\omega\xi^* + Q(t, \xi, \xi^*, h, \varepsilon), \\ \frac{dh}{dt} &= Hh + R(t, \xi, \xi^*, h, \varepsilon). \end{aligned} \qquad (10)$$

where P, Q, R possess properties similar to those of functions $X + \varepsilon X^*$.

At this point the following should be taken into consideration.

In transforming from (2) to (10), the system is written down in the form

$$\frac{dx}{dt} = X_x'(0)x + \bar{X}(x) + \varepsilon X^*(t, x, \varepsilon). \qquad (11)$$

where $\bar{X}(x)$ begins with terms not lower than second order with respect to x. Hence, there will always exist a constant C and a small $\delta < \delta_0$ such that $\bar{X}(x) < Cx^2$ for $|x| < \delta$.

From this point on we shall be considering only a small range of variation of x. In this connection, the manifold, as distinct from N.N. Bogoliubov's [1], [2] and Yu.A.Mitropolsky's [3] theorems, will be of a local character, or in other words, we are considering it in a small neignbourhood of x.

The region U_{ρ_1} for ξ, ξ^* depends on how the choice of δ is effected. A small ρ_1 is chosen such that $x \in U_\delta$ for $\xi, \xi^* \in U_{\rho_1}$, i.e.

$$\rho_1 < \frac{\delta}{|A| + |B| + |D|}.$$

In what follows the local character of the manifold is determined by the fact that ξ_t, ξ_t^* contained in the manifold function (8) are being considered up to those values of t when they begin to leave the region U_{ρ_1}.

Having effected the transformation of system (2) to (10), the existence and uniqueness of a two-dimensional local integral manifold is then proved, and some of its properties established. Then, by charging back to the former variables x and transferring simultaneously the properties of the solutions of system (10) to the solution x, the result stated in theorem I is obtained.

The following lemma holds.

Lemma I. Let the right sides of system (10) possess the following properties:

1. The functions P, Q, R are defined in region $R \times U_{\rho_1} \times \Omega_{\varepsilon_0}$ and belong to class $C^t(\xi, \xi^*, h, \varepsilon)$.

2. In region $R \times U_{\rho_1} \times \Omega_{\varepsilon_0} (\rho < \rho_1)$ these functions belong to class $M(\varepsilon, \rho^2)$.

3. In region $R \times U_{\rho_1} \times \Omega_{\varepsilon_0}$ we have

$$\left.\begin{aligned} &P(t, \xi, \xi^*, h, \varepsilon) \\ &Q(t, \xi, \xi^*, h, \varepsilon) \\ &R(t, \xi, \xi^*, h, \varepsilon) \end{aligned}\right\} \in Lip(\xi, \xi^*, h, \varepsilon) \qquad (12)$$

4. The real parts of the roots of equation $Det \| \gamma_{n-2} E - H \| = 0$ are all negative.

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Then a positive $\varepsilon_1 < \varepsilon_0$ can be given such that for every positive ε less than ε_1 the system of equations (10) will have a unique local integral manifold $\mathcal{M}_t$ [x/] representable by

$$h = f(t, \xi, \xi^*, \varepsilon),$$

where f possesses the following properties:

1. The function $f(t, \xi, \xi^*, \varepsilon)$ is defined in region $R \times U_{\rho_1} \times \Omega_{\varepsilon_1}$ and belongs to class $\mathcal{O}t^r(\xi, \xi^*, \varepsilon)$ (or $\bar{\mathcal{O}}t^r(\xi, \xi^*, \varepsilon$).

2. In region $R \times U_{\rho_1} \times \Omega_{\varepsilon}$ $(\varepsilon < \varepsilon_1)$

$$f(t, \xi, \xi^*, \varepsilon) \in \mathcal{Z}_{1\rho}(\xi, \xi^*; \Delta(\varepsilon, \rho^2)),$$
$$|f(t, \xi, \xi^*, \varepsilon)| \leq D(\varepsilon, \rho^2) < \rho_1.$$

The proof of this lemma is as follows.

Instead of system (10) the system

$$\begin{aligned} \frac{d\xi}{dt} &= i\omega\xi + P_1(t, \xi, \xi^*, h, \varepsilon), \\ \frac{d\xi^*}{dt} &= -i\omega\xi^* + Q_1(t, \xi, \xi^*, h, \varepsilon), \\ \frac{dh}{dt} &= Hh + R_1(t, \xi, \xi^*, h, \varepsilon), \end{aligned} \tag{13}$$

is considered, where the functions P_1, Q_1, R_1 are defined in region

$$t \in R,\ h \in U_{\rho_1},\ \varepsilon \in \Omega_{\varepsilon_1},\ -\infty < \xi < \infty,\ -\infty < \xi^* < \infty. \tag{14}$$

x/ $\mathcal{M}$ is said to be a local integral manifold of system (10) if for every solution $h(t) = f(t, \xi(t), \xi^*(t), \varepsilon)$ of system (10), from the relationship $h(t_0) = f(t_0, \xi(t_0), \xi^*(t_0), \varepsilon) \in \mathcal{M}_{t_0}$, $|\xi(t_0)| < \rho_1$, $|\xi^*(t_0)| < \rho_1$ which is true for $t = t_0$, there will follow $h(t) = f(t, \xi(t), \xi^*(t), \varepsilon) \in \mathcal{M}_t$ for $t > t_0$ as long as $|\xi_t| < \rho_1$, $|\xi^*(t)| < \rho_1$.

possess in this region the same properties as those of the functions P, Q, R in region $R \times U_{\rho} \times \Omega_{\varepsilon_0}$, and in this very region coincide with them.

The necessity of introducing the "extended" system (13) is explained below.

Having fixed the positive numbers D, Δ $(D < \rho_1)$, we consider class $C(D, \Delta)$ of functions $F(t, \xi, \xi^*, \varepsilon)$ with their values from the $n-2$ -dimensional Euclidean space E_{n-2}, defined for $t \in R, -\infty < \xi < \infty, -\infty < \xi^* < \infty$ periodic in t of period 2π, for which $F(t, \xi, \xi^*, \varepsilon) \in \mathcal{Z}_{1\rho}(\xi, \xi^*; \Delta)$ and $|F(t, \xi, \xi^*, \varepsilon)| \leq D$.

We choose some function of this class and for this function we consider the equations

$$\begin{aligned} \frac{d\xi}{dt} &= i\omega\xi + P_1(t, \xi, \xi^*, F(t, \xi, \xi^*, \varepsilon), \varepsilon), \\ \frac{d\xi^*}{dt} &= -i\omega\xi^* + Q_1(t, \xi, \xi^*, F(t, \xi, \xi^*, \varepsilon), \varepsilon) \end{aligned} \tag{15}$$

Assuming arbitrary initial conditions $\xi = \xi_0$, $\xi^* = \xi_0^*$ for $t = t_0$, corresponding to the starting region /hence, also to region (14)/ and taking into consideration Cauchy's theorem, the existence and uniqueness of the solutions ξ_t, ξ_t^* of equations (19) for every $t > t_0$ are established. For these solutions the following notations are introduced:

$$\xi_t = \alpha^r_{z,t_0}(\xi_0, \xi_0^*); \quad \xi_t^* = M^r_{z,t_0}(\xi_0, \xi_0^*) \tag{16}$$

where $z = t - t_0$.

Then the transformation S of the function F from class $C(D, \Delta)$ into the fuction

$$S_{t,\xi,\xi^*}(F) = \int_{-\infty}^{0} \gamma(z) R_1\{t+z, \alpha^r_{z,t}(\xi, \xi^*, \varepsilon), M^r_{z,t}(\xi, \xi^*, \varepsilon), F(t+z, \alpha^r_{z,t}(\xi, \xi^*, \varepsilon), M^r_{z,t}(\xi, \xi^*, \varepsilon), \varepsilon)\} dz \tag{17}$$

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is considered, where $y(t)$ is determined by

$$y(t)=0, \quad t>0,$$
$$y(t)=e^{-\lambda t}, \quad t<0$$

all the roots of equation $\|\gamma_{n-2}z - H\| = 0$ having negative real parts.

Integration in (17) with respect to z is seen to be done between limits from $-\infty$ to ∞, hence the solutions $\bar{x}_t$, $\bar{x}_t^*$ being dependent on z, it is natural to extend the range of variation of $\bar{x}_t$, $\bar{x}_t^*$ over the whole complex plane in proving the lemma. That is the reason for replacing system (10) by system (13) which is being considered in the extended region.

We now consider the functional equation

$$F = SF.$$

By applying the fixed point theorem, the existence and uniqueness of the solution

$$h_t = f_t(t, \bar{x}, \bar{x}^*, \varepsilon) \tag{18}$$

of this equation is established.

Transforming equation (17)

$$f_t(t, \bar{x}, \bar{x}^*, \varepsilon) = \int_{-\infty}^{\infty} y(z) R_t\{t+z; \mathcal{L}_{z,t}^{h}(\bar{x}, \bar{x}^*, \varepsilon), \mathcal{U}_{z,t}^{h}(\bar{x}, \bar{x}^*, \varepsilon);$$
$$f_t(t+z; \mathcal{L}_{z,t}^{h}(\bar{x}, \bar{x}^*, \varepsilon), M_{z,t}^{h}(\bar{x}, \bar{x}^*, \varepsilon)\} dz \tag{19}$$

it is seen after a number of algebraic operations that function (19) satisfies the equation

$$\frac{dh_t}{dt} = Hh_t + R_t(t, \bar{x}_t, \bar{x}_t^*, h_t, \varepsilon).$$

Hence it follows that the manifold as determined by (19) is an integral one for system (13).

System (13) being equivalent to system (10) in the starting region, it follows that in this region

$$f_t(t, \bar{x}, \bar{x}^*, \varepsilon) = f(t, \bar{x}, \bar{x}^*, \varepsilon) \tag{20}$$

represents the local integral manifold for system (10).

We merely state the result which establishes the property of the local integral manifold of system (10) to attract the trajectories of any solutions of system (10) issuing from some $n-2$ -dimensional region U_{ρ_0}.

Lemma II. Positive constants γ, ε', ρ_1 $(\varepsilon' < \varepsilon_1)$ can be pointed out such that if all the characteristic numbers of matrix H corresponding to the equation

$$\|\gamma_{n-2}z - H\| = 0$$

have negative real parts, then for every $\varepsilon < \varepsilon'$ of any real t_0 and any $\bar{x}_0$, $\bar{x}_0^*$ for which the condition

$$|\bar{x}_0| < \rho_1, \quad |\bar{x}_0^*| < \rho_1$$

is fulfilled, there exists and $n-2$ -dimensional region of initial values h of U_{ρ_0} such that if

$$h_t'' \in U_{\rho_0}$$

for $t = t_0$, then

$$|h_t'' - f(t, \bar{x}_t, \bar{x}_t^*, \varepsilon)| < \nu(\varepsilon, \rho_0^2) e^{\gamma(t-t_0)} |h_0'' - f(t_0, \bar{x}_0, \bar{x}_0^*, \varepsilon)|, \tag{21}$$

for all $t > t_0$ where $\bar{x}_0$, $\bar{x}_0^*$, h_0'' represent $\bar{x}_t$, $\bar{x}_t^*$, h_t'' for $t = t_0$; h_t'' is any solution of system (10) not belonging to the integral manifold M_t; $\nu(\varepsilon, \rho_0^2)$ is a positive constant depending on the parameters ε and ρ_0.

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By extending the statement of properties of the solutions of system (10) to the solutions of the starting system, the result is obtained as stated in theorem I.

3. As this shown in the beginning of the paper, in investigating systems with many degrees of freedom by using the method of integral manifolds, the stability property of an approximate two-parametric family of solutions corresponding to the single-frequency process can be established.

Indeed, by taking a number of estimates it can be shown that the difference between the exact two-parametric family of solutions and its m-th approximation /whose construction presents no difficulty/ is a quantity of order ε^m.

The integral surface S /i.e. integral manifold/ which is covered by the curves of the exact two-parametric family of solutions attracts exponentially /as long as $f_t, f_t^* \in U_{\rho_1}$/ the trajectories of any solutions issuing at the initial moment from points lying in the vicinity of this surface.

Thus, in course of time these trajectories will also tend to the approximate two-parametric family of solutions which is in the ε^m-neighbourhood of the exact two-parametric family, or else this tendency will be effected according to the following law:

$$|x(t) - x_{np}^m(t)| \leq C(\varepsilon, \rho^2) e^{\gamma(t-t_0)} + \delta(\varepsilon) = \psi(t, \varepsilon) \to 0 \quad (22)$$

as $t \to \infty$, $\varepsilon \to 0$.

4. As an example we consider the case

$$X(x) = Px$$

where P is a constrant $n \times n$ matrix.

As a result, a system

$$\frac{dx}{dt} = Px + \varepsilon X^*(t, x, \varepsilon) \quad (23)$$

is obtained, which will be considered for any finite values of x as $t \in R$, $\varepsilon \in \Omega_{\varepsilon_0}$.

In the case under consideration the locality condition falls away and theorem I holds for any finite x.

The property of attraction also exists for trajectories of any solutions of system (23) whose initial values belong now not to a small neighbourhood of the manifold but to any finite region.

5. By applying the method of integral manifolds it seems possible to investigate the stability of the trivial solution $x_* = 0$ of system (3), acted upon by the functions $\varepsilon X^*(t, x, \varepsilon)$ in critical cases. We shall consider the critical case only when there are purely imaginary roots. For the sake of simplicity we take up the case of two purely imaginary roots.

It is evident that

$$|x(t) - x_*| \leq |x(t) - x_{MH}(t)| + |x_{MH}(t) - x_*|. \quad (24)$$

where $x_{MH}(t) = \Phi(t, f_t, f_t^*, \varepsilon)$ is a function of the manifold and has the form of (6). For the first term, when the conditions of theorem I are fulfilled, we have the following estimate:

$$|x(t) - x_{MH}(t)| \leq C(\varepsilon, \rho^2) e^{-\beta(t-t_0)} \quad (25)$$

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with $x(t) \in U_{\sigma_1}$ /as long as $|\mathcal{Y}_t| < \rho_1$, $|\mathcal{Y}_t^*| < \rho_1$ /, and C a constant.

Since we are concerned with the stability of $x=0$ for any t, it seems quite natural to obtain first of all suitable estimates for $\mathcal{Y}_t$, $\mathcal{Y}_t^*$ at any t.

For this purpose we introduce a change of variables in equations (5)

$$\mathcal{Y} = C e^{i\lambda t}, \quad \mathcal{Y}^* = C^* e^{-i\lambda t}$$

and reduce them to standard form

$$\frac{dC}{dt} = \varepsilon P_1(t, \lambda t, C, C^*, \varepsilon), \quad \frac{dC^*}{dt} = \varepsilon Q_1(t, \lambda t, C, C^*, \varepsilon). \qquad (26)$$

Consider the corresponding averaged equations with an accuracy to magnitudes of order ε:

$$\frac{d\bar{C}}{dt} = \varepsilon \bar{P}_1(\bar{C}, \bar{C}^*), \quad \frac{d\bar{C}^*}{dt} = \varepsilon \bar{Q}_1(\bar{C}, \bar{C}^*)$$

where

$$\bar{P}_1 = \lim_{T\to\infty} \frac{1}{T} \int_0^T P_1(t, \lambda t, C, C^*, 0)\, dt,$$
$$\bar{Q}_1 = \lim_{T\to\infty} \frac{1}{T} \int_0^T Q_1(t, \lambda t, C, C^*, 0)\, dt$$

and assume that these equations have a zero solution:

$$\bar{C}_0 = 0, \quad \bar{C}_0^* = 0.$$

It is seen from the transformation formulae that from the boundedness of C and C^* follows the boundedness of $\mathcal{Y}$ and $\mathcal{Y}^*$.

Applying the results of N.N.Bogoliubov's well known theorem [1], it is possible to establish that, if in addition to the conditions of the theorem concerning the existence of a manifold, the condition is also fulfilled that for the variational equations set up for the solution $\bar{C}_0 = 0, \bar{C}_0^* = 0$ both characteristic exponents have negative real parts, then a sufficiently small $\bar{\varepsilon} < \varepsilon^*$ and a sufficiently great $\bar{t}$ can be always given such that for all $\varepsilon < \bar{\varepsilon}$, $t > \bar{t}$ the solutions C, C^* and consequently $\mathcal{Y}$, $\mathcal{Y}^*$ whose initial values belong to region U_{ρ_2} will be bounded in modulus by some functions Ω_1, Ω_2 tending to zero as $t \to \infty$, $\varepsilon \to 0$.

In proving theorem I it is established, in particular, that $f(t, \mathcal{Y}_t, \mathcal{Y}_t^*, \varepsilon) \to 0$ as $\mathcal{Y}_t \to 0$, $\mathcal{Y}_t^* \to 0$, $\varepsilon \to 0$ where f is contained in (6).

Consequently, if the above conditions are fulfilled, then the function of the manifold will be bounded in modulus by some function $\Psi(t,\varepsilon) \to 0$ for $t \to \infty$, $\varepsilon \to 0$.

Thus, it is finally obtained

$$|x(t) - x_*| \le C(\varepsilon, \rho^2) e^{-\gamma(t-t_0)} + \Psi(t, \varepsilon) \to 0$$

for $t \to \infty$, $\varepsilon \to 0$.

The latter estimate denotes that if the above conditions are fulfilled, then a sufficiently small positive $\bar{\varepsilon}$ and positive constants σ_1 and δ ($\delta < \sigma_1, \varepsilon < \varepsilon^*$) as well as a sufficiently great $\bar{t}$, can be pointed out such that for every $\varepsilon < \bar{\varepsilon}$, $t > \bar{t}$ any solution of equations (1), whose initial value belongs to enighbourhood $U_{\sigma_1(0)}$, will in course of time within the δ-neighbourhood of the solution $x_* = 0$, which is drawn towards $x_* = 0$ for $t \to \infty$, $\varepsilon \to 0$.

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As is known the phase motion of particles in all the resonance accelerators which are based on the phase-self stability principle is described by the one-type equation which may be generally written as follows*.

$$\frac{d}{dt}[m(t)\dot{\psi}] + f(t)\mathcal{U}(\psi) = 0 \qquad /1/$$

where $\mathcal{U}(\psi) = \frac{d\mathcal{U}}{d\psi} = -\frac{1}{\sin\varphi_s}[\cos(\varphi_s+\psi) - \cos\varphi_s]$, φ_s – is the synchrone phase, ψ – is the difference between the phases of the considered particle and the synchrone one.

One of the principal problems of the theory of accelerators is to determine the capture region, i.e. the region of the initial values of ψ°, $\dot{\psi}^\circ$ from which with $t = t_\circ$ the solutions oscillating with respect to the position of stable equilibrium $\psi = 0$ emerge.

The knowledge of the capture region makes it possible to evaluate the part of the injected particles incident into the synchrone acceleration regime and, therefore, leaving the accelerator with an energy close to the calculated one. Since in equation /1/ the functions $m(t)$ and $f(t)$ vary in the concrete cases rather slowly then in a certain approximation they may be considered what enables to integrate equation /1/ /the conservation approximation/ quite easily. Then the boundary of the capture region is determined by the formula:

$$\dot{\psi}^\circ = \pm\sqrt{2\mathcal{U}_m - 2\mathcal{U}(\psi^\circ)}, \quad \mathcal{U}(\psi^\circ) = \int_0^{\psi^\circ} \mathcal{U}(\xi)\,d\xi \qquad /2/$$

where $\mathcal{U}_m = \mathcal{U}(-2\varphi_s)$ is the maximum of $\mathcal{U}(\psi)$ /it is assumed that $m(t_\circ) = f$ / $t_\circ$ =I that does not violate the generality/. At $\psi^\circ = 0$ the width of the

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* Note that in equation /1/ we neglect the terms of type of external force $F(t)$, and do not take into account the effect of the volume charge.

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capture region is evidently $\Delta\psi_c = 2\varphi_s + \psi_m^\circ$, where ψ_m° and $-2\varphi_s$ are the roots of the equation $U(\psi) = U_m$ /the root $-2\varphi_s$ is two fold/. Particularly, if $\varphi_s \ll 1$ /in fact is quite sufficient to assume that $\varphi_s < 30°$/, then $\psi_m \approx \varphi_s$ and $\Delta\psi_c \approx 3\varphi_s$.

The motion of particles for which the initial values ψ°, $\dot\psi^\circ$ lie inside the egg-shaped region limited by the curves /2/ will be stable.

In papers devoted to the theory of various resonance accelerators untill now the authors restricted themselves only by the determination of the capture region in the conservation approximation according to /2/ /see, e.g.//1/*.

In this paper the calculation of the capture region is given with account of the non-conservation of equation /1/. This calculation is based upon the assumption that the change of $m(t)$ and $f(t)$ is slow. This is understood in the sense that the time τ of the considerable change of $m(t)$ and $f(t)$ is great if compared with the period $2\pi/\Omega$ of the linear phase oscillations. This means that equation /1/ may be written in the form:

$$\frac{d}{dt}\left[m(\varepsilon t)\dot\psi\right] + f(\varepsilon t)\,U'(\psi) = 0 \qquad /1a/$$

where $\varepsilon = \frac{2\pi}{\Omega\tau}$.

Equation /1a/ belongs to the equations with "slow time", to which many papers by N.N.Bogolubov, Yu.A.Mitropolsky [4], and V.M.Volosov [5] are devoted. The methods for investigating these equations developed in [4,5] are, however,

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* Exceptions are the papers /2/-/3/, containing some numerical calculations concerning the role of variability of coefficients $m(t)$, $f(t)$ in eq. /1/.

based upon the assumption that their solutions are oscillating and the problem about the boundary of the region of the initial values to which such stable solutions correspond was not in general set up by these authors.

For the equations of the considered type /1a/ with the increasing $m(\varepsilon t)$ and $f(\varepsilon t)$ one may find the boundary of the capture region with the help of the following physical considerations. It is evident that any particle coming from the right to the vertex of the barrier with the velocity different from zero must be unstable. /see, Fig. 1/.

On the other hand, the particles which move from the right to this vertex and fall to reach it are stable. Therefore, it is natural to define the separatrix limiting the region of stable ψ°, $\dot\psi^\circ$ as a set of such ψ°, $\dot\psi^\circ$ that the trajectories outgoing from them at $t = t_0$ asymptotically approach the unstable equilibrium position $-2\varphi_s$. It is clear that there exists an infinite number of trajectories /we shall call them boundary trajectories which satisfy the condition $\psi \to -2\varphi_s$, whereas $\dot\psi \to 0$ at $t \to \infty$. The problem of further investigation is to find the functional dependence $\dot\psi^\circ = \dot\psi^\circ(\psi^\circ)$ for them. Since equation /1a/ involves a small parameter ε then one may try to find boundary trajectories by means of the formal expansion of the solutions of equation /1a/ in powers of ε. Since the considered region of the change of independent variable is infinite, the Poincaré theorem about the analiticity of the solutions of equations of type /1a/ with respect to ε may be incorrect./see [6]/.

V.K.Melnikov has shown [7], however, that the formal expansion $\psi = \psi_0 + \varepsilon\psi_1 + \dots$ [illegible] auxiliary

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$\psi_0 \to -2\varphi_3$, $\dot\psi_0 \to 0$, $\psi_i \to 0$ $(i=1,2,\dots)$ at $t \to \infty$ and $\psi_i = 0$ $(i=1,2,\dots)$ at $\psi_0 = \overline{\psi}_0$ /$\overline{\psi}_0$ is an arbitrary point lying sufficiently close to $-2\varphi_3$ to the right from $-2\varphi_3$ for the trajectories approaching $\psi = -2\varphi_3$ to the right and from $-2\varphi_3$ for the trajectories approaching $\psi = -2\varphi_3$ from the left /is asymptotic with respect to E and really determines the separatrix in the sense that the initial values of the boundary trajectories divide the plane ψ°, $\dot\psi^\circ$ into the regions from which the solutions of one type emerge*. Evidently, the behaviour of boundary trajectories is essentially different for $\dot\psi^\circ < 0$ /the initial velocity is directed from the right to the left /Fig.I/ or $\dot\psi^\circ > 0$ /the initial velocity is directed from the left to the right /Fig.I/ since in the case $\dot\psi^\circ > 0$ there may exist a turning point /where $\dot\psi = 0$ / absent in the case $\dot\psi^\circ < 0$.

Let us investigate first the boundary trajectories for $\dot\psi^\circ < 0$. We shall assume $t_0 = 0$, whereas $m(0) = f(0) = 1$ /this can be always achieved in the proper definition of t/. All the calculations must be made by restricting by the quantities of the first order over E.

The equations which the functions $\psi_0, \psi_1, \dots, \psi_i, \dots$ of the formal expansion satisfy have the form

$$\ddot\psi_0 + U'(\psi_0) = 0, \qquad /3/$$

$$\ddot\psi_1 + U''(\psi_0)\psi_1 = -f'(0)t\,U'(\psi_0) - m'(0)\frac{d}{dt}(\dot\psi_0 t), \qquad /4/$$

$$\ddot\psi_i + U''(\psi_0)\psi_i = \mathcal{F}_i(t, \psi_0, \dots \psi_{i-1}). \qquad /5/$$

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* Point out that this theorem is proved in [7] also for the equations [illegible] /1a/.

It follows from equation /3/

$$\frac{\dot\psi_0^2}{2} + U(\psi_0) = C.$$

Making use of the conditions $\psi_0 \to -2\varphi_3$ and $\dot\psi_0 \to 0$ at $t \to \infty$ that gives $C = U_m$, we find

$$\frac{\dot\psi_0^2}{2} = U_m - U(\psi_0), \quad t = \int_{\psi_0}^{\chi} \left(\sqrt{2U_m - U(\xi)}\right)^{-1} d\xi \qquad /6/$$

χ is the parameter defining one of the boundary trajectories, ξ is the integration variable.

Equations /4/ and /5/ can be easily integrated if introduce ψ_0 as an independent variable connected with formula /6/. Indeed, as can be easily shown

$$\frac{d}{d\psi_0}\left[2\,\frac{d\psi_i}{d\psi_0}(U_m - U(\psi_0)) + U'(\psi_0)\psi_i\right] = \ddot\psi_i + U''(\psi_0)\psi_i$$

and, therefore, equation /4/ is reduced to the equation of the first order

$$2(U_m - U(\psi_0))\frac{d\psi_1}{d\psi_0} + U'(\psi_0)\psi_1 = Q(\psi_0)$$

where

$$Q(\psi_0) = -m'(0)\int_{-2\varphi_3}^{\psi_0} \dot\psi_0(\xi)\,d\xi + (m'(0) - f'(0))\int_{-2\varphi_3}^{\psi_0} t(\xi)U'(\xi)\,d\xi. \qquad /7/$$

From /7/ using the condition $\psi_1 = 0$ at $\psi_0 = \overline{\psi}_0$ we find

$$\psi_1(\psi_0) = \sqrt{2U_m - 2U(\psi_0)} \cdot \int_{\overline{\psi}_0}^{\psi_0} \frac{Q(\xi)\,d\xi}{(2U_m - 2U(\xi))^{3/2}}. \qquad /8/$$

An analogous formula holds for ψ_i $(i=2,3,\dots)$. Thus, the equation of the boundary trajectories at $\dot\psi^\circ < 0$ is determined with the accuracy up to the terms of the first order E by means of the formulas

$$\psi = \psi_0(t,\chi) + \varepsilon\psi_1(t,\chi) + \dots$$

$$\dot\psi = -\sqrt{2U_m - 2U(\psi_0)} + \varepsilon\left[-\frac{Q(\psi_0)}{\sqrt{2U_m - 2U(\psi_0)}} + U'(\psi_0)\int_{\overline{\psi}_0}^{\psi_0} \frac{Q(\xi)\,d\xi}{(2U_m - 2U(\xi))^{3/2}}\right] + \dots \qquad /9/$$

The connection between the initial values of these trajectories corresponding to $t = 0$ is obtained at $\psi_0 = \chi$, i.e., determined by the parametric formula

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$$\psi_0 = \chi + \varepsilon\psi_1(0,\chi) + \dots$$

$$\dot\psi^\circ = -\sqrt{2U_m - 2U(\chi)} + \varepsilon - \left[\frac{Q(\chi)}{\sqrt{2U_m - 2U(\chi)}} + U'(\chi)\int_{\psi_0}^{\chi}\frac{Q(\xi)\,d\xi}{(2U_m - 2U(\xi))^{3/2}}\right] + \dots \quad /10/$$

where $Q(x) = \frac{m(0)+f(0)}{2}\int_{-2\varphi_s}^{x}\sqrt{2U_m - 2U(\xi)}\,d\xi$.

Let us raise $\dot\psi^\circ$ determined by /10/ to the second power restricting by the terms of the first order E. This gives

$$(\dot\psi^\circ)^2 = 2(U_m - 2U(\chi)) + 2\varepsilon[Q(\chi) - U'(\chi)\psi_1(0,\chi)]. \quad /11/$$

Now pass to the direct dependence between $(\dot\psi^\circ)^2$ and ψ° with the accuracy up to the terms of the first order over E. According to /10/ in this approximation $x = \psi^\circ - \varepsilon\psi_1(0,\psi^\circ)$. Substituting this equation /11/ we find the formula for the separatrix at $\dot\psi^\circ < 0$:

$$(\dot\psi^\circ)^2 = 2(U_m - U(\psi^\circ)) + \bar\varepsilon\int_{-2\varphi_s}^{\psi^\circ}\sqrt{2U_m - 2U(\xi)}\,d\xi, \text{ where } \bar\varepsilon = [m(0)+f(0)]\varepsilon. \quad /12/$$

In particular, the width of the capture region at $\dot\psi^\circ = 0$ is equal to $\Delta\psi = \psi_m^1 + 2\varphi_s$, where ψ_m^1 and $-2\varphi_s$ are the roots of the equation

$$2(U_m - U(\psi^\circ)) + \bar\varepsilon\int_{-2\varphi_s}^{\psi^\circ}\sqrt{2U_m - 2U(\xi)}\,d\xi = 0.$$

It can be easily seen that

$$\psi_m^1 = \psi_m^\circ + \frac{\bar\varepsilon}{2U(\psi_m^\circ)}\int_{-2\varphi_s}^{\psi_m^\circ}\sqrt{2U_m - 2U(\xi)}\,d\xi = \psi_m^\circ + \delta\psi, \quad /13/$$

where $\delta\psi$ is the increment of the capture region due to the nonconservation.

At sufficiently small φ_s one may assume that $U = \frac{\psi^2}{2} + \frac{\psi^3}{6\varphi_s}$. Introducing new quantities $\dot x^\circ = \frac{\dot\psi^\circ}{\varphi_s}$, $x^\circ = \frac{\psi^\circ}{\varphi_s}$ represent the equation of separatrix /12/ at $\dot x^\circ < 0$ in the form

$$(\dot x^\circ)^2 = 2\left(\frac{2}{3} - U(x^\circ)\right) + \bar\varepsilon\int_{-2}^{x^\circ}\sqrt{\frac{4}{3} - 2U(x)}\,dx \quad /14/$$

where $U(x) = \frac{x^2}{2} + \frac{x^3}{6}$, $2(U_m - U) = \frac{1}{3}(1-x)(2+x)^2$,

$$\int_{-2}^{x_0}\sqrt{\frac{4}{3} - 2U(x)}\,dx = \frac{12}{5} - \frac{2}{5\sqrt{3}}(1-x^\circ)^{3/2}(4+x^\circ).$$

At $x^\circ = 0$ the width of the capture region

where

$$\delta x = \frac{\bar\varepsilon}{2U(1)}\int_{-2}^{1}\sqrt{(1-x)(2+x)^2}\,dx = \frac{4}{5}\bar\varepsilon.$$

In Fig. 2 the graphs of the separatrix are given at $\bar E = 0{,}1$ and $\bar E = 0{,}2$ in the variables x°, $\dot x^\circ$ found by the formula /14/.

Now let us investigate the behaviour of the boundary trajectories at $\dot\psi^\circ > 0$ /the initial velocity is directed from left to right/. In this case two possibilities should be indicated:

1/ The particles start to move on the left from the barrier $U_m(\psi^\circ < -2\varphi_s)$ and come to the vertex of the barrier with zero velocity. Evidently, geometrical place of initial values ψ°, $\dot\psi^\circ$ for such trajectories is obtained from the curve $\dot\psi^\circ = \sqrt{2U_m - 2U(\psi^\circ)}$ by some shift due to the nonconservation.

2/ the particles start to move to the left from the barrier and come to it with the velocity different from zero $\dot\psi > 0$ or start to move on the left from the barrier U_m with $\dot\psi^\circ > 0$. In this case the boundary trajectory must have at a certain moment of time t_1 at $\psi > 0$ a turning point where $\dot\psi = 0$, and further it moves with $\dot\psi < 0$ along trajectory determined by the formula found earlier /12/.

Let us consider the first from the mentioned possibilities. The boundary trajectory is described by the formal expansion $\psi = \psi_0 + \varepsilon\psi_1 + \dots$, where ψ_i obey the same

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auxiliary conditions and is determined as in the case in the case $\dot{\psi}^\circ < 0$. The difference is that the point lying to the left from $-2\varphi_s$ should be taken as $\overline{\varphi_0}$. Then $t = \int_{\chi}^{\psi_0} \frac{d\xi}{\sqrt{2U_m - 2U(\xi)}}$ since at $t > 0$, $\dot{\psi}_0 > 0$ and $\psi_0 > \chi$. The boundary trajectories are described by the formulas:

$$\psi = \psi_0(t,\chi) + \varepsilon\psi_1(t,\chi) + \dots$$

$$\dot{\psi} = \sqrt{2U_m - 2U(\psi_0)} + \varepsilon\left[\frac{Q(\psi_0)}{\sqrt{2U_m - 2U(\psi_0)}} - U(\psi_0)\int_{\psi_0}^{\psi_0} \frac{Q(\xi)\,d\xi}{(2U_m - 2U(\xi))^{3/2}}\right] \quad /15/$$

whereas their initial are obtained from /15/ at $t = 0$, i.e., at $\psi_0 = \chi$.

From here, reasoning in the manner as in the derivation of /12/ we obtain:

$$(\dot{\psi}^\circ)^2 = 2\left(U_m - 2U(\psi_0)\right) + \varepsilon\int_{\psi_0}^{-2\varphi_s} \sqrt{2U_m - 2U(\xi)}\,d\xi \,. \quad /16/$$

In particular, at $\varphi_s \ll 1$, introducing the quantities $x^\circ = \frac{\psi^\circ}{\varphi_s}$, $\dot{x}^\circ = \frac{\dot{\psi}^\circ}{\varphi_s}$, we rewrite /16/ in the form:

$$(\dot{x}^\circ)^2 = 2\left(\frac{2}{3} - U(x)\right) + \bar{\varepsilon}\int_{x^\circ}^{-2} \sqrt{\frac{2}{3} - 2U(x)}\,dx \quad /17/$$

where $U(x)$ and $\int_{x^\circ}^{-2}\sqrt{\frac{2}{3} - 2U(x)}\,dx$ are defined in /14/.

Finally, let us investigate the last case. We shall look for the motion of a particle in the interval (t_1, ∞) , i.e. after the turning point just as in case $\dot{\psi}^\circ < 0$. Using the expansion obtained earlier /9/ we can calculate $\psi(t_1)$ /i.e., the value of ψ in the turning point/ which as can be easily checked with the accuracy up to the terms of the first order over ε is defined by /14/ inclusive. After this small look for the solution $\psi(t)$ on the interval $(0, t_1)$ in the form of a series

$$\psi = \psi_0(t, t_1) + \varepsilon\psi_1(t, t_1) + \dots$$

where the functions ψ_0 , ψ_1 , ψ_2 satisfy the equations /3/, /4/ and /5/ with the initial conditions $\psi_0(t_1, t_1) = \psi(t_1)$, $\dot{\psi}_0(t_1, t_1) = 0$ and $\psi_i(t_1, t_1) = \dot{\psi}_i(t_1, t_1)$ at $i = 1, 2, \dots$ /For this expansion the Poincare theorem about the analytic dependence upon the parameter ε /see, /8/ /is correct/. It can be easily verified that up to the solution with the accuracy up to the term of the first order over ε is of the form

$$\psi = \psi_0 + \varepsilon\sqrt{2\mathcal{E}_m - 2U(\psi_0)}\int_{\psi_m'}^{\psi^\circ} \frac{Q(\xi)\,d\xi}{(2\mathcal{E}_m - 2U(\xi))^{3/2}} + \dots ,$$

$$\dot{\psi} = \sqrt{2\mathcal{E}_m - 2U(\psi_0)} + \varepsilon\left[\frac{Q(\psi_0)}{\sqrt{2\mathcal{E}_m - 2U(\psi_0)}} - U(\psi_0)\int_{\psi_m'}^{\psi_0} \frac{Q(\xi)\,d\xi}{(2\mathcal{E}_m - 2U(\xi))^{3/2}}\right] \quad /18/$$

where $\mathcal{E}_m = U(\psi_m'')$. From /16/ reasoning in the same manner as in the derivation of /12/ we obtain

$$(\dot{\psi}^\circ)^2 = 2\left(\mathcal{E}_m - U(\psi_0)\right) + \bar{\varepsilon}\int_{\psi^\circ}^{\psi_m'} \sqrt{2\mathcal{E}_m - 2U(\xi)}\,d\xi \,. \quad /19/$$

Remind, that $\mathcal{E}_m = U(\psi_m'')$ depends upon ε, i.e., in /19/ the terms of higher orders over ε are taken into account in an indirect way. However, as follows from the derivation of this formula the error has the order over ε not lower than the second. For small φ_s restricting in the expansion $U(\psi)$ by the terms not higher than the second order and substituting $x^\circ = \frac{\psi^\circ}{\varphi_s}$, $\dot{x}^\circ = \frac{\dot{\psi}^\circ}{\varphi_s}$ we may reduce expression /19/ to the form:

$$(\dot{x}^\circ)^2 = \frac{2}{3} - 2U(x^\circ) + \frac{12}{5}\bar{\varepsilon} + \bar{\varepsilon}\int_{x^\circ}^{x_1^\circ} \sqrt{\frac{2}{3} + \frac{12}{5}\bar{\varepsilon} - 2U(x)}\,dx \,. \quad /20/$$

The set of formulas /19/, /18/ and /19/ makes it possible to construct on the plane $\psi, \dot{\psi}$ the curve which gives the boundary of the phase region. This curve in the variables x°, $\dot{x}^\circ$ is given in Fig. 3 for $\bar{\varepsilon} = 0.2$ and for $\bar{\varepsilon} = 0$ using the formulas /14/, /17/ and /20/.

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The obtained results make it possible to draw some interesting conclusions which in our opinion may be of technical interest:

As is seen from Fig. 2 at $\bar{\varepsilon} > 0$ the capture region looses its symmetrical form with respect to the straight line $\dot{\psi}^\circ = 0$. Now the greatest width of the capture is obtained /at sufficiently small $\bar{\varepsilon} > 0$ / not at $\dot{\psi}^\circ = 0$ but according to the formula /I0/ at

$$\dot{\psi}_{min} = \dot{\psi}^\circ(-2\varphi_s) = \sqrt{2(\mathcal{E}_m - \mathcal{U}_m) + \bar{\varepsilon}\int_{-2\varphi_s}^{\psi_m^*} \sqrt{2\mathcal{E}_m - 2U(\xi)}\, d\xi}\,.$$

It is defined by the root of the equation

$$(\dot{\psi}_{min})^2 = 2(\mathcal{E}_m - U(\psi)) + \bar{\varepsilon}\int_{\psi}^{\psi_m^*} \sqrt{2\mathcal{E}_m - 2U(\xi)}\, d\xi$$

nearest to ψ_m^* and by the root of the equation.

$$(\dot{\psi}_{min})^2 = 2(\mathcal{U}_m - U(\psi)) + \bar{\varepsilon}\int_{\psi}^{-2\varphi_s} \sqrt{2\mathcal{U}_m - 2U(\xi)}\, d\xi$$

nearest to $-2\varphi_s$. We shall look for the root of the first equation in the form $\psi = \psi_m^* + \delta_1\psi$, the root of the second one - in the form $\psi = -2\varphi_s + \delta_2\psi$. Substituting the obtained expressions into our we may easily find that in the first term of the expansion over E

$$\delta_1\psi = -\frac{\bar{\varepsilon}}{U(\psi_m^*)}\int_{-2\varphi_s}^{\psi_m^*} \sqrt{2\mathcal{E}_m - 2U(\xi)}\, d\xi \qquad \text{and}$$

$$\delta_2\psi = -\sqrt{-\frac{2\bar{\varepsilon}}{U'(-2\varphi_s)}}\int_{-2\varphi_s}^{\psi_m^*} \sqrt{2\mathcal{E}_m - 2U(\xi)}\, d\xi\,.$$

It can be easily seen that with the accuracy up to the terms of the first order over E $\delta_1\psi = -2\delta\psi$ where $\delta\psi$ is determined by the formula /I3/.

Thus, at $\bar{\varepsilon} > 0$ the width of the capture increases in the first term of the expansion over E by $-\delta_2\psi$

In the variables x°, $\dot{x}^\circ$, i.e., at small synchrone phases in the units φ_s the increment of the capture region is equal to $\Delta_1 x = -\frac{\delta_2\psi}{\varphi_s} = \sqrt{\frac{2\bar{\varepsilon}}{5}}$. At $E = 0{,}2$, $\delta_1 x = 1{,}0$, i.e., the width of the capture increases in comparison with its magnitude in the conservation case by 20%.

These conclusions may be of great interest if applied to the proton linear accelerator with the drift tubes. The equation of phase oscillations in this case may be written as follows /see, for instance, /II/ /:

$$\frac{d}{dt}(v_s^2\dot{\psi}) = \frac{e E \omega v_s}{m}\left[\cos(\varphi_s + \psi) - \cos\varphi_s\right] \qquad /21/$$

where $\psi = \frac{\omega}{v_s} q$, q is the small distance between the considered an the synchrone particles at the given stage, $E = \frac{1}{L}\int_L E_z \cos\frac{2\pi}{\lambda} z\, dz \;/\; E_z$ - is the strength of the accelerating field on the axis of the accelerating section with the length L, along which the integration is made/, $\omega = \frac{2\pi}{T}$, T is the period of high-frequency oscillations, e, m is the charge and mass of the particle, v_s is the velocity of the synchrone particle, t - is the time in sec. Note that the increment of its velocity on one accelerating section is determined by the formula $\Delta v_s = \frac{e}{m} E T \cos\varphi_s$. For simplicity we assume this quantity to be constant, i.e., $v_s = v_{s0} + \frac{\Delta v_s}{T} t = v_{s0}\left(1 + \frac{t}{\tau}\right)$, v_{s0} is the initial velocity of the synchrone particle $\tau = \frac{v_{s0}}{\Delta v_s} T$. Let us introduce a dimensionless variable $t' = \Omega t$, $\Omega^2 = \frac{e\omega E \sin\varphi_s}{m v_{s0}}$ /it is evident, that Ω coincides with the frequency of the linear phase oscillations at $v_s = v_{s0}$ /. Then equation /21/ may be written with the accuracy up to the magnitudes of the first order over ε in the form:

$$\frac{d}{dt'}\left[(1 + 2\varepsilon t')\dot{\psi}\right] + (1 + \varepsilon t')\, U'(\psi) = 0, \qquad /22/$$

where

$$U(\psi) = -\frac{1}{\sin\varphi_s}\left[\cos(\varphi_s + \psi) - \cos\varphi_s\right], \quad \varepsilon = \frac{1}{\Omega\tau} = \frac{1}{2\pi}\sqrt{\frac{\Delta\varphi_s}{v_{s0}}},$$

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$$\mathcal{E}_0 = \frac{m v_{s0}^2}{2} = 4 mev \;,\; \varphi_s = 20^\circ,\; \frac{\Delta v_s}{v_{s0}} = 0,04 \quad *)$$

... that $\bar{\varepsilon} = 0,4$. According to /14/ the increment of the width of the capture δx at $\dot{\psi}^\circ = 0$ is /in the units of φ_s/ ... r the considered case $\delta x = \frac{4}{5}\bar{\varepsilon} = 0,3$, i.e., 10%.

If the initial energy of particles exceeds the energy ... by the quantity, corresponding to $(\frac{d\varphi}{dt^2})_{min} = \frac{1}{\sqrt{2}}\dot{\psi}_{min}$ i.e., by the quantity $\delta\mathcal{E}_0 = \mathcal{E}_0 \varphi_s \sqrt{\frac{\pi\varphi_s}{2\pi}} \frac{\Delta v_s}{v_{s0}} (\frac{dx}{dt^2})_{min}$ /at small synchrone phases/ then the increment of the width of capture is $\sqrt{\frac{24}{5}\bar{\varepsilon}}$. In other words, in the considered case the energy increase of the injected particles from 4 MeV up to 100 MeV must lead to the increase of the capture approximately by 1.4, i.e., almost by 50% if compared with the conservation approximation corresponding to $\dot{\psi}^\circ = 0$.

It is interesting to note that in paper /11/ devoted to the description of the linear accelerator meant for the injection into the Berkeley Bevatron it is indicated that the greatest capture is obtained if the initial proton energy is not the designed one /450 KeV/, but exceeds it by 10 kV**). It is possible that this phenomenon is explained by the above mentioned facts, i.e., by the increase of the capture with the increase of the initial energy if compared with the designed one due to the nonconservation of the phase ... tion.

---

* These data concern the Alvarez accelerator /10/.

** According to our formulas the maximum width of the capture concerning the data of this accelerator is obtained at the injection ... exceeding the designed one by 30 KeV.

In conclusion we should like to emphasize that ... abovementioned technical considerations take place not only for linear accelerators but also for the cycle accelerators of phasotron and synchrophasotron type. In particular one may believe that the increase of the field amplitude with time in the accelerating sections of the synchrophasotron and a certain increase of the initial proton energy injected if compared with the designed one may lead to a considerable increase of the number of particles ... into the regime of the synchrone accelerator.

The autors express their gratitude to L.A.Kulev, ...Shwartz and to Professor A.S.Pompaneetz for the discussion of the results.

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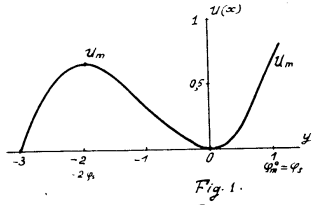

Fig. 1.
Рис.1.

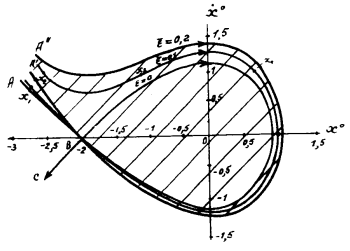

Fig. 2.
Рис.2

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ЭНЕРГЕТИЧЕСКИЕ СООТНОШЕНИЯ В ЭЛЕКТРИЧЕСКИХ ЦЕПЯХ С ФЕРРОМАГНИТНОЙ СВЯЗЬЮ

We know the difficulties emerging in the estimation of the electric circuits, where non-linear coupling and the means of connecting elements create conditions for the appearance of currents and voltages with different harmonics in the sections of such circuits.

As an example of such circuits we may take a static frequency changer in which gate tubes or magnetic cores are non-linear coupling elements.

This work describes the method of estimation of frequency changers with ferromagnetic coupling. This method is based on the addition of some outer e.m.f. or current sources into the circuits. The equations of power balance considering these outer sources are composed for the changer subsequently applying various conditions to the power state of these sources, we may consider all various duties of a given frequency changer.

The application of this method for the estimation of a frequency changer (inverter) with gas gate tube is shown in other published works of the author [1, 2].

The investigation and estimation of processes in a frequency changer with non-linear

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B-H curve of ferromagnetic cores may be reduced to the investigation of processes in one of the elements of such a frequency changer. This element (fig.1) in a general view represents a core with windings supplied from the sources (of e.m.f. or current) with different frequencies. For the elimination of the influence of currents (or e.m.f.) induced in the windings by the sources of other frequencies, the ideal filters of harmonics are applied to the circuit windings, permitting a single frequency in the circuits of corresponding sources. A similar element circuit applied to the theory of magnetic amplifier was discussed by M. Rosenblatt [3].

Consideration of processes in an element is carried out with simplified assumptions, usual in calculations of such circuits: the resistance of the windings, the dissipation fluxes and the hysteresis of magnetization characteristics are not taken into account. Non-hysteresis characteristic of core (the mean line of hysteresis loop) is analytically expressed by a hyperbolic sine (in the case of switching e.m.f. sources into circuit windings)

$$H = \alpha Sh\beta B,$$

or by means of dimensionless quantities

$$h = Shb$$

In accordance with fig.1 the core induction may be given as a sum of three components: a constant component and the components of two different frequencies

$$b = b_0 + b_p + b_q.$$

Assume the [illegible] these components

$$b_p = B_p \cos x, \qquad b_q = B_q \cos y$$

where

$$x = pt + \varphi, \qquad y = qt + \psi.$$

Let us take p [illegible] so let us consider such ratio equal to the integer (the frequencies the multiple) and more than a unit.

Expanding the hyperbolic sine into series including the Bessel functions of imaginery argument (see appendix 1) and singling out from the series intensity harmonics, corresponding to the voltage frequencies of the sources we shall have

For even values s

$$h_0 = Sh\beta_0 [I_0(B_p) I_0(B_q) + 2\sum_{r}^{\infty} I_{2rs}(B_p) I_{2r}(B_q) \cos 2rx] +$$
$$+ Ch B_0 \, 2\sum_{r}^{\infty} I_{2r-1,s}(B_p) I_{2r-1}(B_q) \cos[(2r-1)x],$$

$$h_p = Sh B_0 \, 2\sum_{r}^{\infty} I_{(2r-1)s+1}(B_p) I_{2r-1}(B_q) \cos[x + (2r-1)x] +$$
$$+ Ch B_0 \, 2\sum_{0}^{\infty} I_{2rs+1}(B_p) I_{2r}(B_q) \cos[x + (2r-1)x],$$

$$h_q = Sh B_0 \, 2\sum_{r}^{\infty} I_{(2r-1)s}(B_p) I_{(2r-1)s+1}(B_q) \cos[y + (2r-1)x] +$$
$$+ Ch B_0 \, 2\sum_{r}^{\infty} I_{2rs}(B_p) I_{2r-1}(B_q) \cos(y + 2rx),$$

where $x = s\varphi - \psi$.

For odd values s

$$h_0 = Sh B_0 [I_0(B_p) I_0(B_q) + 2\sum_{r}^{\infty} I_{rs}(B_p) I_r(B_q) \cos rx],$$

$$h_p = Ch B_0 \, 2\sum_{0}^{\infty} I_{rs+1}(B_p) I_r(B_q) \cos(x + rx),$$

$$h_q = Ch B_0 \, 2\sum_{0}^{\infty} I_{rs}(B_p) I_{r+1}(B_q) \cos(y + rx).$$

Note that the upper sign of Bessel functions indexes corresponds to the upper sign of the

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cosine arguments. Bessel function indexes are always integer and positive.

Let us define power supplied to the core by the e.m.f. sources using the formulas.

Active power

$$P = V\frac{1}{T}\int_0^T H\frac{dB}{dt}dt,$$

or in relative units (see appendix II)

$$P^* = \frac{P}{C_s} = \frac{2}{\rho}\frac{1}{T}\int_0^T h\frac{dB}{dt}dt.$$

Reactive power

$$Q_p = V_p\frac{1}{T}\int_0^T H_p B_p dt, \quad Q_q = V_q\frac{1}{T}\int_0^T H_q B_q dt,$$

or in relative units

$$Q_p^* = \frac{Q_p}{C_s} = 2\frac{1}{T}\int_0^T h_p B_p dt, \quad Q_q^* = 2S\frac{1}{T}\int_0^T h_q B_q dt.$$

In these expressions

$$C_s = \frac{\alpha\rho V}{2\beta},$$

V is the volume of the core ferromagnetic material.

We shall have power expressions.

For even values S

$$P_p^* = -P_q^* = -4S[Sh\beta_o\sum_1^{\infty}(2\kappa-1)I_{(2\kappa-1)S}(B_p)I_{2\kappa-1}(B_q)Sin(2\kappa-1)\chi + Ch\beta_o\sum_1^{\infty}2\kappa I_{2\kappa S}(B_p)I_{2\kappa}(B_q)Sin\,2\kappa\chi],$$

$$Q_p^* = 2B_p\{Sh\beta_o\cdot 2\sum_1^{\infty}I'_{(2\kappa-1)S}(B_p)I_{2\kappa-1}(B_q)Cos(2\kappa-1)\chi + Ch\beta_o[I'_o(B_p)I_o(B_q)+2\sum_1^{\infty}I'_{2\kappa S}(B_p)I_{2\kappa}(B_q)Cos\,2\kappa\chi]\},$$

$$Q_q^* = 2SB_q\{Sh\beta_o\cdot 2\sum_{\kappa=1,3,5}^{\infty}I_{2\kappa}(B_p)I'_{\frac{2\kappa}{S}}(B_q)Cos\frac{2\kappa}{S}\chi + Ch\beta_o[I_o(B_p)I'_o(B_q)+2\sum_{\kappa=2,4}^{\infty}I_{2\kappa}(B_p)I'_{\frac{2\kappa}{S}}(B_q)Cos\frac{2\kappa}{S}\chi]\}$$

For odd values S

$$P_o^* = -P_q^* = -4SCh\beta_o\sum_1^{\infty}\kappa I_{\kappa S}(B_p)I_{\kappa}(B_q)Sin\,\kappa\chi,$$

$$Q_p^* = 2B_pCh\beta_o[I'_o(B_p)I_o(B_q)+2\sum_1^{\infty}I'_{\kappa S}(B_p)I_{\kappa}(B_q)Cos\,\kappa\chi],$$

$$Q_q^* = 2SB_qCh\beta_o[I_o(B_p)I'_o(B_q)+2\sum_1^{\infty}I_{\kappa}(B_p)I'_{\frac{\kappa}{S}}(B_q)Cos\frac{\kappa}{S}\chi]$$

In the power expressions

$I'_{\kappa}(B)$ - Bessel functions derivatives

$$2I'_{\kappa}(B) = I_{\kappa-1}(B) + I_{\kappa+1}(B),$$

$$I'_o(B) = I_1(B).$$

From the received expressions it may be concluded that the examined element with ferromagnetic coupling may work in two directions depending on the sign of the angle χ :

a) active power flux is directed from the lower frequency side (p) to the higher frequency side (q) i.e. the element works in the frequency multiplication duty;

b) the reverse direction of real power flux takes place i.e. the element works in the dividing frequency duty.

Consider, for instance, the frequency multiplication duty.

Let us add a load with admittance Y_q connected in parallel with the outer source U_q (fig.2) to the element circuit.

The apparent power consumed by this load

$$S_{qH} = U_q^2 Y_q.$$

Let us compose the power balance equation

$$S_{q\,acm} = S_q + S_{qH},$$

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[illegible] $S_{q\,ucm}$ [illegible] S_q [illegible] is the apparent power of the source and the power that must be applied to the element to assure the given operating duty.

In the multiplication frequency duty we may be interested in the case, when

$$S_{q\,ucm} = 0$$

and the necessary [illegible] power is given off by a [illegible] i.e.

$$S_{qk} = -S_q$$

This will be possible if the load ad[illegible] according to the following condition

$$[illegible]$$

It is [illegible] connection of such a load takes off from the [illegible] the problem of supplying [illegible] with active and reactive power required to assure the given duty. However the full disconnection of the source from the element circuit becomes possible only in that case [illegible] the circuit itself can ensure the magnetic induction of field necessary for keeping the duty. The control of this possibility is achieved by analyzing of no-load and short circuit conditions of the source U_q.

In these duties the following condition must be kept

$$P_q = 0,$$

from which the values of angle $\chi = 0^\circ$ or 180° are determined, one of them corresponding to short circuit duty (0°) and another to no-load condition.

The outer source may be disconnected from the circuit, if with the given induction value b_p and with certain induction value $b_q = b_{q0}$ the condition

$$Q_q \big|_{\chi = 180^\circ} = 0$$

will be fulfilled.

Short circuit current (magnetic intensity) is defined from the equation for the intensity h_q by substitution of the induction value $b_{qk} = 0$ and the value $\chi = 0^\circ$ into it.

The difference of this current from zero also points out the possibility of disconnecting the outer source from the circuit.

Without these conditions the outer source must be left in the circuit, even if it does not carry any load and furthermore, the inner power losses may be covered by means of power transformation. Thus this source may be a machine of the synchronous compensator type.

Similar conclusions may be made for the dividing frequency duty. For this purpose let us use a circuit with load Y_p connected in parallel to the source U_p (fig.3). The value of this load admittance providing the keeping of a given duty with source power equal to zero is

$$Y_p = \frac{S_p}{U_p^2}.$$

Let us consider some relations characteristic for the work of the element as a [illegible]

[illegible]

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ber of interrelative quantity which may be chosen the parameters determining the duty or, for short determinating parameters". For the investigated element there are five such parameters: relative inductions B_o, B_p and B_q, relative load admittances Y_p^* or Y_q^*, phase angle induction $\chi = S\varphi - \psi$.

We have no possibility to represent graphically the relations between all parameters. It is necessary to take at least two parameters as basic or given ones. It is convenient to take as given parameters the induction values of B_p and B_o.

Then the relations between other parameters may be graphically represented by curves

$$g^* = -\frac{P^*}{U^{*2}} = f_1(B_q, \chi),$$

$$B^* = -\frac{Q^*}{U^{*2}} = f_2(B_q, \chi).$$

Using these curves we may choose the duty and plot changer characteristics, in particular, the outer one.

These relations for certain values of determinating parameters are represented graphically. For all relations the value of relative induction of the „fundamental frequency" $B_p = 6$ is assumed.

In determining the relations we must consider the constant induction components of the core, which can arise due to magnetic flux rectification, this having been noted by M.A.Rosenblatt [3,4].

The possibility of such rectification is seen from the received difference of intensity h_o from zero even in the absence of the outer source U_o, that is physically impossible. The dependance of this intensity from phase angle χ is also physically impossible. As a matter of fact the intensity value h_o must be equal to zero, without outer magnetization of the core by a constant field. It must be equal to

$$h_o = Sh B_o$$

with outer magnitization.

Here B_o' is the core induction constant component depending on m.m.f. of the winding, connected to the constant voltage supply.

From these conditions the "inner" induction component B_o' is determined.

In the general case B_o'' is determined from the expression

$$B_o'' = B_o - B_o'$$

where

$$th B_o = -C_1 \pm \sqrt{C_1^2 - C_2^2},$$

$$C_1 = \frac{A_{o1} A_{o2}}{A_{o1}^2 + Sh^2 B_o'}, \quad C_2^2 = \frac{A_{o2}^2 - Sh^2 B_o'}{A_{o1}^2 + Sh^2 B_o'}.$$

A_{o1} and A_{o2} - Bessel functions series.

For even values S

$$A_{o1} = I_o(B_p) I_o(B_q) + 2\sum_{\tau}^{\infty} I_{2\kappa S}(B_p) I_{2\kappa}(B_q) \cos 2\kappa\chi,$$

$$A_{o2} = 2\sum_{\tau}^{\infty} I_{(2\kappa-1)S}(B_p) I_{2\kappa-1}(B_q) \cos(2\kappa-1)\chi$$

For odd values S

$$A_{o1} = I_o(B_p) I_o(B_q) + 2\sum_{\tau}^{\infty} I_{\kappa S}(B_p) I_{\kappa}(B_q) \cos \kappa\chi,$$

$$A_{o2} = 0.$$

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The sign in the expression $th\,\beta_o$ depends upon the direction of the induction β_c. With the positive sign β_o' it is necessary to take the plus sign and vice versa.

Fig.4 represents the relations between multiplication frequency duty determining parameters and for the even $3 = 2$. Without outer magnetization of the core by a constant field the change is performed within the angle values

$$90° < |-\chi| < 180° \quad (g_q^* > 0).$$

The load of this duty must have a capacity component ($\beta_q^* < 0$).

During this duty the supply source U_q cannot be disconnected from the circuit, as the power Q_q^* with $\chi = 180°$ is reduced to zero only with $\beta_q = 0$, i.e. no-load duty is not accomplished.

With outer magnitization by a constant field ($\beta_o' = 2$) the changing zone is expanded and the connecting of active and induction load ($\beta_q^* \geq 0$) with some values of β_q becomes possible. The outer supply source U_q in this duty may be disconnected from the circuit, as with a certain induction value $\beta_q = \beta_{qo}$ both admittance components and consequently the power components are reduced to zero. In the case in question the no-load induction

$$\beta_{qo} = 0{,}26 .$$

In this magnetization conditions the core represents a well known Jolie-Epstein frequency doubler circuit element.

In fig.5 the relations for the same values of determining parameters are given, but specifically for the dividing frequency duty. In this case the phase angle χ is positive. The nature of relations in these conditions is kept the same, the load, however, must have a capacity component i.e. the changer in these conditions cannot work as a reactive power generator. The source U_p cannot be disconnected from the circuit and must provide the value β_p required to keep the duty.

Figures 6 and 7 represent similar relations for the odd value $3 = 3$.

The phenomenon of magnetic flux rectification in this case can take place only with the outer magnetization of the core by a constant field. The constant magnetic flux component, arising as a result of rectification, decreases the "outer" magnetic flux and in consequence a change of admittances is negligible. For instance, with induction "outer" field value $\beta_o' = 5$ the maximum admittances change in comparison with the duty $\beta_o' = 0$ amounts to about 15%. It corresponds to the published data that the outer magnetization by a constant field has no advantages of output power as compared to the fundamental Spinelli circuit, an element of which is the core in question in given magnetization conditions.

In multiplication frequency duty of this circuit the no-load conditions are fulfilled (in our case $\beta_{qo} = 1.1$) and the source U_q may be

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disconnected from the circuit. In the dividing frequency duty the source $1/\rho$ is required.

It is of some interest to compare the power values, that may be transformed during the performance in multiplication and dividing frequency duties. If we take the maximum power of changer equal to 100% in the multiplication frequency duty, then in the dividing duty the maximum power makes 27,5% for accepted determining parameters values. As for reactive power the percentage of different sides in supplying the changer with reactive power is characterized by the following figures:

Fundamental frequency side	80%
Higher frequency side	20%

The examined core is an element of many different frequency changers, in which nonlinear magnetization characteristic features are used.

Appendix I

hyperbolic sine expansion into series.

Let us use the well known expansions into series including the Bessel functions of imaginery argument

$$e^{x\cos\varphi} = I_0(x) + 2\sum_{\tau}^{\infty} I_{2\kappa}(x)\cos 2\kappa\varphi,$$

$$e^{x\sin\varphi} = I_0(x) + 2\sum_{\tau}^{\infty}(-1)^{\kappa-1} I_{2\kappa-1}(x)\sin(2\kappa-1)\varphi + 2\sum_{\tau}^{\infty}(-1)^{\kappa} I_{2\kappa}(x)\cos 2\kappa\varphi$$

From these series we derive hyperbolic function expansions with arguments, containing circular functions of the variables.

The function expansion

$$h = Sh\,\mathcal{B}$$

where $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2 = \mathcal{B}_0 + \mathcal{B}_p\cos x + \mathcal{B}_q\cos y$

is derived by preliminary function trasformation to the form of

$$h = Sh\,\mathcal{B}_0\,Ch(\mathcal{B}_1+\mathcal{B}_2) + Ch\,\mathcal{B}_0\,Sh(\mathcal{B}_1-\mathcal{B}_2)$$

The hyperbolic sine full series

$$h = Sh\,\mathcal{B}_0\{I_0(\mathcal{B}_p)I_0(\mathcal{B}_q) + 2I_0(\mathcal{B}_p)\sum_{\tau}^{\infty} I_{2\kappa}(\mathcal{B}_q)\cos 2\kappa y + 2I_0(\mathcal{B}_q)\sum_{\tau}^{\infty} I_{2\kappa}(\mathcal{B}_p)\cos 2\kappa x + 2\sum_{\tau}^{\infty}\sum_{\tau}^{\infty} I_{2n}(\mathcal{B}_p)I_{2m}(\mathcal{B}_q)\times\cos 2(nx \mp my) + 2\sum_{\tau}^{\infty}\sum_{\tau}^{\infty} I_{2n-1}(\mathcal{B}_p)I_{2m-1}(\mathcal{B}_q)\cos[(2n-1)x \mp (2m-1)y]\} + Ch\,\mathcal{B}_0\{2I_0(\mathcal{B}_p)\sum_{\tau}^{\infty} I_{2\kappa-1}(\mathcal{B}_q)\cos(2\kappa-1)y + 2\sum_{\tau}^{\infty}\sum_{\tau}^{\infty} I_{2n-1}(\mathcal{B}_p)I_{2m}(\mathcal{B}_q)\cos[(2n-1)x + 2my] + 2\sum_{\tau}^{\infty}\sum_{\tau}^{\infty} I_{2n}(\mathcal{B}_p)\times I_{2m-1}(\mathcal{B}_q)\cos[2nx \mp (2m-1)y] + 2I_0(\mathcal{B}_q)\sum_{\tau}^{\infty} I_{2\kappa-1}(\mathcal{B}_p)\cos(2\kappa-1)x\}$$

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Appendix II

The system of relative units taken in this work

The dimentionless induction and magnetic field intensity values of the core are taken as basic of relative units system.

$$h = \frac{H}{\alpha}, \quad b = \beta B.$$

Then the r.m.s. of circuit voltages and current is expressed

$$U_p = \frac{V_p}{\sqrt{2}\beta\ell} W_1 b_p = C_U U_p^*, \quad U_q = \frac{V_p}{\sqrt{2}\beta\ell} S W_2 b_q = C_U U_q^*,$$

$$I_p = \frac{\alpha\ell h_{pm}}{\sqrt{2}\, W_1} = C_I I_p^*, \quad I_q = \frac{\alpha\ell h_{qm}}{\sqrt{2}\, W_2} = C_I I_q^*.$$

The admittance and power relative values

$$Y = \frac{I}{U} = \frac{\alpha\beta\ell^2}{V_p} \frac{I^*}{U^*} = C_Y Y^*,$$

$$S = UI = \frac{V\alpha p}{2\beta} U^* I^* = C_S S^*.$$

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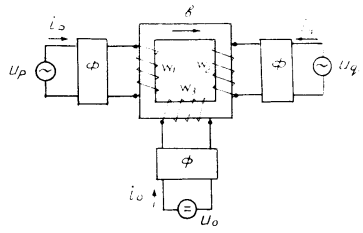

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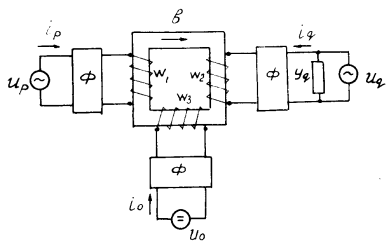

Fig. 2

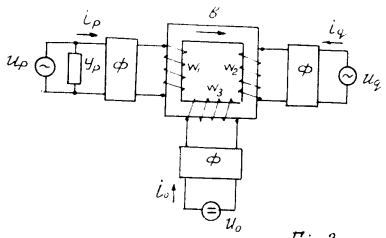

Fig. 3.

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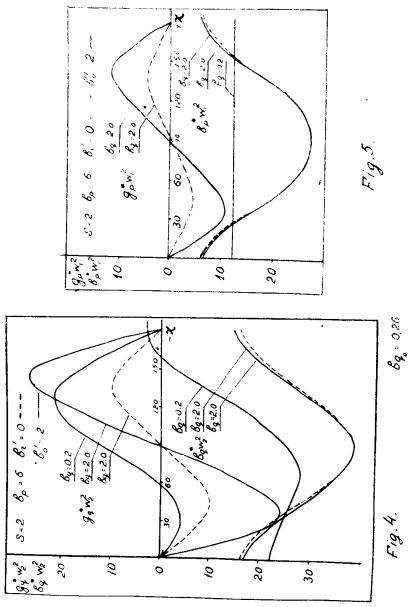
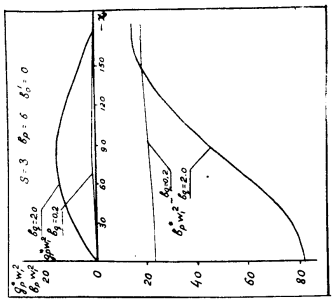


Fig. 4.

Fig. 5.



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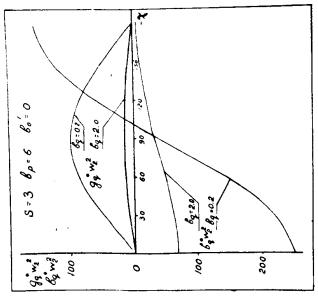

Fig. 6

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1. INTRODUCTION

This paper deals with some problems in the theory of non–linear oscillations of a fluid mass having gravitational forces. The problems to be considered are ones concerned with the essentially unstable flows (those which cannot be reduced to stable ones by a coordinate transformation).

Thus in this report all the classical problems of standing waves will not be touched upon and nothing will be said about them, although the methods developed in such investigations are of definite inte–rest in the oscillations theory.

Despite large value in applications of the theory of non–linear oscillations of fluid masses, there are few papers connected with this theory.

Evidently the first investigation of non–linear oscillations in a fluid was the work by Professor Sekerge–Zenkovich ("To the theory of standing waves of a finite amplitude on the surface of heavy fluid". Reports of the USSR Academy of Sciences, Vol.58, N 4, 1947). His work has direct application in the theory of seches.

In the theory of oscillations of a bounded volume of ideal fluid still another series of essentially non–linear problems arises. Among these, for example, is the question of resonance oscillations of water in a port basin produced by waves arriving from the ocean. In recent years scientists of Grenoble school headed by Professor U. Akrav–chenko have considered a series of problems connected with this

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theory. However these investigations were concerned only with linear problems and attempts to study the dependence of the amplitude of induced waves upon the frequency of the waves inducing them under conditions near to resonance were unsuccessful.

Various problems of auto-oscillations in the theory of a viscous fluid are very interesting, for example, the problem of free oscillations of a spherical layer of a sphere viscous fluid with gravitational forces having a fixed angular velocity. A number of similar examples could be considered.

The central question in considering these problems is the question of mathematical apparatus since the problems mentioned in their strict formulation lie in most complicated classes of boundary value problems in mathematical physics, classes for which there are no general methods of investigation.

Taking into consideration everything that has been said, at first it is natural to find out to what extent the methods of non-linear mechanics of a system with a finite number of degrees of freedom may be applied to the theory of oscillations of a fluid mass.

It appears that in the case of an ideal fluid such analogy may be rather fruitful and be the base of effective methods of analysis.

The situation is much more complicated in the case of a real fluid. Even linear problems are very difficult in this case. Due to this fact in investigating oscillations of a real fluid, the problems of constructing different asymptotic methods must be attacked first.

Since in one report, even in a review report, it is impossible to give a detailed exposition of the theory of fluid oscillations, in this paper I shall touch upon only three questions, which, it seems to me, should be considered first of all: the problem of free oscillations of bounded mass of heavy liquid with gravitational forces, the problem of resonance oscillations of the same mass, and the methods of asymptotic analysis in the problem of linear oscillations of a viscous fluid. It seems, that if we have the solution of these three problems we shall be able to consider a fairly wide class of problems in the theory of oscillations of a fluid.

2. FREE OSCILLATIONS OF FLUID

Assuming the flow in potential and in incomassible fluid we may formulate the problem of free oscillations of a fluid mass as follows:

to define periodic functions of t

$$\varphi(P,t)\,(P\in\tau)$$

and

$$z=\zeta(P,t)\,(P\in S)$$

satisfying the following equations and boundary conditions (see fig.1):

$$\left.\begin{aligned}
&\Delta\varphi=0; \quad P(x,y,z)\in\tau;\\
&\frac{d\zeta}{dt}=\zeta_t+\zeta_x\varphi_x+\zeta_y\varphi_y=\varphi_z; \quad z=\zeta\\
&\frac{\partial\varphi}{\partial n}=0; \quad P\in\Sigma;\\
&\varphi_t+g\zeta+\frac{1}{2}(\nabla\varphi)^2=0, \quad z=\zeta
\end{aligned}\right\} \qquad (2,1)$$

here $z=\zeta(x,y,t)$ is a free surface, $\bar{g}=-g\bar{z}^0$ is the vector of gravity field tensity, S – is the plane figure which coincides with the free surface in the position of equilibrium.

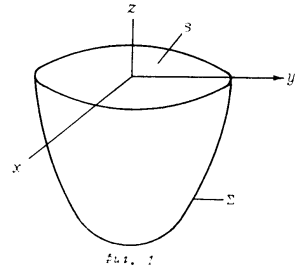

Fig. 1

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We suppose that we can solve the corresponding linear problem. At present there are already different methods of solving the linear problem (above all variational methods). Therefore this hypothesis makes sense beyond those cases, when the solution of linear problem may be obtained by elementary methods (for instance, by means of dividing variables).

We shall express this fact in the following way:

$$\left.\begin{aligned} &\varphi(P) = H\zeta_t(Q); \\ &P \in \tau_0; \quad Q \in S, \end{aligned}\right\} \tag{2,2}$$

here τ_0 the region occupied by the fluid in the equilibrium position. It is not difficult to show, that H is a completely continuous symmetric operator. Let λ_n and ψ_n be its eigen-values and normalized eigen functions. Supposing the operator H is effectively constructed, we use the last condition from (2,1) to define its characteristic frequencies. To do this it is necessary to linearize and to put

$$\zeta = \psi_n \cos \sigma_n t .$$

Then for $P \in S$

$$\varphi_n(P, t) = -\sigma_n \sin \sigma_n t \cdot H\psi_n = -\frac{\sigma_n}{\lambda_n} \sin \sigma_n t \cdot \psi_n(P) \tag{2,3}$$

and hence

$$\varphi_t + g\zeta = -\frac{\sigma_n^2}{\lambda_n} \cos \sigma_n t \cdot \psi_n + g\psi_n \cos \sigma_n t = 0,$$

from which it follows that $\sigma_n^2 = g\lambda_n$.

Having these results, we can obtain the solution of the problem following the Lyapunov's method that is finding the solution of the period close to $T_m = \frac{2\pi}{\sigma_m}$.

For this we write

$$\theta = \frac{\sigma_m t}{1 + \Sigma h_n \varepsilon^n}; \quad \varphi = \varepsilon \Sigma \varphi_n \varepsilon^n; \quad \zeta = \varepsilon \Sigma \zeta_n \varepsilon^n, \tag{2,4}$$

here ε is a small parameter. Its physical sense is a wave amplitude. The h_n are unknown numbers. The functions φ_1 and ζ_1 satisfy the linear problem

$$\left.\begin{aligned} &\Delta\varphi_1 = 0 \quad \text{in} \quad \tau_0 \\ &\varphi_{1z} = \zeta_{1t}; \\ &\varphi_{1\theta} + \frac{g}{\sigma_m}\zeta_1 = 0 \end{aligned}\right\} \quad P \in S. \tag{2,5}$$

It is required to find the periodic solutions in t with period 2π. It is easily verified that these solutions are given by the formulas

$$\zeta_1 = f_1 \psi_m;$$

$$\varphi_1(P) = f_1' H\psi_m; \quad P \in \tau_0;$$

$$\varphi_1(P) = \frac{f_1' g \psi_m}{\sigma_m^2}; \quad P \in S.$$

The function $f_1(\theta)$ satisfies the equation

$$f_1'' + f_1 = 0. \tag{2,6}$$

The functions φ_2 and ζ_2 also satisfy the linear problem, but the conditions (2,5) will be not homogeneous

$$\left.\begin{aligned} &\varphi_{2z} = \zeta_{2t} + A_2(\varphi_1, \zeta_1, \theta, h); \\ &\varphi_{2\theta} + \frac{g}{\sigma_m}\zeta_2 = B_2(\varphi_1, \zeta_1, \theta, h). \end{aligned}\right\} \tag{2,7}$$

here A_2 and B_2 are functions θ, x and y which are determined by the first approximation.

The solution of the problem (2,7) may be given in the following way

$$\zeta_2 = \Sigma f_{2k} \psi_k; \quad \varphi(P, \theta) = \Sigma f_{2k}' H\psi_k. \tag{2,8}$$

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The functions f_{2k} satisfy the following system of ordinary differential equations:

$$\left.\begin{aligned} k \neq m;\quad & f_k'' + \frac{\sigma_k^2}{\sigma_m^2} f_k = A_{2k}(\theta, h); \\ k = m;\quad & f_m'' + f_m = A_{2m}(\theta, h). \end{aligned}\right\} \qquad (2,9)$$

The functions A_{2k} have the form

$$A_{2k} = a_{2k} \cos \theta + b_{2k} \sin 2\theta, \qquad (2,10)$$

where a_{2k} and b_{2k} are numbers.

In order that the systems (2,9) and (2,10) had have periodic solutions with the period 2π it is necessary and sufficient, that

$$a_{2m} = 0. \qquad (2,11)$$

The equation (2,11) determines the number h_1 the first correction to the frequency. It appears that $h_1 = 0$.

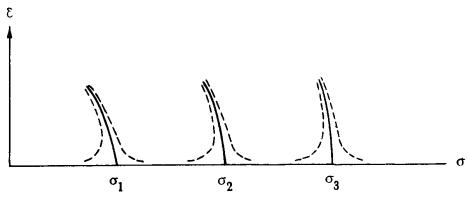
It is easy to show that the mentioned scheme allows to calculate the functions ζ_n and φ_n of any index n. It should be noted that numbers $h_i = 0$ for all odd i. Hence, the determination of the first correction to the frequency requires calculation of the third approximation. Thus, m-th free oscillation has the following free surface

$$\zeta = \varepsilon \psi_m \sin \frac{\sigma_m t}{1 + h_2 \varepsilon^2 + \cdots} + \varepsilon^2 (\cdots) + \cdots \qquad (2,12)$$

In this way the Lyapunov – Poincare method may be formally extended to the case of free oscillations of a fluid.

Remarks. 1. The theory developed is of a formal character. We know nothing about the convergence of the series used. At the same time there is reason to believe we may be sure that the formula, defining the dependence of the amplitude on the frequency defines the dependence which exists in nature with good accuracy. Therefore a rigorous analysis of the boundary layer problem (2,1) represents a very important mathematical problem.

2. The problem (2,1) is a problem about eigen–values. The theory described above allows one to define the structure of the spectrum and its dependence on the amplitude. The spectrum is an infinite union of finite intervals (see fig. 2).



фиг. 2

In spite of the existence of the analogy between fluid oscillations and free oscillations of conservative system whith a finite number of degrees of freedom there is an essential singularity in the fluid oscillation theory. Periodic motions of a fluid can exist only in the case of sufficiently low (small) amplitudes. This fact is connected with the fact that the waves of a large slope destroy themselves.

3. FORCED OSCILLATIONS

Let us consider the problem of the oscillations of a fluid mass under the action of an external force periodic in t and directly proportional to the mass and the tensity of the fluid. Let us designate by means of $U(t, x, y)$ the potential of the perturbing force. Then it is necessary to replace the last to the conditions (2,1) by the following one

$$\varphi_t + g\zeta + -(\nabla\varphi)^2 = U(t, x, y). \qquad (3,1)$$

Let us consider the most important case, when

$$U = \frac{\mu}{\omega} \sin \omega t F(x, y), \qquad (3,2)$$

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and rise the problem of finding periodic solutions of the period $T = \frac{2\pi}{\omega}$. For $\mu = 0$ the problem will describe oscillations of some conservative system and therefore a usual quasilinear treatment of the problem cannot be sufficient. It is natural to consider our system as the system close to the Lyapunov system and to see the solutions which at $\mu \to 0$ become the solutions of the problem (2,1). For $\mu = 0$ in the system (2,1) there exist periodic solutions, the period of which depends on the amplitude.

In this case under consideration the period is given

$$T = T_n = \frac{2\pi}{n\omega},$$

where n is any integer. According to (2,12) the period of oscillation is connected with the amplitude ε by the following formula:

$$T = \frac{2\pi(1 + h_2\varepsilon^2 + h_4\varepsilon^4 + \cdots)}{\sigma_m}. \tag{3,3}$$

Thus for the sought for amplitude ε we obtain the equation:

$$\left.\begin{aligned} 1 + h_2\varepsilon^2 + h_4\varepsilon^4 + \cdots &= \frac{\sigma_m}{n\omega} \\ \text{or} \\ \varepsilon^2 h_2 + \varepsilon^4 h_4 + \cdots &= \frac{\sigma_m - n\omega}{n\omega}. \end{aligned}\right\} \tag{3,4}$$

This expression makes it clear that the problem posed definitely does not have a unique solution. The physical sense of this circumstance is following. There can exist solutions, at $\mu \to 0$ becoming trivial, but there can exist solutions which at $\mu \to 0$ become free oscillations, which were mentioned in the previous section.

The most interesting (and the most difficult) case is the resonance case when the frequency of the disturbing force ω is close to one of the eigen frequencies of the free oscillations.

Let us show, how the Lyapunov method can be used for finding solutions which become trivial at $\mu \to 0$ under conditions that the numbers $\sqrt{\lambda_n}$ are irrational. Assuming the frequency ω is close to σ_m let us put

$$\sigma_m^2 = \omega^2 - \mu\alpha;$$
$$\xi = \frac{\sigma_m}{\lambda_m} = \frac{\omega^2}{\lambda_m} - \mu\frac{\alpha}{\lambda_m}.$$

It is natural to seek solutions of the problem in the form of series

$$\varphi = \Sigma\varphi_n\mu^{na/b}; \quad \zeta = \Sigma\zeta_n\mu^{na/b}. \tag{3,5}$$

It appears that the series of the form (3,5) define a non–trivial solution only in the case when $\frac{a}{b} = \frac{1}{3}$.

Thus

$$\varphi = \mu^{1/3}\varphi_1 + \mu^{2/3}\varphi_2 + \mu\varphi_3 + \cdots;$$
$$\zeta = \mu^{1/3}\zeta_1 + \mu^{2/3}\zeta_2 + \mu\zeta_3 + \cdots$$

The functions φ_1 and ζ_1 will satisfy the linear problem

$$\varphi_{1t} + \frac{\omega^2}{\lambda_m}\xi_1 = 0; \quad \zeta_{1t} = \varphi_{1\tau}.$$

Supposing, as we did earlier, that

$$\zeta_1 = \Sigma f_{n1}\psi_n, \tag{3,6}$$

we obtain the following equation for the function f_{n1}

$$f''_{n1} + \frac{\lambda_n}{\lambda_m}\omega^2 f_{n1} = 0.$$
$$n = 1, 2, 3 \ldots$$

The only periodic solution of this equation having the period $\frac{2\pi}{\omega}$ is

$$f_{n1} = 0, \text{ if } n \neq m;$$
$$f_{m1} = N_1 \sin\omega t + N_1 \cos\omega t.$$

The numbers N_1 and N_1 are unspecified constants.

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The functions φ_1 and ζ_1 will also satisfy a linear problem, but it will not be homogeneous one. If we look solutions of this problem in the form of sums of the type (3,6), we shall obtain the following equations for the functions f_2:

$$f''_{n2} + \frac{\lambda_n}{\lambda_m}\omega^2 f_{n2} = \pi_1^{(n)} \sin 2\omega t + \pi_2^{(n)} \cos 2\omega t \quad , \qquad (3,7)$$

where $\pi_1^{(n)}$ and $\pi_2^{(n)}$ are second degree polynomials in N_1 and M_1.

By hypothesis there do not exist eigen-values λ_n and λ_m, so that $\sqrt{\frac{\lambda_n}{\lambda_m}} = 2$, thus we can always construct periodic solutions with the required period

$$f_n = A_n \sin 2\omega t + B_n \cos 2\omega t; \quad n \neq m;$$
$$f_m = A_m \sin 2\omega t + B_m \cos 2\omega t + M_2 \sin \omega t + N_2 \cos \omega t ,$$

here A_n and B_n are constants, which depend upon N_1 and M_1. The constants M_2 and N_2 are not specified.

Equations of the third approximation are formed in exactly the same way. These equations will be analogous to the equations (3,7), but in their right members they will have terms of the kind

$$\pi_{11}^{(n)} \sin 3\omega t + \pi_{12}^{(n)} \cos 3\omega t + R_1^{(n)} \cos \omega t + R_2^{(n)} \sin \omega t ,$$

where $\pi_{11}^{(n)}$, $\pi_{12}^{(n)}$, $R_1^{(n)}$, $R_2^{(n)}$ are third degree in N_1 and M_1 and where $R_1^{(n)}$ and $R_2^{(n)}$ do not depend on the numbers N_2 and M_2.

In order that this system have a solution it is necessary and sufficient that

$$\left.\begin{aligned} R_1^{(m)}(N_1, M_1) &= 0; \\ R_2^{(m)}(N_1, M_1) &= 0. \end{aligned}\right\} \qquad (3,8)$$

The system (3,8) is a system of two cubic equations in N_1 and M_1. It is not difficult to see that we can continue this process indefinitely.

The present method has been used in several problems. In particular, Cherkasov, a graduate student at the Steklov Mathematical Institute of the AN USSR calculated the resonance oscillations of water induced by waves coming from the open sea into a part of rectangular shape. Besides that he made calculations of stationary waves which can be produced by a wave inducer in a canal of finite length if the oscillation frequency of the wave inducer is close to one of the characteristic frequencies of the canal.

4. VISCOUS FLUID OSCILLATIONS

1. In previous sections of this paper the possibilities which are offered by the use of methods, generalizing, in the proper way, classical methods of non-linear mechanics in investigations of ideal fluid oscillations. The possibilities of these methods are based on our ability to solve corresponding linear problems. It is extremely difficult to solve these auxiliary problems in the case of the viscous fluid oscillation theory. Therefore the investigation of the linear problem should procede the reduction of the non-linear problem to the linear one.

Apparantly, from the applied point of view it is more interesting to study oscillations of a fluid of small viscousity. In this case essential simplifications may appear.

In this paper the problem of oscillations of a gravitating fluid sphere will be used as an example to show what simplification may arise from this supposition. The oscillation of a fluid sphere under the action of gravitational forces may be described by the following equations

$$\bar{v}_t = -\frac{1}{\rho}\nabla p + \nabla U_1 + \nabla U_2 + \nu \Delta \bar{v}; \qquad (4,1)$$

$$\nabla \bar{v} = 0, \qquad (4,2)$$

where $U_1(x, y, z)$ is the potential of gravitational forces; and where $U_2 = U^*(x, y, z)\cos \omega t$ is the potential of disturbing forces.

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In the equilibrium position $U^* = 0$, $v = 0$, $p = p_0$. The function $\bar{v}$ must satisfy the conditions of absence of strain on the surface of the sphere. To write these conditions we shall introduce a local system of Cartesian coordinates s, s, r. Then the conditions of absence of strain will be of the following form ($H_r \equiv 1$):

$$p_{31} = \mu\{\frac{1}{H_1} v_{3s_1} + v_{1r} - \frac{v_1}{H_1} H_{1s_2}\} = 0; \qquad (4,3)$$

$$p_{32} = \mu\{\frac{1}{H_2} v_{3s_2} + v_{2r} - \frac{v_2}{H_2} H_{2s_2}\} = 0; \qquad (4,4)$$

$$p_{33} = -p + 2\mu v_{3r}. \qquad (4,5)$$

An unknown form of the free boundary

$$r = f(t, s_1, s_2)$$

is defined by the equation

$$r_t = v_3 \qquad (4,6)$$

2. Let us assume $\bar{v} = \nabla\varphi + \bar{u}$, where

$$\Delta\varphi = 0; \quad \psi_t = \nu\Delta\psi; \quad \chi_t = \nu\Delta\chi; \qquad (4,7)$$

$$u_1 = \psi_z - \chi_y; \quad u_2 = \chi_z + \chi_x; \quad u_3 = -\psi_x - \chi_y. \qquad (4,8)$$

By such a choice of the functions φ, ψ and χ we shall satisfy the equation of continuity (4,2).

Remark. In the case of the arbitrary ortogonal system of the coordinates s_1, s_2, s_3 the equation of continuity has the form

$$(v_1 H_2 H_3)_{s_1} + (v_2 H_1 H_3)_{s_2} + (v_3 H_1 H_2)_{s_3} = 0;$$

therefore the representation (4,8) should be written in the form

$$\left.\begin{aligned} H_2 H_3 u_1 &= \psi_{s_3} - \chi_{s_2}; \\ H_1 H_3 u_2 &= \chi_{s_1} + \chi_{s_3}; \\ H_1 H_2 u_3 &= -\psi_{s_1} - \chi_{s_2}. \end{aligned}\right\} \qquad (4,8')$$

If we subject the function p to the condition

$$\frac{1}{\rho} p = -\varphi_t + U_1 + U_2, \qquad (4,9)$$

we shall also satisfy the equation of motion (4,1).

The problem of finding the three unknown functions φ, ψ and χ affords an opportunity to develop an asymptotical theory for small ν.

3. We shall seek solutions in the following form:

$$v = v' e^{\sigma t}; \quad p = \rho U_1 + p' e^{\sigma t}; \quad f - r_0 = F e^{\sigma t}, \qquad (4,10)$$

where r_0 is the boundary of the sphere in the position of equilibrium. When the functions ψ and χ will satisfy the equations

$$\left.\begin{aligned} \sigma\psi &= \nu\Delta\psi; \\ \sigma\chi &= \nu\Delta\chi. \end{aligned}\right\} \qquad (4,11)$$

If ν is small, then the functions ψ and χ will noticeably differ from zero only in the neighbourhood of the boundary $r = f$. In this neighbourhood we cannot neglect the guaranties (values) ψ and χ since the potential component of the velocity vector can satisfy only one of the conditions (4,3) and (4,5), for instance, the condition (4,5). Hence, on the boundary of the oscillating sphere the functions ψ and χ will compensate the fact that there are too many boundary conditions.

Since ν is small, then it is reasonable to replace ψ and χ by their asymptotic representations. The form of the equation (4,11) permits application of the standart methods for constructing asymptotics.

In the neighbourhood of a point P lying on the boundary, we shall perform a stretching of the coordinate r For this we shall set

$$r - r_0 = \xi\sqrt{\nu}$$

in equations (4,11).

Then disregarding terms, containing ν as a multiplier we shall bring the equations (4,11) to the form

$$\left.\begin{aligned} \sigma\psi &= \psi_{\xi\xi}; \\ \sigma\chi &= \chi_{\xi\xi}. \end{aligned}\right\} \qquad (4,12)$$

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Particular solutions of these equations are of the following form:

$$\left.\begin{aligned} \varphi &= C(s_1, s_2)\exp \pm \sqrt{\sigma}\xi; \\ \chi &= D(s_1, s_2)\exp \pm \sqrt{\sigma}\xi, \end{aligned}\right\} \qquad (4,13)$$

where C and D are arbitrary functions s_1 and s_2.

We must find solutions, disappearing at $\xi \to -\infty$. Let us agree to take that value $\sqrt{\sigma}$ for which

$$Re\ \sqrt{\sigma} > 0.$$

Then the sought for representations will be the following:

$$\left.\begin{aligned} \varphi &= C(s_1, s_2)\exp \sqrt{\sigma/\nu}\,(r - r_0); \\ \chi &= D(s_1, s_2)\exp \sqrt{\sigma/\nu}\,(r - r_0). \end{aligned}\right\} \qquad (4,14)$$

In order to define the functions C and D we use the conditions (4,3) and (4,4), which will give us two equations in two unknown functions. It is easy to see, that

$$C = 0(\nu); \quad D = 0(\nu).$$

Therefore it is natural to write them in the form

$$C = \nu C^*; \quad D = \nu D^*$$

and look for C^* and D^*.

Using the expression (4,8′), we bring the conditions (4,3) and (4,4) to the form

$$\left.\begin{aligned} &\frac{\sigma C^*}{H_2} + \varphi_{rs_1}\left(\frac{1}{H_1} + 1\right) - \varphi_{s_1}(\ln H_1)_r + 0(\sqrt{\nu}) = 0; \\ &\frac{\sigma D^*}{H_1} + \varphi_{rs_2}\left(\frac{1}{H_2} + 1\right) - \varphi_{s_2}(\ln H_2)_r + 0(\sqrt{\nu}) = 0. \end{aligned}\right\} \qquad (4,15)$$

From these equations we immediately find the functions C^* and D^*. Thus, the formulas (4,14) may be rewritten as

$$\left.\begin{aligned} \varphi &= \nu\left[a_{11}\varphi_{rs_1} + a_{12}\varphi_{s_1}\right]_{r=r_0}\exp \sqrt{\frac{\sigma}{\nu}}(r - r_0) + 0(\nu^{3/2}); \\ \chi &= \nu\left[a_{21}\varphi_{rs_2} + a_{22}\varphi_{s_2}\right]_{r=r_0}\exp \sqrt{\frac{\sigma}{\nu}}(r - r_0) + 0(\nu^{3/2}), \end{aligned}\right\} \qquad (4,16)$$

where a_{ij} is some known function of s_1 and s_2.

Hence, the functions φ and χ are expressed by way of boundary values of derivatives of φ. This makes it possible to exclude the functions φ and χ from the boundary condition (4,5). Using (4,9), where $f_1(s_1, s_2, t)$ is replaced by

$$f_0 + (t - r_0)f_r(r_0)$$

we shall bring the condition (4,5) to the form:

$$\left.\begin{aligned} &\varphi + \lambda F + 2\nu\varphi_{rr} + 0(\nu^{3/2}) = 0, \\ &\text{where} \\ &F = \varphi_r - \nu\frac{1}{H_1H_2}\left(a_{21}\varphi_{rs_2} + a_{22}\varphi_{s_2}\right)_{s_2} - \\ &\quad - \nu\frac{1}{H_1H_2}\left(a_{11}\varphi_{rs_1} + a_{12}\varphi_{s_1}\right)_{s_1} \end{aligned}\right\} \qquad (4,17)$$

here λ is a constant.

In this way the problem is reduced to satisfying Laplace equation together with a nonself–adjoint boundary condition (4,17) and thus a principal simplification of the problem is made. In the case we have considered, oscillations of a sphere, the problem may be completely solved since in this case it is possible to separate variables and the solution may be explicitly obtained.

CONCLUSION

We aim in this paper was not to consider the solution of specific concrete problems. I only wanted to indicate some possible approaches

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for investigations. It is clear that difficulties of the developed theory are such that many years will pass before the theory of oscillations of a continuous medium reaches the level of the oscillation theory of systems with a final number of degrees of freedom.

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Б. И. МОСЕЕНКОВ

ИССЛЕДОВАНИЕ НЕСТАЦИОНАРНЫХ ОДНОЧАСТОТНЫХ РЕЖИМОВ КОЛЕБАНИЙ В СИСТЕМАХ С РАСПРЕДЕЛЕННЫМИ ПАРАМЕТРАМИ

The theory of [illegible] machines of to[illegible] lighter constructions requires an exact dynamical analysis as to the strength of the elements of the constructions and separate units, taking into account all the phenomena, appearing in the process of vibration of these constructions, units and details.

When investigating non-stationary vibrations of such systems with distributed parameters (if the distributed masses are taken into account) great mathematical difficulties arise as corresponding partial differential equations /linear and non-linear/ contain variable coefficients depending upon time. As the obtaining of general solutions of the indicated equations is impossible in most cases, so it is of great interest to pay attention to the construction by the asymptotical methods of the two-parametrical families of the partial solutions, which correspond to the solo-frequent regimes of non-stationary vibrations in a definite form of the dynamical equilibrium.

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1. We shall consider briefly the procedure of investigation of the solo-frequent vibrations in the systems with N degrees of freedom. For the convenience of the application of the obtained results to the solution of practical problems, the method is worked out by Yu.A.Mitropolsky[4] directly for the systems of Lagrange equations of the second order, which can be written as:

$$\frac{d}{dt}\left(\sum_{i=1}^{N} a_{ij}(\tau)\dot{q}_i\right)+\sum_{i=1}^{N} b_{ij}(\tau)q_i=\varepsilon Q(\tau,\theta,q_1,\dots,q_N,\dot{q}_1,\dots,\dot{q}_N,\varepsilon). \quad (j=1,2,\dots,N) \tag{1.1}$$

In the equations (1.1) ε is a small positive parameter, t - time, $\tau=\varepsilon t$ - slow time, $q_1,\dots,q_N$ generalized coordinates, $a_{ij}(\tau)=a_{ji}(\tau)$ and $b_{ij}(\tau)=b_{ji}(\tau)$ infinitely derivable functions for all finite values of τ, $Q_j(\tau,\theta,q_1,\dots,q_N,\dot{q}_1,\dots,\dot{q}_N,\varepsilon)$ are periodic in θ (with the period 2π) functions which are infinitely derivable for all finite values of the variables and sufficiently small ε, and $\frac{d\theta}{dt}=\gamma(\tau)$ non-negative infinitely-derivable function for all finite values of τ.

For the perturbated system (1.1) asymptotic solution is being built, corresponding to solo-frequent vibrations, close for ε sufficiently small to one of the normal vibrations of the unperturbated system /system (1.1) for $\varepsilon=0$ /. If one assumes that the considered vibrations are close to the first normal unperturbated vibration and in unperturbated system for all values of the parameter τ $(0\le\tau\le L)$ undamped vibrations

of frequency $\omega_1(\tau)$ are possible and internal resonances $(\omega_1(\tau)\neq p_\kappa\omega_\kappa(\tau);\ \kappa=2,3,\dots,N)$ are absent then the solution of the system (1.1) is expressed in the form of asymptotic series:

$$q_i=\varphi_i^{(1)}(\tau)a\cos(s\varphi+\psi)+\varepsilon U_i^{(1)}(\tau,a,\theta,s\varphi+\psi)+\varepsilon^2U_i^{(2)}(\tau,a,\theta,s\varphi+\psi)+\dots \quad (i=1,2,\dots,N). \tag{1.2}$$

In the series (1.2) the functions $U_i^{(s)}(\tau,a,\theta,s\varphi+\psi)\Big|_{s=1,2,\dots}^{i=1,2,\dots,N}$ are periodic in θ and $s\varphi+\psi$ of period 2π, $\varphi=\frac{1}{r}\theta$, s and r are mutually prime small integers, the choice of which depends on the resonance being investigated, $\varphi_j^{(1)}(\tau)$ are nontrivial solutions of the homogeneous algebraic equations of the system

$$\sum_{i=1}^{N}\left[-a_{ij}(\tau)\omega_\kappa^2(\tau)+b_{ij}(\tau)\right]\varphi_i^{(\kappa)}(\tau)=0, \quad (j,\kappa=1,2,\dots,N) \tag{1.3}$$

a and ψ are determined by the system of differential equations:

$$\frac{da}{dt}=\varepsilon A_1(\tau,a,\psi)+\varepsilon^2A_2(\tau,a,\psi)+\varepsilon^3\dots$$
$$\frac{d\psi}{dt}=\omega_1(\tau)-\frac{s}{r}\gamma(\tau)+\varepsilon B_1(\tau,a,\psi)+\varepsilon^2B_2(\tau,a,\psi)+\varepsilon^3\dots, \tag{1.4}$$

so the finding of two-parametrical family of private solutions of the equations (1.1) corresponding to solo-frequent vibrations of the perturbated system is reduced to the determining of the functions.

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$$U_i^{(\ell)}(\tau,a,\theta,s\varphi+\psi),\ A_\ell(\tau,a,\psi)\ \text{and}\ B_\ell(\tau,a,\psi)\ \begin{pmatrix}\ell=1,2,3,\dots\\ i=1,2,\dots,N\end{pmatrix} \tag{1.5}$$

and to the integration of system of only two equations (1.5). The procedure of determining the unknown functions (1.5) offers no difficulty.

According to the proposed method it is possible to construct any approximation of the asymptotic solution of the system (1.1). Thus, the first approximation has the form:

$$q_j = \varphi_j^{(1)}(\tau)\,a\cos(s\varphi+\psi),$$
$$(j=1,2,\dots,N) \tag{1.6}$$

where a and ψ are termined by the first approximation equations:

$$\begin{cases}\dfrac{da}{dt}=\varepsilon A_1(\tau,a,\psi),\\[2ex] \dfrac{d\psi}{dt}=\omega_1(\tau)-\dfrac{s}{r}\gamma(\tau)+\varepsilon B_1(\tau,a,\psi).\end{cases} \tag{1.7}$$

The expressions for the functions $A_1(\tau,a,\psi)$ and $B_1(\tau,a,\psi)$ being obtained as partial periodic solutions in ψ from the following system:

$$\begin{cases}(\omega_1-\dfrac{s}{r}\gamma)\dfrac{\partial A_1}{\partial\psi}-2a\omega_1 B_1=\dfrac{1}{2\pi^2 m_1}\sum_{s=-\infty}^{\infty}e^{is\theta+\gamma}\int_0^{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}Q_{js}^{(1)}\varphi_j^{(1)}e^{-is\theta-\gamma}\cos(s\varphi+\psi)\,d\theta\,d(s\varphi+\psi),\\[2ex] (\omega_1-\dfrac{s}{r}\gamma)a\dfrac{\partial B_1}{\partial\psi}+2\omega_1 A_1=-\dfrac{a}{m_1}\dfrac{d(m_1\omega_1)}{d\tau}-\dfrac{1}{2\pi^2 m_1}\sum_{s=-\infty}^{\infty}e^{is\theta+\gamma}\times\\ \times\int_0^{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}Q_{js}^{(1)}\varphi_j^{(1)}e^{-is\theta-\gamma}\sin(s\varphi+\psi)\,d\theta\,d(s\varphi+\psi).\end{cases} \tag{1.8}$$

where

$$m_1=\sum_{j=1}^{N}a_{ij}\varphi_i^{(1)}\varphi_j^{(1)}=2T\left[\varphi^{(1)}\right]. \tag{1.9}$$

$$Q_{js}^{(1)}=Q_j^{(1)}\left[\tau,\theta,\varphi_i^{(1)}a\cos(s\varphi+\psi),\ \dots,\ \varphi_i^{(1)}a\omega_1\sin(s\varphi+\psi),\ \dots\right]. \tag{1.10}$$

The first approximation equations and of cause the higher approximations equations are not generally integrable in quadratures and so the numerical methods of integration are required if not to limit with qualitative investigations.

The application of numerical integration methods is simplified considerably because of the proportionality of the right-hand sides of these equations to small parameter ε on the resonance interval of the change of τ.

2. The discussed method of constructing the first approximation equations and also higher approximations equations permits a simple energetical interpritation [4].

Indeed, let us consider the expression of the virtual work δW which were accomplished by the perturbative forces Q_j $(j=1,2,3,\dots,N)$ in the regime of sinusoidal vibrations.

$$q_i=\varphi_i^{(1)}(\tau)\,a\cos(s\varphi+\psi),\ \dot q_i=-\varphi_i^{(1)}(\tau)\,a\omega_1\sin(s\varphi+\psi) \tag{2.1}$$

on virtual deflections

$$\delta q_i=\varphi_j^{(1)}(\tau)\cos(s\varphi+\psi)\delta a-\varphi_j^{(1)}(\tau)\,a\sin(s\varphi+\psi)\delta\psi. \tag{2.2}$$

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corresponding to the variations of amplitude and to the phase of the first "normal" vibration.

Then, to within the first order terms,

$$\delta W=\sum_{j=1}^{N} Q_{j0}^{(1)}\left[\varphi_j''(\tau)\cos(s\varphi+\psi)\delta a-\varphi_j''(\tau)a\sin(s\varphi+\psi)\delta\psi\right]. \tag{2.3}$$

Let us designate by $\overline{\delta W}$ the mean value of this work in the whole cycle of vibration, that is

$$\overline{\delta W}=\frac{1}{2\pi}\int_0^{2\pi}\delta W\,d(s\varphi+\psi). \tag{2.4}$$

Having expanded $Q_{j0}^{(1)}$ $(j=1,2,\ldots,N)$ into Fourier double series, we have from (2.3) and (2.4)

$$\overline{\delta W}=\frac{1}{4\pi^2}\sum_{\sigma=-\infty}^{\infty}e^{i\sigma r\psi}\int_0^{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}Q_{j0}^{(1)}\varphi_j''e^{-i\sigma r\psi}\cos(s\varphi+\psi)\,d\theta\,d(s\varphi+\psi)\,\delta a-$$

$$-\frac{a}{4\pi^2}\sum_{\sigma=-\infty}^{\infty}e^{i\sigma r\psi}\int_0^{2\pi}\int_0^{2\pi}\sum_{j=1}^{N}Q_{j0}^{(1)}\varphi_j''e^{-i\sigma r\psi}\sin(s\varphi+\psi)\,d\theta\,d(s\varphi+\psi)\,\delta\psi. \tag{2.5}$$

Let us use the symbolic notations $\overline{\delta W}/\delta a$ and $\overline{\delta W}/\delta\psi$ for the coefficients $\overline{\delta W}$ in the variations δa and $\delta\psi$. Then the system of equations (1.8) determining the functions $A_1(\tau,a,\psi)$ and $B_1(\tau,a,\psi)$ looks like:

$$\begin{cases}\left(\omega_1-\dfrac{s}{r}\gamma\right)\dfrac{\partial A_1}{\partial\psi}-2a\omega_1B_1=\dfrac{2}{m_1}\dfrac{\overline{\delta W}}{\delta a},\\ a\left(\omega_1-\dfrac{s}{r}\gamma\right)\dfrac{\partial B_1}{\partial\psi}+2\omega_1A_1=-\dfrac{a}{m_1}\dfrac{d(m_1\omega_1)}{d\tau}+\dfrac{2}{m_1}\dfrac{\overline{\delta W}}{\delta\psi}\end{cases} \tag{2.6}$$

Now it is not difficult to give the energetical interpretation directly for the equations of the first approximation. For this purpose let us represent the mean value of virtual work in the form of the following sum:

$$\overline{\delta W}=\sum_{\sigma=-\infty}^{\infty}\overline{\delta W}_\sigma=\sum_{\sigma=-\infty}^{\infty}\left\{\frac{\overline{\delta W}_\sigma}{\delta a}\delta a+\frac{\overline{\delta W}_\sigma}{\delta\psi}\delta\psi\right\}, \tag{2.7}$$

where $\overline{\delta W}_\sigma$ designates the mean value of virtual work which σ-term of the Fourier expansion for excitative force in the sinusoidal regime would carry out at cycle of vibrations on the virtual deflections, corresponding to the variation of the amplitude and phase of the vibration.

Taking into consideration the notations (2.7) the functions A_1 and B_1 may be easily obtained from the system (2.6) as partial solution periodic in ψ. As a result of this, the equations of the first approximation are as follows:

$$\frac{da}{dt}=-\frac{\varepsilon a}{2m_1\omega_1}\frac{d(m_1\omega_1)}{d\tau}+\frac{2\varepsilon}{m_1}\sum_{\sigma=-\infty}^{\infty}\frac{(r\omega_1-s\nu)\sigma_1\dfrac{\delta\overline{W}_\sigma}{\delta a}+2\omega_1\dfrac{\delta\overline{W}_\sigma}{\delta\psi}\dfrac{1}{a}}{4\omega_1^2-(r\omega_1-s\nu)^2\sigma^2},$$

$$\frac{d\psi}{dt}=\omega_1-\frac{s}{r}\gamma+\frac{2\varepsilon}{m_1a}\sum_{\sigma=-\infty}^{\infty}\frac{(r\omega_1-s\nu)\sigma_1\dfrac{\delta\overline{W}_\sigma}{\delta\psi}\dfrac{1}{a}-2\omega_1\dfrac{\delta\overline{W}_\sigma}{\delta a}}{4\omega_1^2-(r\omega_1-s\nu)^2\sigma^2}. \tag{2.8}$$

Thus, when forming equations of the first approximation it is necessary to find $\overline{\delta W}$ and to expand the obtained expression into Fourier series and, finally, to put the "partial derivatives" of σ-term into the equations (2.8).

For the case when the oscillatory system is

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only under the action of potential excitative forces

$$\delta W = \sum_{j=1}^{N} Q_j \delta q_j = -\delta V, \tag{2.9}$$

where V is the excitative potential energy. Hence it follows that it is necessary to find the mean value of the variation of the excitative potential energy δV in order to form the equations of the first approximation and to expand the obtained expression into Fourier series, and after which those "partial derivatives" of the δ -term taken with the opposite signs should be substituted into the equations (2.8) instead of $\frac{\delta W_6}{\delta a}$ and $\frac{\delta W_6}{\delta \varphi}$.

With the help of the energetical interpritation the equations of the first (and second) approximation are being formed when using directly the form of work (or potential energy) and kinetic energy. This gives the possibility of obtaining the approximate solutions without preliminary forming the exact differential equations of the problem, besides the possibility of the formal extension of the worked out method for obtaining the approximate solutions of the partial differential equations.

For linear oscillatory systems with finite and infinite number of degrees of freedom the principle of superposition of the vibrations takes place. The principle of superposition of the linear vibrations is clearly eliminated in the form of general solutions. As it is well

known the latter have the form of functions which are finite and infinite sums (according to the number of the degrees of freedom) of harmonic terms of the normal vibrations of the corresponding configurations or the form of the dynamical equilibrium.

Besides the principle of superposition of vibrations for the linear oscillatory systems there is a principle of solo-frequency of the definite regimes of vibrations. Really, in the presence of the external excitation of the definite frequency and the dissipative forces it is practically very soon establishes the forced vibrations of this frequency in the resonance as well as in the non-resonance case. Consequently, all points of the system will perform solo-frequent vibrations in definite configuration (for N degrees of freedom) or in the definite form of the dynamical equilibrium (for infinitely many degrees).

The principle of superposition does not take place for non-linear oscillatory systems, but the principle of solo-frequency of the regimes of vibrations in many cases takes place and is being used in the investigations.

The validity of the principle of solo-frequency of the regimes of vibrations for non-linear vibrational systems besides of the linear ones testifies to its universality.

Analogically with the systems with N degrees of freedom, the oscillatory systems with the distributed parameters described by the partial dif-

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ferential equations permit under definite conditions the solo-frequent regimes of vibrations in corresponding forms of the dynamical equilibrium and in the case of their stability the investigation of such regimes is of great physical interest.

That is why while investigating solo-frequent regimes of the vibrations of the systems with distributed parameters it is quite logically to use the method of forming the approximate solutions having been worked out for the oscillatory systems with N degrees of freedom and to extend it to oscillatory systems with the infinite number of degrees of freedom.

In the problems discussed below the procedure of the construction of the approximate solutions in its energetical interpritation is formally applied to the investigation of the solo-frequent vibrations of systems with distributed parameters.

3. The problem of the diametral vibrations of the pivot which is under the action of the longitudinal sinusoidal force with the variable frequency is investigated by Yu.A.Mitropolsky[3] who was the first to apply the above mentioned method while investigating non-stationary solo-frequent vibrations of the systems with distributed parameters.

This problem deals with the diametrical vibrations of the pivot of the length ℓ, the axis force of the following form acting on the free end of the pivot:

$$S = S_0 + \varepsilon f(t), \tag{3.1}$$

where

$$F(t) = A + B\sin\theta \tag{3.2}$$

and $\frac{d\theta}{dt} = \gamma(t)$. (fig.1)

Let us designate the rigidity of the pivot by EJ. Let γ be the weight of the unity of volume, g – the acceleration of gravity, Ω is the area of the diametrical section. Then for the potential energy of the curve disregarding the inertia of rotation and the cutting force we have the following expression:

$$V = \frac{1}{2}\left(E - \frac{S}{\Omega}\right)J\int_0^\ell \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx - \frac{1}{2}S\int_0^\ell \left(\frac{\partial y}{\partial x}\right)^2 dx \tag{3.3}$$

and for kinetic energy

$$T = \frac{1}{2}\frac{\gamma\Omega}{g}\int_0^\ell \left(\frac{\partial y}{\partial t}\right)^2 dx . \tag{3.4}$$

Let us define

$$V = V_0 + \varepsilon V_1 , \tag{3.5}$$

where

$$V_0 = \frac{1}{2}\left(E - \frac{S_0}{\Omega}\right)J\int_0^\ell \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx - \frac{1}{2}S_0\int_0^\ell \left(\frac{\partial y}{\partial x}\right)^2 dx \tag{3.6}$$

is the potential energy of the non-perturbated system

$$\varepsilon V_1 = -\frac{\varepsilon}{2}F(t)\left[\frac{J}{\Omega}\int_0^\ell \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx + \int_0^\ell \left(\frac{\partial y}{\partial x}\right)^2 dx\right] \tag{3.7}$$

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is the "excitative" potential energy.

In order to form the differential equations of the "non-perturbated" movement taking into consideration the expressions (3.4) and (3.6) we use the principle of Ostrogradsky-Camilton

$$\delta\int_{t_1}^{t_2}(T-V_o)dt=0 .$$

As a result of this we have non-excited equation

$$\left(E-\frac{S_o}{\Omega}\right)J\frac{\partial^4 y}{\partial x^4}+S_o\frac{\partial^2 y}{\partial x^2}+\frac{\delta\Omega}{g}\frac{\partial^2 y}{\partial t^2}=0 \qquad (3.8)$$

with the natural boundary conditions:

$$y\big|_{x=0}=0\ ,\quad \frac{\partial y}{\partial x}\Big|_{x=0}=0\ ,\quad \frac{\partial^2 y}{\partial x^2}\Big|_{x=\ell}=0;$$

$$\left(E-\frac{S_o}{\Omega}\right)J\frac{\partial^3 y}{\partial x^3}\Big|_{x=\ell}=-S_o\frac{\partial y}{\partial x}\Big|_{x=\ell}\ . \qquad (3.9)$$

Having solved the equation (3.8) with boundary conditions (3.9) we find the normal functions

$$\varphi^{(\kappa)}(x)=\left(\lambda_1^{(\kappa)2}\sin\lambda_1^{(\kappa)}\ell+\lambda_1^{(\kappa)}\lambda_2^{(\kappa)}\mathrm{Sh}\lambda_2^{(\kappa)}\ell\right)\left(\cos\lambda_1^{(\kappa)}x-\mathrm{Ch}\lambda_2^{(\kappa)}x\right)-$$
$$-\left(\lambda_1^{(\kappa)2}\cos\lambda_1^{(\kappa)}\ell+\lambda_2^{(\kappa)2}\mathrm{Ch}\lambda_2^{(\kappa)}\ell\right)\left(\sin\lambda_1^{(\kappa)}x-\frac{\lambda_1^{(\kappa)}}{\lambda_2^{(\kappa)}}\mathrm{Sh}\lambda_2^{(\kappa)}x\right), \qquad (3.10)$$

where $\lambda_1^{(\kappa)}$ and $\lambda_2^{(\kappa)}$ are determined from the characteristic equation

$$\left(E-\frac{S_o}{\Omega}\right)J\lambda^{(\kappa)4}+S_o\lambda^{(\kappa)2}-\frac{\omega_\kappa^2\gamma\Omega}{g}=0 \qquad (3.11)$$

ω_κ ($\kappa=1,2,3,\ldots$) are the frequences of the κ-th normal vibration from the corresponding frequence equation.

Going on to the investigation of the "excited" movement, that is with the accounting of the "excited" potential energy εV_1, it is necessary to note that the boundary conditions (3.9) don't give any possibility by the corresponding substitution of the variables to transform the equation of the "excited" movement to the ordinary differential equation. On the base of the above-stated method of forming the equations of the first approximation for the amplitude and phase coming directly from the expression of the "excited" energy εV, the solution of this problem causes no difficulties.

Under the action of the periodic axis force with the frequency which is approximately two times more than the fundamental main partial frequency ω_1, the intense vibrations can appear (parametrical resonance).

Starting to the investigation of these vibrations one should seek the solution for the "excited" movement in the form:

$$y=\varphi^{(1)}(x)a\cos\left(\tfrac{1}{2}\theta+\psi\right), \qquad (3.12)$$

where a and ψ must satisfy the first approximation equations (2.8).

For their construction we find the variation of the "excited" potential energy δV_1 and then its mean value for the whole cycle of vibrations

$$\overline{\delta V_1}\equiv\frac{1}{2\pi}\int_0^{2\pi}\delta V_1\,d\left(\tfrac{1}{2}\theta+\psi\right)=-\Phi^{(1)}\frac{b_1}{2}\delta a-\Phi^{(1)}\frac{a_0 B}{8i}e^{2i\psi}\delta a+$$
$$+\Phi^{(1)}\frac{b_1 B}{8i}e^{2i\psi}\delta a+\Phi^{(1)}\frac{a^2 B}{8}e^{2i\psi}\delta\psi+\Phi^{(1)}\frac{a^2 B}{8}e^{2i\psi}\delta\psi, \qquad (3.13)$$

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where

$$\Phi''' = \frac{J}{\Omega}\int_0^{\ell}\left(\frac{d^2\varphi'(x)}{dx^2}\right)^2 dx + \int_0^{\ell}\left(\frac{d\varphi''(x)}{dx}\right)^2 dx \qquad (3.14)$$

Having substituted $\frac{\delta\bar{V}_{1\sigma}}{\delta a}$ and $\frac{\delta\bar{V}_{1\sigma}}{\delta\psi}$ with the opposite signs into the equations (2.8) we obtain the first approximation equations

$$\begin{cases} \frac{da}{dt} = -\frac{\varepsilon\Phi''' aB}{2m_1\nu(\tau)}\cos 2\psi \\ \frac{d\psi}{dt} = \omega_1 - \frac{\nu(\tau)}{2} - \frac{\varepsilon\Phi''' A}{2m_1\omega_1} + \frac{\varepsilon\Phi''' B}{2m_1\nu(\tau)}\sin 2\psi, \end{cases} \qquad (3.15)$$

where

$$m_1 = \frac{\gamma\Omega}{g}\int_0^{\ell}\left[\varphi''(x)\right]^2 dx. \qquad (3.16)$$

From the equations (3.15) we can investigate the qualitative character of the solo-frequent regime of the vibrations. Particularly while investigating stational regimes of the vibrations and their stability, it is easy to find a zone of the parametric resonance defined by the inequality

$$\left|\omega_1 - \frac{\nu(\tau)}{2} - \frac{\varepsilon\Phi''' A}{2m_1\omega_1}\right| < \frac{\varepsilon\Phi''' B}{2m_1\nu(\tau)} \qquad (3.17)$$

For the investigation of non-stationary oscillatory process of the pivot oscillations with variable frequency of external force and passing the system through parametrical resonance a law of change $\nu(\tau)$ and numerical values of parameters of the oscillatory system are considered as data.

Under some numerical values the first approximation equations (3.15) are integrated by the numerical method and the resonance amplitude curve is built in the coordinates $\frac{a}{a_0}, t$ where a_0 is the initial value of the amplitude. This curve is characteristic for non-stationary process /fig.2/.

4. The problem about the diametrical vibrations of the pivot which is under the action of the moving load and pulsative force, was investigated by B.I.Moseenkov [7].

This work deals with the investigations of non-stationary vibrations of the pivot (beam) which are fastened swivelly on the ends, ℓ being the length of the beam and Ω the constant diametrical section, with the assumption that some mass εM is moving along the beam (small in comparison with the mass of the beam). Besides, let us assume that the beam is under the action of the moving vertical quasi-periodical force $\varepsilon F'(\theta) = \varepsilon F_0 \sin\theta$, the point of application of which coincides with the centre of gravity εM /fig.3/.

Let us introduce the following notations: ρ — the density of the material of the beam, E — the Young modulus, J — the moment of inertia of the diametrical section.

While examining the diametrical vibrations which are being accomplished in vertical plane xOy we ignore the inertia of rotation of the diametrical sections and cutting forces.

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Kinetic and potential energies of the examined system (the beam and the load) are as follows:

$$T=\frac{\rho\Omega}{2}\int_0^\ell\left(\frac{\partial y}{\partial t}\right)^2dx+\varepsilon M\left(\frac{\partial y}{\partial t}\right)^2_{x=\xi}=T_0+\varepsilon T_1 \tag{4.1}$$

and

$$V=\frac{EJ}{2}\int_0^\ell\left(\frac{\partial^2 y}{\partial x^2}\right)^2dx-\varepsilon Mg(y)_{x=\xi}=V_0+\varepsilon V_1 \tag{4.2}$$

The equation of proper diametrical vibrations of the beam, kinetic and potential energies of which are equal T_0 and V_0 respectively is the equation of "non-excited" movement:

$$\frac{\partial^4 y}{\partial x^4}+\frac{\rho\Omega}{EJ}\frac{\partial^2 y}{\partial t^2}=0 \tag{4.3}$$

and the boundary conditions have the form

$$y|_{x=0}=0,\ \frac{\partial^2 y}{\partial x^2}\Big|_{x=0}=0,\ y|_{x=\ell}=0,\ \frac{\partial^2 y}{\partial x^2}\Big|_{x=\ell}=0. \tag{4.4}$$

The forms of the normal (principal) vibrations of "non-excited" movement are determined from the equations (4.3) and the conditions (4.4).

$$\varphi^{(\kappa)}(x)=\sin\frac{\kappa\pi}{\ell}x \qquad (\kappa=1,2,3,\dots) \tag{4.5}$$

and their proper frequencies

$$\omega_\kappa=\frac{\kappa^2\pi^2}{\ell^2}\sqrt{\frac{EJ}{\rho\Omega}}, \qquad (\kappa=1,2,3,\dots) \tag{4.6}$$

that are necessary for constructing asymptotic solutions of "excited" movement.

The excited force consists of the external pulsative force and others which are put on it with the account of the "excited" potential energy εV_1 and kinetic energy εT_1. It has the form.

$$\varepsilon\Phi(\tau,\theta,\ddot{y})=\varepsilon\left[F_0\sin\theta+Mg-M\frac{\partial^2 y}{\partial t^2}\right]_{x=\xi}, \tag{4.7}$$

where $\xi=v\tau$ is the current coordinate of its point of application and v is the velocity of movement of load and pulsative force along the beam.

If we consider the sag of the beam y as the generalized coordinate, the first approximation of the solution corresponding to solo-frequent vibrations close to the first normal vibration in the presence of the principal resonance in the "excited" movement will be as follows:

$$y^{(1)}=\sin\frac{\pi}{\ell}x\,a\cos(\theta+\psi) \tag{4.8}$$

The amplitude a and the phase ψ are termined by the first approximation equations (2.8).

In order to form them we find the expression of the virtual work

$$\delta W=\Phi_0^{(1)}\delta y^{(1)}=\left[F_0\sin\theta+Mg+M\omega_1^2\sin\frac{\pi}{\ell}\xi a\cos(\theta+\psi)\right]\times$$
$$\times\sin\frac{\pi}{\ell}\xi\left[a\cos(\theta+\psi)\delta a-a\sin(\theta+\psi)\delta\psi\right] \tag{4.9}$$

This being done, we find the mean value of virtual work δW and its δ -th "partial derivatives" with the help of the formula (2.4) and we also find the value

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Kinetic and potential energies of the examined system (the beam and the load) are as follows:

$$T=\frac{\rho\Omega}{2}\int_0^\ell\left(\frac{\partial y}{\partial t}\right)^2dx+\varepsilon M\left(\frac{\partial y}{\partial t}\right)^2_{x=\xi}=T_0+\varepsilon T_1,$$

and (4.1)

$$V=\frac{EJ}{2}\int_0^\ell\left(\frac{\partial^2 y}{\partial t^2}\right)^2dx-\varepsilon Mg(y)_{x=\xi}=V_0+\varepsilon V_1, \tag{4.2}$$

The equation of proper diametrical vibrations of the beam, kinetic and potential energies of which are equal T_0 and V_0 respectively is the equation of "non-excited" movement:

$$\frac{\partial^4 y}{\partial x^4}+\frac{\rho\Omega}{EJ}\frac{\partial^2 y}{\partial t^2}=0 \tag{4.3}$$

and the boundary conditions have the form

$$y|_{x=0}=0,\ \frac{\partial^2 y}{\partial x^2}|_{x=0}=0,\ y|_{x=\ell}=0,\ \frac{\partial^2 y}{\partial x^2}|_{x=\ell}=0. \tag{4.4}$$

The forms of the normal (principal) vibrations of "non-excited" movement are determined from the equations (4.3) and the conditions (4.4).

$$\varphi^{(k)}(x)=\sin\frac{k\pi}{\ell}x \qquad (k=1,2,3,\dots) \tag{4.5}$$

and their proper frequencies

$$\omega_k=\frac{k^2\pi^2}{\ell^2}\sqrt{\frac{EJ}{\rho\Omega}}, \qquad (k=1,2,3,\dots) \tag{4.6}$$

that are necessary for constructing asymptotic solutions of "excited" movement.

The excited force consists of the external pulsative force and others which are put on it with the account of the "excited" potential energy εV_1 and kinetic energy εT_1. It has the form:

$$\varepsilon\Phi(\tau,\theta,\ddot{y})=\varepsilon\left[F_0\sin\theta+Mg-M\frac{\partial^2 y}{\partial t^2}\right]_{x=\xi}, \tag{4.7}$$

where $\xi=v\tau$ is the current coordinate of its point of application and v is the velocity of movement of load and pulsative force along the beam.

If we consider the sag of the beam y as the generalized coordinate, the first approximation of the solution corresponding to solo-frequent vibrations close to the first normal vibration in the presence of the principal resonance in the "excited" movement will be as follows:

$$y^{(1)}=\sin\frac{\pi}{\ell}x\,a\cos(\theta+\psi) \tag{4.8}$$

The amplitude a and the phase ψ are terminated by the first approximation equations (2.8).

In order to form them we find the expression of the virtual work

$$\delta W=\Phi_0^{(1)}\delta y^{(1)}=\left[F_0\sin\theta+Mg+M\omega^2\sin\frac{\pi}{\ell}\xi a\cos(\theta+\psi)\right]x$$
$$x\sin\frac{\pi}{\ell}\xi\left[a\cos(\theta+\psi)\delta a-a\sin(\theta+\psi)\delta\psi\right] \tag{4.9}$$

This being done, we find the mean value of virtual work δW and its δ-th "partial derivatives" with the help of the formula (2.4) and we also find the value

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$$m_1 = \rho\Omega\int_0^{\ell} \sin^2\frac{\pi}{\ell}x\,dx = \frac{\rho\Omega\ell}{2}, \tag{4.10}$$

which are substituted into the equations (2.8). As a result of this we have the first approximation equations for the solution (4.8):

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{2\varepsilon F_0 \sin\frac{\pi}{\ell}\xi(\tau)}{\rho\Omega\ell(\omega_1+\nu(\tau))}\cos\psi \\ \dfrac{d\psi}{dt} = \omega_1 - \nu(\tau) - \dfrac{c\omega_1 M\sin^2\frac{\pi}{\ell}\xi(\tau)}{\rho\Omega\ell} + \dfrac{2\varepsilon F_0 \sin\frac{\pi}{\ell}\xi(\tau)}{a\rho\Omega\ell(\omega_1-\nu(\tau))}\sin\psi \end{cases} \tag{4.11}$$

This system is integrated in quadratures

$$z = u + iv = ae^{i\psi} \tag{4.12}$$

and by the substitution is reduced to the linear equation for a complex-valued function z

$$\frac{dz}{dt} - i\left[\omega_1 - \nu(\tau) - \frac{c\omega_1 M\sin^2\frac{\pi}{\ell}\xi(\tau)}{\rho\Omega\ell}\right]z = -\frac{2\varepsilon F_0 \sin\frac{\pi}{\ell}\xi(\tau)}{\rho\Omega\ell(\omega_1+\nu(\tau))}. \tag{4.13}$$

Integrating (4.13) and then separating the real and imaginary parts of the solution we find the expression for a and ψ according to (4.12).

Having found the relation $a=a(t)$ we plotted the resonance amplitude curve of the principal tone of vibrations in non-stationary regime /fig.4/ for some numerical values.

Non-stationarity of the oscillatory process while crossing the principal resonance is characterized by the finiteness of amplitudes of the transitional regime and by the displacement

of maximum of the resonance curve to the right hand.

5. The problem about the bending vibrations of the pivot of double rigidity in transitional regime of rotation is examined by B.I.Moseenkov [8], where the examined method is extended to more complicated case of spatial diametrical vibrations of the pivot. Besides ordinary assumptions about the rectilinearity of the pivot in an unstrained state and egnoring the influence of cutting forces and the inertia of rotation of the diametrical sections we assume the pivot does not twist in the process of vibrations. We think also that the pivot is unbalanced statically, that is the line of centres of gravity of elements of the non-rotation pivot is displaced according to the rectilinear axis and is a plane curve, the form of the change of unbalance along the pivot is considered as datum.

While examining the diametrical oscillations we disregard the longitudinal vibrations. We suppose the conditions of fastening to be of isotropic swivel type.

Let us introduce two coordinate systems $Ox_1y_1z_1$ and $Oxyz$ /fig.5/, the first being immovable, the second rotating with the angular velocity ω. Axes Ox_1 and Ox are directed along the axis of rotation, the direction of which coincides with rectilinear axis of the non-rotating pivot. The origin of the coordinates O is placed on the left support.

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The otner axes of the movable coordinate system Oy and Oz coincide by their directions with the principal directions of the diametrical section of the pivot, the axis of the movable system Oy_1 is directed vertically below.

Then kinetic energy of bending vibrations and potential energy of deformation without taking into account their proper weight will take the following form according to the movable coordinate system:

$$T=\frac{m}{2}\int_0^\ell\left\{\left[\frac{\partial y}{\partial t}-\omega(z+\rho_1)\right]^2+\left[\frac{\partial z}{\partial t}+\omega(y+\rho_2)\right]^2\right\}dx,$$

$$V=\frac{E}{2}\int_0^\ell\left[J_1\left(\frac{\partial^2 y}{\partial x^2}\right)^2+J_2\left(\frac{\partial^2 z}{\partial x^2}\right)^2\right]dx \qquad (5.1)$$

(5.2)

We have used the following notations in the expressions (5.1) and (5.2): m the linear density of the pivot of the constant section, ℓ — the length of the pivot, $\rho_1=\rho_z$ and $\rho_2=\rho_y$ — the projections of vector of eccentricity of the centre of gravity, $\omega=\omega(\tau)$ — the angular velocity of rotation, E — the Young modulus, $J_1=J_z$ and $J_2=J_y$ — the principal moments of inertia of the area of the diametrical section, y and z — the projections of the vector of displacement of the geometrical centre of the pivot's element.

Due to the fact that such factors as the asymmetry of the diametrical section or the

"lack of coordination" of rigidity, the influence of statical unbalancity are small, let us choose in the expressions T and V those corresponding parts which take into account these small factors and they will correspond to the "excited" kinetic and potential energies.

Then the expressions (5.1) and (5.2) will take the form:

$$T=\frac{m}{2}\int_0^\ell\left\{\left[\frac{\partial y}{\partial t}-\omega(\tau)z\right]^2+\left[\frac{\partial z}{\partial t}+\omega(\tau)y\right]^2\right\}dx+\frac{m}{2}\int_0^\ell\left\{\rho_1^2\omega^2(\tau)+\rho_2^2\omega^2(\tau)\right.$$

$$\left.-2\omega(\tau)\left[\rho_1\left(\frac{\partial y}{\partial t}-\omega(\tau)z\right)-\rho_2\left(\frac{\partial z}{\partial t}+\omega(\tau)y\right)\right]\right\}dx=T_0+\varepsilon T_1,$$

$$V=\frac{E(J_1+J_2)}{4}\int_0^\ell\left[\left(\frac{\partial^2 y}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial x^2}\right)^2\right]dx+\frac{E(J_1-J_2)}{4}\int_0^\ell\left[\left(\frac{\partial^2 y}{\partial x^2}\right)^2-\right. \qquad (5.3)$$

$$\left.-\left(\frac{\partial^2 z}{\partial x^2}\right)^2\right]dx=V_0+\varepsilon V_1. \qquad (5.4)$$

For the purpose of forming the differential equations of the "non-excited" movement let us use again the principle of Ostrogradsky-Gamilton.

$$\delta\int_{t_1}^{t_2}(T_0-V_0)dt=0.$$

As a result of this we have the system of two equations concerning y and z.

Having introduced complex-valued function

$$\xi(x,t)=y(x,t)+iz(x,t) \qquad (5.5)$$

and having used the formula of transition from the coordinates of the movable system to the

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coordinates of the immovable system

$$\xi(x,t)=\rho(x,t)e^{-i\theta}, \tag{5.6}$$

where

$$\rho(x,t)=y_1(x,t)+iz_1(x,t), \tag{5.7}$$

we obtain the equation of the "non-excited" movement in the immovable system of coordinate from the above-mentioned system of the two equations

$$C^2\frac{\partial^4\rho}{\partial x^4}+\frac{\partial^2\rho}{\partial t^2}=0, \tag{5.8}$$

where

$$C^2=\frac{E(J_1+J_2)}{2m}. \tag{5.9}$$

The natural boundary conditions concerning the complex-valued function ρ have the form:

$$\rho|_{x=0}=\rho|_{x=\ell}=0,\quad \frac{\partial^2\rho}{\partial x^2}\Big|_{x=0}=\frac{\partial^2\rho}{\partial x^2}\Big|_{x=\ell}=0. \tag{5.10}$$

From the equations (5.8) and conditions (5.10) we deduce:

$$\rho_\kappa(x,t)=\sin\frac{\kappa\pi}{\ell}x\left[a_{1\kappa}e^{i(\omega_\kappa t+\psi_{1\kappa})}+a_{2\kappa}e^{-i(\omega_\kappa t+\psi_{2\kappa})}\right], \tag{5.11}$$

where

$$\omega_\kappa=\frac{\kappa^2\pi^2}{\ell^2}\sqrt{\frac{E(J_1+J_2)}{2m}}. \qquad (\kappa=1,2,3,\ldots) \tag{5.12}$$

These solutions can be interpreted as the vibrations in the κ-th forms of the dynamic equilibrium which are the superpositions of the two circular motions - the direct and the inverse - with the angular velocity ω_κ.

So the form of the normal vibrations of "unperturbed" movement are determined by the real functions

$$\varphi^{(\kappa)}(x)=\sin\frac{\kappa\pi}{\ell}x, \qquad (\kappa=1,2,3,\ldots) \tag{5.13}$$

and angular velocities are their proper frequences (5.12).

Going on to the investigation of the "excited" movement, we form the expression of exciting force, corresponding to the "excited" kinetic energy εT_1, the "excited" potential energy εV_1 and small dissipating forces of external friction. These exciting forces which act on the element of the pivot dx have the following complex form:

$$\varepsilon Q(\bar\rho,\theta,\dot\rho,\dot{\bar\rho},\dot\theta)=\left[m\rho\omega^2(\tau)e^{i\theta}+\frac{E(J_1-J_2)}{2}\frac{\partial^4\bar\rho}{\partial x^4}e^{2i\theta}-\varkappa\frac{\partial\rho}{\partial t}\right]dx, \tag{5.14}$$

where $\rho=\rho_1+i\rho_2$, $\frac{d\theta}{dt}=\omega(\tau)$, $\varkappa$ is the coefficient of the external friction and $\bar\rho(x,t)$ is conjugated valued function to $\rho(x,t)$.

We consider that the angular velocity of rotation $\omega(\tau)$ on the interval of change τ $(0\le\tau\le L)$ have the following value

$$\omega_1=\frac{\pi^2}{\ell^2}\sqrt{\frac{E(J_1+J_2)}{2m}},$$

that is the coincidence of frequences of the inert excited force $\frac{d\theta}{dt}=\omega(\tau)$ with the principal circular frequency ω_1 or the passing through

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$$-a_1\rho_0^*\delta\psi_1)\left(e^{i\psi_1}+e^{-i\psi_1}\right)+\left(\rho_2^*\delta a_1+a_1\rho_1^*\delta\psi_1\right)i\left(e^{i\psi_1}-e^{-i\psi_1}\right)\Big]+$$

$$+\frac{E(J_2-J_1)\varkappa^4}{4\ell}a_1\Big[\left(e^{2i\psi_1}+e^{-2i\psi_1}\right)\delta a_1+a_1 i\left(e^{2i\psi_1}-e^{-2i\psi_1}\right)\delta\psi_1\Big]-$$

$$-\varkappa\omega_1 a_1^2\delta\psi_1\Big\}-\frac{\ell}{4}\varkappa\omega_1 a_2^2\delta\psi_2 \,. \tag{5.19}$$

At last having found the value

$$m_1=m\int_0^{\ell}\sin^2\frac{\pi}{\ell}x\,dx=\frac{m\ell}{2} \tag{5.20}$$

and also non-vanishing δ-th "partial derivatives" of the expressions δW_1 and δW_2 and substituting them into the equations (2.8) we get the first approximation equations for a_1, ψ_1 and a_2, ψ_2.

Thus we have:

$$\begin{cases}\dfrac{da_1}{dt_1}=-\alpha a_1-\gamma_1(\tau)\sin\psi_1-\gamma_2(\tau)\cos\psi_1-\beta(\tau)a_1\sin 2\psi_1\\ \dfrac{d\psi_1}{dt}=\varepsilon(\tau)-\delta_1(\tau)\dfrac{1}{a_1}\cos\psi_1-\gamma_2(\tau)\dfrac{1}{a_1}\sin\psi_1-\beta(\tau)\cos 2\psi_1\end{cases}, \tag{5.21}$$

$$\begin{cases}\dfrac{da_2}{dt_1}=-\alpha a_2\\ \dfrac{d\psi_2}{dt_1}=0\end{cases}, \tag{5.22}$$

where

$$t_1=\omega_1 t\ ,\quad \Delta(\tau)=\frac{\omega(\tau)}{\omega_1}\ ,\quad \delta=\frac{J_2-J_1}{J_2+J_1}\ ,\quad \alpha=\frac{\varkappa}{2m\omega_1}\ ,$$

$$\beta(\tau)=\frac{\delta}{2\Delta(\tau)}\ ,\quad \varepsilon(\tau)=1-\Delta(\tau)\ ,\quad \gamma_i(\tau)=\frac{P^*\Delta^2(\tau)}{1+\Delta(\tau)}\quad (i=1,2) \tag{5.23}$$

The first approximation equations permit an exident solution and the equations (5.21) are not integrated in closed form but permit the possibility of investigation of qualitative character. Examining stationary regime of vibrations $\left(\frac{da_1}{dt_1}=0, \frac{d\psi_1}{dt}=0\right)$ and their stability, the limits of the parametrical resonance in the first approximation are being established

$$\frac{\omega_1}{2}\left(1+\sqrt{1-2\delta}\right)\le\omega(\tau)\le\frac{\omega_1}{2}\left(1+\sqrt{1+2\delta}\right), \tag{5.24}$$

and the error of the first approximation is also being established depending on asymmetry of the diametrical section.

Examining the transitional rate of rotation of the pivot for the various laws of change of angular velocity and with various lacks of coordination of rigidities and with the help of numerical methods of integration the resonance amplitude curves are built corresponding to the passing through resonance zones [8].

Amplitude curves of passing through resonance are shown in figures 6 and 7 by the continuous thin lines concerning the statically balanced pivot and statically unbalanced pivot.

Accordingly the first are placed below and the second are placed above them.

Furthermore, resonance curves of stationary amplitudes are depicted in these figures (for comparison) with heavy (rich) lines, their stable branches are depicted by the continuous lines, the unstable - by dotted lines.

Let us note that the resonance curve of the stationary amplitudes has two branches (continuous lines) the appearance of which is caused by the presence of statical unbalancity.

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the principal resonance takes place. Furthermore, it is necessary to note that the presence of the second term in the expression of the excited forces [illegible] which is the result of asymmetry of the diametrical section leads to the parametrical (demultiplicative) resonance [8] for the same relation for frequences $\frac{\omega(t)}{\omega_1}=1$ in the same (first) form of the dynamical equilibrium.

Taking into consideration the form of the solution (5.11) the first approximation of the asymptotic solution corresponding to the excited vibrations close by their form to the normal vibrations of the first form of the dynamical equilibrium taking into account all possible resonances of the oscillatory system will be expressed by the complex-valued function

$$\rho^{(1)}=\sin\frac{\pi}{\ell}x\left[a_1 e^{i(\theta+\psi_1)}+a_2 e^{-i(\omega_1 t+\psi_2)}\right], \qquad (5.15)$$

where the amplitudes and the phases a_1,ψ_1 and a_2,ψ_2 must be obtained from the corresponding first approximation equations (2.8).

For their forming we separate at first the real and the imaginary parts of the expressions (5.14) and (5.15) and obtain small forces of excitation εQy_1 ad εQz_1 acting on the element of the pivot in the direction of axes Oy_1 and Oz_1 and also the deflections $y_1^{(1)}$ and $z_1^{(1)}$ of the geometrical centre of element of the pivot in the first approximation. Then

we obtain the expressions of the excited forces εQy_1 and εQz_1 in the regime of the resulting sinusoidal vibrations (when the variables y_1 and z_1 are replaced by the expressions $y_1^{(1)}$ and $z_1^{(1)}$ that is we obtain $\varepsilon Q_{y_1 0}^{(1)}$ and $\varepsilon Q_{z_1 0}^{(1)}$. This being done we work out the expressions of the virtual work of the excited force $\varepsilon Q_{y_1 0}^{(1)}$ and $\varepsilon Q_{z_1 0}^{(1)}$ which they would have accomplished on the virtual displacements $\delta y_1^{(1)}$ and $\delta z_1^{(1)}$, corresponding to the variations of amplitudes a_1, a_2 and phases ψ_1,ψ_2 in the first form of the dynamical equilibrium.

$$\delta W_{y_1}=\int_0^{\ell}\varepsilon Q_{y_1 0}^{(1)}\delta y_1^{(1)}, \quad \delta W_{z_1}=\int_0^{\ell}\varepsilon Q_{z_1 0}^{(1)}\delta z_1^{(1)}. \qquad (5.16)$$

Averaging the last expressions over angular arguments $\theta+\psi_1$ and $\omega_1 t+\psi_2$ we find the mean values

$$\overline{\delta W_{y_1}}=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\delta W_{y_1}\,d(\theta+\psi_1)\,d(\omega_1 t+\psi_2)$$

$$\overline{\delta W_{z_1}}=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\delta W_{z_1}\,d(\theta+\psi_1)\,d(\omega_1 t+\psi_2), \qquad (5.17)$$

that are equal each other in the first approximation, that is

$$\overline{\delta W_{y_1}}=\overline{\delta W_{z_1}}=\overline{\delta W}$$

If we take for simplicity the sinusoidal law of change of statical unbalancity, that is

$$\rho=(\rho_1^*-i\rho_2^*)\sin\frac{\pi}{\ell}x \qquad (5.18)$$

then we shall have the explicit expression:

$$\overline{\delta W}=\overline{\delta W_1}(a_1,\psi_1)+\overline{\delta W_2}(a_2,\psi_2)=\frac{\ell}{4}\left[\frac{m\omega^2 r_0}{2}\right]\left(\rho_1^*\delta a_1-\right.$$

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There is a zone of parametrical resonance between these stable branches, the presence of which is caused by the double regidity of the pivot taking into account the asymmetry of the diametrical section.

The analysis of graphs in forms of 6 and 7 permits us to draw the conclusions about the influence of the unbalancity and the width of the zone of the parametrical resonance, that is the lack of coordination of regidities of the pivot, and also the velocity of passing through resonance zone over the value of the maximal amplitudes while passing through resonance over the displacement of the maximum and so on. So, when the lack of coordination of rigidities is small /fig.6/ the amplitudes of vibrations of statically unbalanced pivot develop chiefly at the expense of static unbalancity generating principal resonances at the bounds of zones of the parametrical resonance.

The lack of coordination being significant /fig.7/ tney develop chiefly at the expense of asymmetry of the diametrical section generating the parametrical resonance in particular zone of angular velocities of the rotation of the pivot.

6. In conclusion let us dwell on the investigations of non-stationary regime of the vibrations of the turbine blade taking into account the dispersion of energy in the material [9] and curve-torsional vibrations of pivots with asymmetrical section with the account of the internal friction [2]. In those two above-mentioned problems which are essentially non-linear it is possible to work out rather simply and by one scheme the first approximation equations using their energetical interpretation i.e. to construct the first approximation equation in accordance with the considered method [6].

The differential equation of the "excited" movement in the first problem in relative variables ($\xi=\frac{x}{\ell}$, $v=\frac{U}{\ell}$) have the form:

$$\frac{\partial^4 v}{\partial \xi^4}-\frac{\ell^2}{EJ}\left(P_0-\alpha r_0 \ell \xi-\frac{\alpha \ell^2}{2}\xi^2\right)\frac{\partial^2 v}{\partial \xi^2}+\frac{\alpha \ell^3}{EJ}\left(r_0+\ell\xi\right)\frac{\partial v}{\partial \xi}+$$
$$+\frac{\mu \ell^4}{EJ}\frac{\partial^2 v}{\partial t^2}=\frac{\varepsilon q_0 \ell^3}{EJ}\sin\theta-\frac{\varepsilon \ell}{EJ}\frac{\partial^2}{\partial \xi}\left[\overline{\overline{\Phi}}\left(\frac{\partial^2 v}{\partial \xi^2}\right)\right] \qquad (6.1)$$

where u is sagging and ℓ is the length of the pivot.

Under the corresponding boundary conditions:

$$v(0,t)=0\ ,\quad \frac{\partial v(0,t)}{d\xi}=0\ ,$$

$$\frac{\partial^2 v(1,t)}{\partial \xi^2}=\rho\frac{\partial v(1,t)}{\partial \xi}\ ,\quad \frac{\partial^3 v(1,t)}{\partial \xi^3}=d_0\frac{\partial v(1,t)}{\partial \xi}\ . \qquad (6.2)$$

The functional $\varepsilon\overline{\overline{\Phi}}\left(\frac{\partial^2 v}{\partial \xi^2}\right)$ involved in the equation (6.1) is determined by the curving moment in sections of the pivot-blade:

$$\varepsilon\overline{\overline{\Phi}}\left(\frac{\partial^2 v}{\partial \xi^2}\right)=\overline{\overline{M}}-\frac{EJ}{\ell}\frac{\partial^2 v}{\partial \xi^2}\ , \qquad (6.3)$$

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$\overleftarrow{M}$ and $\overrightarrow{M}$ being corresponding curving moments in sections for ascending and descending branches of the loop of hysteresis.

In the second problem the differential equations of the "excited" movement have the form:

$$\frac{EJ}{\ell^4}\frac{\partial^4 y}{\partial \xi^4} + m\frac{\partial^2 y}{\partial t^2} + mz_c\frac{\partial^2 \theta}{\partial t^2} = \varepsilon q(\xi)\cos\gamma t - \varepsilon\overleftrightarrow{\Phi_1}\left(\frac{\partial^2 y}{\partial \xi^2}\right)$$

$$-\frac{GJ_d}{\ell^2}\frac{\partial^2 \theta}{\partial \xi^2} + J_m\frac{\partial^2 \theta}{\partial t^2} + mz_c\frac{\partial^2 y}{\partial t^2} = \varepsilon m_\xi(\xi)\cos\gamma t - \varepsilon\overleftrightarrow{\Phi_2}\left(\frac{\partial \theta}{\partial \xi}\right), \quad (0 \le \xi \le 1) \qquad (6.4)$$

where $\xi = \frac{x}{\ell}$ is the relative absciss, y is the sagging of the pivot in the principal plane of the least rigidity curvature, θ is the angle of torsion of the pivot.

The corresponding boundary conditions for cantilevel pivot will be as follows:

$$y(0,t) = \frac{\partial y(0,t)}{\partial \xi} = \frac{\partial^2 y(1,t)}{\partial \xi^2} = \frac{\partial^3 y(1,t)}{\partial \xi^3} = 0,$$

$$\theta(0,t) = \frac{\partial \theta(1,t)}{\partial \xi} = 0 \qquad (6.5)$$

Consequently, the examined method of constructing the first approximation equations for amplitude and phase of the solo-frequent vibrations to our opinion may be successively used while investigating rather complicated oscillatory systems, both linear and non-linear, with the distributed parametres.

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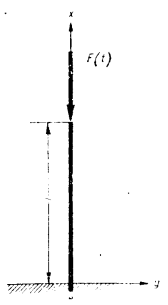

Fig. 1

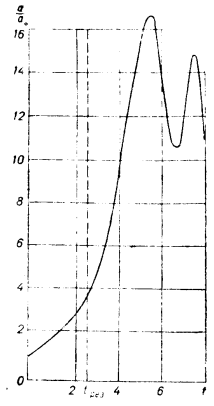

Fig. 2

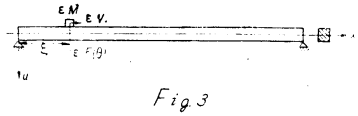

Fig. 3

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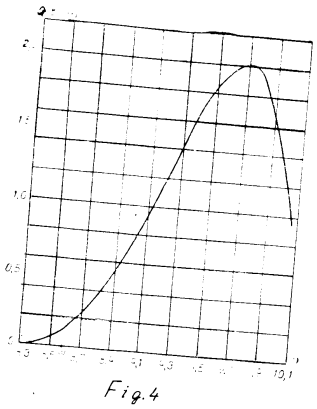

Fig.4

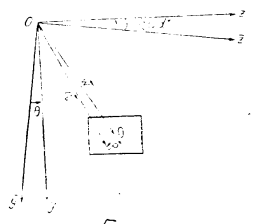

Fig.5

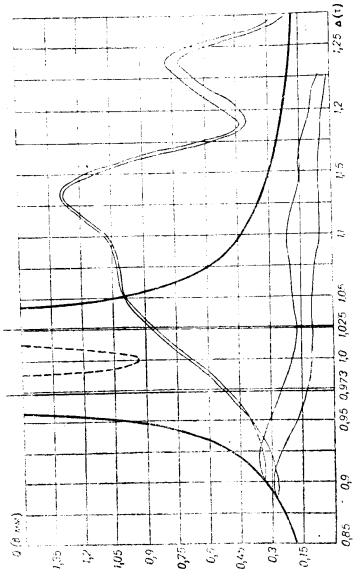

Fig.6

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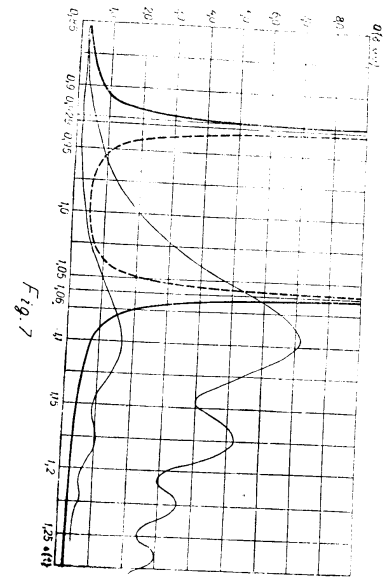

Fig. 7

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CHAPTER I

Historic remarks and introduction

The method of point mappings is an effective instrument of exact investigation of certain types of non-linear systems and a number of general questions of the theory of non-linear oscillations. This method was distinctly formulated by A.A.Andronov as mathematical instrument for the investigation of non-linear systems and, in particular, for systems of automatic control. The development and various applications of this method are due to, in the main, the Gorky school of the theory of non-linear oscillations.

Historically the method of point mappings in the theory of non-linear oscillations has originated from the so-called "method of "adjustment" (*приспособления*) whose essence is as follows: "...we replace a non-linear differential equation by a few linear equations and "adjust"

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( ) constants of integration at the points of conjugation of different equations [illegible] the requirement of continuity of solu[illegible] For the first time the method of "adjustment" was used by N.D.Papaleksi in the problem of [illegible] (1917), then by A.Sommerfeld in the theory of forced oscillations of an arc (1914), [illegible] by the author.

In 1923 L.I.Mandelstam proposed that A.A. [illegible] should investigate stability of motions [illegible] by the method of "adjustment" and in general should make an attempt to lay a serious mathematical foundation under this method". "So it was, - L.I.Mandelstam said, - those who had used the method - N.D.Papaleksi, A.Sommerfeld, and others - in fact had solved individual casuistical problems and confined themselves to finding periodic solutions, leaving aside questions of stability and, in general, questions of the behaviour of solutions under other initial conditions than those which correspond to the periodic motion".

The first concrete results, obtained in this direction, were published in the monograph by A.A.Andronov, A.A.Witt and S.E.Chaikin "Theory of oscillations" issued in 1937. These results contain an exhaustive investigation of dynamics of certain simplest non-linear systems which was performed by means of the method of point mappings of a straight line into itself (the problem of the excitation in a valve generator with 2-characteristic and the simplest model of the clock).

The general theoretical foundations of the method of point mappings in the theory of non-linear oscillations were stated by A.A.Andronov in 1934 in the paper "The theory of point mappings of Poincare-Birkhoff and the theory of non-linear oscillations" which he read at the session of the physics and mathematics section of the Academy of sciences of the USSR.

The method of point mappings is an immediate development of the method of sticking (*raccordement*). Thanks to the application of the theory of the cutting surface of Poincare-Birkhoff the method of point mappings permitted to solve a number of non-linear problems, remained unsolvable for many years in spite of repeated attempts of a series of well-known investigators. In 1944-1947 A.A.Andronov together with N.N.Pautin and A.G.Mayer solved completely the problem of Mises P [II],

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application of the mathematical apparatus of the theory of point mappings and the theory of the

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.he problem of stabilization of a plane by the autopilot [12], [13], the problem of Vishnegradsky [14], [15], and a number of others [16-20].

The possibility of obtaining point mappings by direct integration and the consideration together with the phase space of systems their space of parameters immensely contributed to an effective application of the method of point mappings to piecewise linear systems, particularly in the questions of global behaviour.

It would be wrong to consider the method of point mappings to be solely an effective method of the investigation of only piece-wise systems or a method based upon the possibility of step by step integration. It cannot also be considered to be a method which although is, in fact, the only method of the global investigation of dynamic systems of more than second order, but is a special method and therefore it has a very limited region of application.

The present paper aims at showing first of all wide possibilities and efficiency of the method of point mappings for the solution of various problems of the theory of non-linear oscillation, wide possibilities of the method of point mappings

not only for the investigation of concrete non-linear systems, but for the study of many general questions of the theory of non-linear dynamic systems.

The description of the method of point mappings has to include the description of a collection of mathematical means, the methods of concrete investigation, and the types of non-linear systems which have been studied and can be studied, the description of possible applications to the investigation of general questions of the theory of oscillations and a number of general questions which have been treated in one measure or another. This description represent considerable difficulties not only because of vastness and variety of data, but also due to the fact that the method of point mappings keeps on developing. The method of point mappings is based upon the fact that it is possible to reduce the investigation of the motion of a dynamic system to the investigation of some point mappings, in particu r, the theorem of the relationship of states o[illegible] periodic motions with the fixed s of properly constructed point mappings. A general method by which the investigation of mo-

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tion of a dynamic system can be reduced to the treatment of point mappings is described in §2, Chap.II. When the method of point mappings is employed in practice, it is very important to choose properly the way of reduction to a point mapping and the way of its setting. The success and simplicity of solution of a problem depend largely on these ways. A general recommendation here is as follows: on the basis of the treatment of the phase space of a system one should try to employ the method of reduction to point mappings of the least dimension and to set them in a parametric form. However, since these recommendations alone explain too little, I should like to cite a few examples.

Example I. A relay control system is described by a non-linear differential equation of the second order in the form:

$$\frac{d^2x}{dt^2} + 2h\frac{dx}{dt} + x = F(\psi), \quad \psi = x + \kappa\frac{dx}{dt}, \qquad (I.I)$$

where

$$F(\psi) = \begin{cases} +1 & \text{when } \psi > \delta \text{ and when } -\delta < \psi < \delta \text{ if up to this there was } \psi > \delta, \\ 1 & \text{when } \psi < -\delta \text{ and when } -\delta < \psi < \delta \text{ if up to this there was } \psi < \delta, \end{cases}$$

κ is any positive number, and $1 < 1$.

The phase space of the system represents two partially laid on each other half-plane which are

stuck along the parts of straight lines $\psi = \delta$ and $\psi = -\delta$ (Fig.I). When the phase point appears on these lines there occurs relay switching, and the phase point passes to the other half-plane.

It is not difficult to notice that during its motion any phase point crosses again and again the straight lines $\psi = \pm\delta$. This circumstance permits to reduce the investigation of the phase point to the one of its successive intersections with these straight lines.

Let the phase point start its motion from the point (x_0, y_0) ~~of the line $\psi = \delta$~~ 9. Integrating the equation (I.I) with $F(\psi) = +1$ we shall find that the coordinates x_0 and x_1 are in relations which are written in a parametric form (τ is a parameter) in the following way:

$$x_0 = \frac{\omega}{c}\,\frac{(1+\delta)e^{hT} - (1-\delta)\cos\omega T}{\sin\omega T} + \delta + \frac{(1-\delta)(\kappa-h)}{c},$$

$$x_1 = \frac{\omega}{c}\,\frac{-(1-\delta)e^{-hT} + (1+\delta)\cos\omega T}{\sin\omega T} - \delta + \frac{(1+\delta)(\kappa-h)}{c}.$$

Due to the symmetry of the phase trajectories relative to the origin of co-ordinates the study of the behaviour of the phase trajectories of the system (I.I) is reduced to the treatment of the

of the line $\psi = \delta$ and in time τ come to the point 9 (x_1, y_1) on the line $\psi = -\delta$. I

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point mapping T which takes a point with an abscissa x_0 of the straight line $y=\delta$ into a point with an abscissa x_1 of the same straight line.

Example 2. The motion of an electromagnetic chopper is described by the following system of non-linear differential equations of the third order.

$$\frac{d^2x}{dt^2}+2h\frac{dx}{dt}+x=y,\ \frac{dy}{dt}+\rho y=\rho \quad \text{when } x\le a,$$
$$\frac{d^2x}{dt^2}+2h\frac{dx}{dt}+x=0,\ y=0 \quad \text{when } x>a. \qquad (I.2)$$

The phase space of the system $x,y,z\ (z=\dot{x})$ represents a sticking of a three-dimensional half-space $x\le a$ with a half-plane $y=0,\ x\ge a$ (Fig.2). In the three-dimensional half-space $x\le a$ the motion of the system is described by a system of equations:

$$\dot{x}=z,\ \dot{y}=-\rho y+\rho,\ \dot{z}=-x+y-2hz \qquad (I.3)$$

and in the half-plane $y=0$, $x\ge a$ by a system

$$\dot{x}=z,\ y=0,\ \dot{z}=-x-2hz \qquad (I.4)$$

Let the phase point at the beginning be on a line $L\ (x=a,\ y=0,\ z<0)$. Its further motion takes place in the three-dimensional half-space on the phase trajectory of the system of equations (I.3) untill it comes onto the plane $x=a$.

Then the representative point instantly moves along a straight line parallel to the axis y to a half-plane $y=0$, for according to idealization adopted at the moment of the break of the electric circuit there occurs an instantaneous drop of the current up to zero. The representative point moves along one of the trajectories of the system (I.4) on the half-plane $y=0$ untill it comes to the line L.

Thus it follows from the treatment of the structure of the phase space and the motion of the phase point that the problem of the investigation of the partition of the phase space into trajectories is reduced to that of the point mapping of the ray L into itself.

Example 3. The problem of vibro-separation and vibro-transport is reduced to the investigation of a non-linear equation in the form:

$$\dot{y}=\sin\tau-\rho-\begin{cases} q+f\sin\tau & \text{when } y>0,\\ \text{any intermediate value} & \text{when } y=0,\\ -q-f\sin\tau & \text{when } y<0,\end{cases}$$

where τ is non-dimensional time, and ρ, q and f are parameters of the system. The phase space of this system is a cylinder. When $f<q$ all the phase trajectories again and again cross

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a line $y=0$, and therefore the treatment of motions of the system is reduced to that of the mapping of a circumference $y=0$ into itself.

For systems, the investigation of which can be reduced to transformation of a straight line into itself [2,3,11,21-43] (a circumference into itself), the method of point mappings is a practically regular one. Success has been achieved in a number of problems which are reduced to transformation a plane into a plane [II-I8,44-47]. In more complicated cases the application of the method of point mappings to a global investigation of the structure of the partition of the phase space into trajectories involves considerable difficulties. However it does not refer to the local investigation of individual and the most important types of motions, such as states of equilibrium periodic motions in application to which the method of point mappings permits to obtain general results which are not restricted by the order of the system under consideration.

Note that the point mapping can be regarded as a difference equation and conversely. The distinction lies in the fact that the usage of the term "point mapping" implies the application of geometric notions.

Upon reducing the considered problem to a point mapping there arises the most difficult and main question to investigate this point mapping. In addition there are used not only general results of the well-known method of the fixed point, constricted mappings, the theory of indeces etc., but a number of special theorems which arose in a great extent in connection with the theory of non-linear oscillations.

The review of these last results is given in Chapter II. The application of the method of point mappings to some questions of the theory of non-linear oscillations is stated in Chapter III.

The size of the present paper does not permit to give a complete review of actual non-linear systems investigated by the method of point mappings. A list of works which includes rather completely literature on this question is supplemented at the end of the paper.

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CHAPTER II

FUNDAMENTAL THEOREMS ON POINT MAPPINGS THEORY USED FOR THE INVESTIGATION OF NON-LINEAR DYNAMIC SYSTEMS

I. Theorems on fixed point stability of a point mapping

A. Fixed points of mapping. The determination of stability and instability

Let every point $M(x_1, x_2, \ldots, x_n)$ of a certain geometric image m is mapped uniquely into its other point $M(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ by the formulas

$$\bar{x}_i = f_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n) \qquad (2.I.I)$$

In this case the formulas (2.I.I.) are said to determine a point mapping T of the geometric image m into itself. The point M^* is called a fixed point of the mapping T if $TM^* = M^*$. The fixed point M^* is called asymptotically stable if for any point M belonging to sufficiently small ε of the neighbourhood M^* there

holds

$$\rho(T^m M, M^*) < \varepsilon_m \qquad (2.I.2)$$

where $\varepsilon_m \to 0$ when $m \to \infty$ and $\max \varepsilon_m \to 0$ when $\varepsilon \to 0$.

The fixed point M^* is called unstable if for a certain $\varepsilon > 0$ in any arbitrary small neighbourhood of the point M^* there exist points M which after successive application to them of the mapping T leave ε-neighbourhood of the fixed point M^*.

B. Liapunov's direct method

Let us formulate a theorem which is the extension of Liapunov's well-known theorems [77] to point mappings.

Theorem 2.I. [75]. Let $M^*(x_1^*, \ldots, x_n^*)$ be a fixed point of a mapping T, and let a certain continuous function V vanishing at the fixed point M^* possess such a property that in a certain neighbourhood of the point M^* the difference

$$V(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n) - V(x_1, x_2, \ldots, x_n) \qquad (2.I.3)$$

is definitely negative. Then the fixed point M^* of the mapping T is asymptotically stable if V is definitely positive in the neighbourhood of this fixed point, and it is unstable if the

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function V is alternate or definitely negative.

According to this theorem after the construction of the function V for which the difference (2.I.3), written in the form

$$V(TM) - V(M) = V(\bar{M}) - V(M),$$

will be definitely negative (such a function will be called Liapunov's function) the problem of fixed point stability is to define whether the function V constructed in such a way is positive or admits negative values too.

It is essential to note in the case of stability it is possible not only to determine the existence of stability by Liapunov's function, but to estimate the region of attraction of the fixed point.

C. A canonical form of a point mapping in the neighbourhood of the fixed point

Let the point M^* with coordinates $x_1^* = x_2^* = \ldots = x_n^* = 0$ be a fixed point of the mapping (2.I.I) which admits continuous derivatives of the first and the second order in the neighbourhood of the point M^*. After the separation of the linear terms in the formulas (2.I.I) and after an appropriate linear change of variables $x_1, x_2, \ldots, x_n$ (in a general case with complex coefficients) the formulas of the mapping (2.I.I) can be reduced to the form

$$\bar{y}_i = \lambda_i y_i + S_i \quad (i = 1, 2, \ldots, n) \qquad (2.I.4)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of the characteristic polynomial:

$$\chi(z) = \| a_{ij} - \delta_{ij} z \| = 0 \qquad (2.I.5)$$

where δ_{ij} is the Kroneker symbol.

It should be noted that the representation in the form (2.I.4) is possible only in such a case if all the roots of the polynomial $\chi(z)$ are different. When there are equal roots, the notation (2.I.4) does not hold. In this particular case it is necessary to resort to the notation of the linear side of the mapping (2.I.I) in the so-called normal form which coincides with a diagonal form in the case of the lack of multiple roots.

The notation of the mapping T in the neighbourhood of the fixed point in the form (2.I.4) is called canonic [78].

D. Theorems on fixed point stability of a point mapping

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The conditions of stability and instability of the fixed point of a point mapping in uncritical [79-82, 78] and critical cases [81-84, 74] are stated below.

Theorem 2.2. For the stability of the fixed point of the mapping (2.I.I) it is sufficient that all the roots have moduli less than unit.

Theorem 2.3. The fixed point of the point mapping (2.I.I) is unstable if at least one root of the characteristic polynomial (2.I.5) has modulus more than unit.

Thus the problem of fixed point stability is reduced to a pure algebraic problem of the location of the roots of the characteristic polynomial (2.I.5) with respect to a unit circle.

Note that when there are roots lying on the boundary of the unit circle the theorem does not permit to judge about the stability of the fixed point. This case is critical and requires further investigation.

it is necessery that all the roots of the charac
I8 teristic polynomial have moduli more than unit, and it is sufficient that

Let a point mapping T_n of n-dimensional Euclidean space R_n into itself taking the point $M(x_1, x_2, \ldots, x_m, \ldots, x_n)$ into the point $\bar{M}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ be represented in the form:

$$\bar{x} = f(x,y), \quad \bar{y} = y^*(x) + g(x,y) \qquad (2.I.6)$$

where x is an m-dimensional vector whose components are the first m co-ordinates of the point M, y is (n-m)-dimensional vector whose components are n-m last co-ordinates of the point M, and

$$|g(x,y)| < q\,|y - y^*(x)|. \qquad (2.I.7)$$

Consider the mapping T_m of m-dimensional space into itself which is defined as follows:

$$\bar{x} = f[x, y^*(x)]. \qquad (2.I.8)$$

It is obvious, if x^* is the fixed point of the mapping T_m, $[x^*, y^*(x^*)]$ will be the fixed point of the mapping T_n.

Theorem 2.4. [83]. If $V(x)$ in the neighbourhood of the fixed point x^* is Liapunov's function [ ] for the mapping T_m and in a certain neighbourhood of the fixed point $[x^*, y^*(x^*)]$ of the mapping T_n

$$|g(x,y)| < \gamma < 1, \; |V[f(x,y)] - V[f(x,y^*(x))]| < \beta |y - y^*(x)|, \qquad (2.I.9)$$

the functi

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$$\Omega(x,y)=V(x)+A|y-y^*(x)| \qquad (2.I.10)$$

will be Liapunov's function for the mapping T_n in the neighbourhood of the point $[x^*, y^*(x^*)]$ when $A<(1-q)^{-1}$.

Let $\lambda_1,\lambda_2,\dots,\lambda_n$ be the roots of the characteristic equation of a mapping T which can be differentiated sufficient number of times and let $\lambda_1,\lambda_2,\dots,\lambda_m$ be distinct from $\lambda_{m+1},\dots,\lambda_n$. Then after an appropriate linear change of variables in the neighbourhood of the fixed point $x^*=y^*=0$ the mapping T can be written in the form:

$$\bar{x}=Lx+\Omega(x,y),\quad \bar{y}=My+\Delta(x,y) \qquad (2.I.11)$$

where the matrix L has characteristic roots $\lambda_1,\lambda_2,\dots,\lambda_m$ and the matrix M the roots $\lambda_{m+1},\dots,\lambda_n$; $|\Omega|$ and $|\Delta|$ are not less than the second order of smallness ( ) with regard to $|x|$ and $|y|$. By the theorem if $|\lambda_{m+1}|,\dots,|\lambda_n|<1$ and if the problem of the mapping stability

$$\bar{x}=Lx+\Omega(x,y^*) \qquad (2.I.12)$$

where y^* is determined uniquely in the neighbourhood of the fixed point through x according to the equation $y^*=My^*+\Delta(x,y^*)$, can be solved by the continuously differentiable function of Liapunov, the fixed point of the mapping T is stable when and only when the fixed point of the mapping (2.I.12) is stable. It follows, in particular, that the consideration of fixed point stability of the mapping of n-dimensional Euclidean space into itself in critical cases of one root is equal to +I or -I, or of two complex-conjugate roots $e^{i\varphi}$ and $e^{-i\varphi}$ is reduced to the same problem for a mapping of a straight line into a straight line or consiquently of a surface into a surface.

The case $\lambda_1=-1$ upon squaring the mapping T is reduced to the case $\lambda_1=1$. Therefore it is sufficient to consider the mapping

$$\bar{x}=x+a_s x^s+O(x^{s+1}) \qquad a_s\neq 0 \qquad (2.I.13)$$

If s is odd, Liapunov's function to be found will be $V=-a_s x^s$, if s is even, such a function will be $V=a_s x^s$. In accordance with these results when s is even, we always have instability, and when s is odd, there is stability if $a_s<0$ and instability if $a_s>0$.

The case $\lambda_1=e^{i\varphi}$, $\lambda_2=e^{-i\varphi}$, where $0<\varphi<\pi$,

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is reduced, according to the preceding discussion, to the investigation of the point mapping stability of a plane into a plane of the form:

$$\bar{u}=e^{i\varphi}u+au^2+\beta uv+cv^2+du^3+ku^2v+fuv^2+gv^3+\dots$$

$$\bar{v}=e^{-i\varphi}v+c'u^2+\beta'uv+a'v^2+g'u^3+f'u^2v+k'uv^2+d'v^3+\dots \quad (2.I.I4)$$

The fixed point of this point mapping is stable if

$$B=2Re\left\{ke^{-i\varphi}+a\beta\frac{e^{i\varphi}(1-e^{-i\varphi})}{1-e^{i\varphi}}+2\beta'\frac{1}{e^{i\varphi}-1}-\frac{2cc'}{1-e^{2i\varphi}}\right\}>0 \quad (2.I.I5)$$

and is unstable if $B<0$.

Let us now formulate the conditions of the stability of the fixed point $M^*(0,0,\dots,0)$ of the point mapping of a form

$$\bar{x}_i=x_i+\tau(x_1,\dots,x_n)\left\{A_i+\sum a_{ij}x_j+O(x_1^2+\dots+x_n^2)\right\} \quad (2.I.I6)$$

where $\tau(x_1,x_2,\dots,x_n)$ are non-negative functions, vanishing at the point M^*, and a_{ij} are constant.

Theorem 2.5. [84]. The fixed point of the mapping (2.I.I6) is stable if all $A_i=0$ and if a state of equilibrium of the system of linear differential equations

$$\frac{dx_i}{dt}=\sum a_{ij}x_j \quad (i=1,2,\dots,n) \quad (2.I.I7)$$

is asymptotically stable.

Under the requirements stated the fixed point M^* will be asymptotically stable if in a certain neighbourhood of the point M^*

$$\tau(x_1,x_2,\dots,x_n)>C(x_1^2+\dots+x_n^2) \quad (2.I.I8)$$

where $C(z)$ is a continuous function z, vanishing only when $z=0$.

If we assume that (2.I.I8) holds or there are points in any neighbourhood near M^* whose sum of values τ in successive mappings (if these points do not leave a certain fixed neighbourhood of the point M^*) infinitely increases, we have the following theorem:

Theorem 2.6. The fixed point M^* of the mapping (2.I.I6) is unstable if at least one $A_i\neq 0$ or at least one root of the characteristic equation of the system (2.I.I7) has a positive real part.

Let the mapping (2.I.I) have a set J of invariant points of dimension m which canbe represented in the form:

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$$x_i = \varphi_i(u_1, u_2, \ldots, u_m) \quad (i = 1, 2, \ldots, n) \qquad (2.I.19)$$

where $u_1, u_2, \ldots, u_m$ are real parameters. For the points J the following expression is identical with respect to $u_1, u_2, \ldots, u_m$

$$\varphi_i(u_1, u_2, \ldots, u_m) = f_i(\varphi_1, \varphi_2, \ldots, \varphi_n). \qquad (2.I.20)$$

Differentiating (2.I.20) with respect to u_s, we obtain the following relations:

$$\frac{\partial \varphi_i}{\partial u_s} = \left(\frac{\partial f_i}{\partial x_1}\right)\frac{\partial \varphi_1}{\partial u_s} + \cdots + \left(\frac{\partial f_i}{\partial x_n}\right)\frac{\partial \varphi_n}{\partial u_s} \qquad (2.I.21)$$

which imply that the mapping T being linearized at the point $x_1 = \varphi_1, x_2 = \varphi_2, \ldots, x_n = \varphi_n$ takes the tangents to J of the vectors $\left(\frac{\partial \varphi_1}{\partial u_s}, \frac{\partial \varphi_2}{\partial u_s}, \ldots, \frac{\partial \varphi_n}{\partial u_s}\right)$ into themselves. Hence, the characteristic equation of the mapping T in the points J has m unit roots.

Theorem 2.7. /78.74/. For stability of the fixed point belonging to the set J it is sufficient that $n - m$ roots lie inside a circle of a unit radius.

2. Theorems on the relationship between periodic and fixed points

A. The description of the class of dynamic systems under consideration. /76/

Let the motion of a dynamic system be described as dependent on its state (and probably on its previous motion) by one of N systems of differential equations of the form:

$$\frac{dx_i^j}{dt} = X_i^j(t; x_1^j, x_2^j, \ldots, x_{n_j}^j) \quad \begin{pmatrix} i = 1, 2, \ldots, n_j \\ j = 1, 2, \ldots, N \end{pmatrix} \qquad (2.2.1)$$

The transition from the description by means of p-th system of equations to the description by means of q-system takes place when the following conditions have been fulfilled:

$$S_{pq}(x_1^p, x_2^p, \ldots, x_{n_p}^p, t) = 0 \qquad (2.2.2)$$

and, perhaps, certain inequalities of the form:

$$\Omega_{pq}^s(x_1^p, x_2^p, \ldots, x_{n_p}^p; t) > 0 \quad (s = 1, 2, \ldots, \gamma_{pq}). \qquad (2.2.3)$$

While proceeding from the description by means of the p-th system of equations (2.2.I) to the description by means of the q-th system the values of new variables $x_1^q, x_2^q, \ldots, x_{n_q}^q$ at the moment of the transition t are defined

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through the values of the former variables x_1^ρ, x_2^ρ, ..., $x_{n_\rho}^\rho$ at the same moment t by means of the following relations

$$x_j^\gamma = f_j^{\rho\gamma}(x_1^\rho, x_2^\rho, \ldots, x_{n_\rho}^\rho, t) \quad (j = 1,2,\ldots,n_\gamma) \qquad (2.2.4)$$

The simplest and very important from practical point of view class of systems, subject to the scheme which has been described just now, is the so-called piecewise linear systems and, in particular, systems with piecewise linear characteristics of non-linearities [3,II-47.. 7, sys - tems with impact interactions [3,5I-56].and systems which in the case of idealization are led to the so-called discontinuous oscillations [3,85-91].

The phase space of the system described above consists of the phase spaces $\Phi_1, \Phi_2, \ldots, \Phi_N$ of the systems (2.2.I). The transition from one phase space Φ_ρ to the other space Φ_γ is performed when the conditions (2.2.2) and (2.2.3) hold according to the mapping (2.2.4). Thus, we shall identify the points of the surface $S_{\rho\gamma}$ of the phase subspace Φ_ρ satisfying the conditions (2.2.3) with the corresponding points of the phase subspace Φ_γ and in accordance with

this result we shall consider the expression (2.2.4) as transition formulas from variables x_i^ρ to x_i^γ. Such identification for systems with small number of degrees of freedom can be accomplished by means of geometrically illustrative "sticking" (склеивание), which sometimes (dry friction, loop characteristic of a relay etc.) leads to the so-called many-sheet phase planes [3,63,30] or phase spaces, constituted, for example, from a halfspace and a halfplane [22]. Fig. shows the "sticking" of the phase subspaces Φ_ρ and Φ_γ on the surface $S_{\rho\gamma}$. In this case one should have in mind that the points of the surface $S_{\rho\gamma}$ can be internal points of the spaces Φ_ρ and Φ_γ. Fig. shows the "sticking" of the phase subspaces Φ_ρ and Φ_γ when the dimension of the space Φ_γ is a unit less than the one of the space Φ_ρ.

The system described above is called autonomous if none of the functions X_i^j, $S_{\rho\gamma}$, $\Omega_{\rho\gamma}^s$, $f_j^{\rho\gamma}$ contains explicitly time t, otherwise the system is called non-autonomous. By the phase space of an autonomous or a non-au-

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tonomous system is meant the "stuck" (as it was described above) phase spaces of the systems (2.2.3). Moreover, below in the case of a non-autonomous system we shall assume that X_i^j, $S_{\rho\gamma}$, $\Omega_{\rho\gamma}^s$, $f_j^{\rho\gamma}$ are periodic functions of time t with a period τ, and that according to this the identification of the points with the same coordinates x_i^j and times which differ by the whole number of periods τ has been done in this phase space of the system. Hence, closed phase trajectories will correspond to the periodic motions of both the autonomous and non-autonomous systems in their phase spaces. We shall assume that the function X_i^j of each system of equations (2.2.3) and the functions $f_j^{\rho\gamma}$, $S_{\rho\gamma}$, and $\Omega_{\rho\gamma}^s$ too are differentiable sufficient number of times and even analytic. Besides, the only phase trajectory will pass through every point of the phase space Φ (with the exception of singular points), since under given initial conditions by the equation (2.2.I) and the conditions of the transition (2.2.4), (2,2,2) and (2.2.I) a further change of phase variables is found uniquely. In this case a continuous dependence of solutions on the initial conditions, generally speaking, does not exist. The violation of the continuous dependence on the initial conditions is possible when the trajectory L is in contact with one of the surfaces $S_{\rho\gamma}$*). However, if $L(t_0 \leqslant t \leqslant t_1)$ is a segment of the phase trajectory which does not cross the boundary points of the surface $S_{\rho\gamma}$, the condition (2.2.2) being satisfied in the points of its intersection with the surfaces $S_{\rho\gamma}$, a continuous dependence of solution on the initial conditions will exist for the points of a certain sufficiently small neighbourhood L and $t_0 \leqslant t \leqslant t_1$; and a twice continuously differentiable point mapping can be constructed in the neighbourhood L for any two surfaces S and $\bar{S}$ cutting L.

As an example of the system whose description is led to the scheme discussed above, let us consider the system of differential equations [70]

*) By the surface $S_{\rho\gamma}$ is meant a part of the surface (2.2.2) satisfying the inequalities (2.2.3).

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$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n) \qquad i = 1, 2, \ldots, n \tag{2.2.5}$$

whose functions have discontinuities of the first kind on some smooth surfaces S_α cutting the space $\mathcal{D}$ of variables x_1, x_2, ..., ..., x_n and t into the domains $\mathcal{D}_1$, ..., $\mathcal{D}_K$. In each of these domains $\mathcal{D}_j$ equations of motion are written in the form :

$$\frac{dx_i}{dt} = f_i^j(t; x_1, \ldots, x_n) \qquad i = 1, 2, \ldots, n \tag{2.2.6}$$

where f_i^j are sufficiently smooth functions in $\mathcal{D}_j$ including the boundary too. Thus, the motion of a representative point is determined in any domain $\mathcal{D}_j$. In order to have a complete description of the phase point it is necessary to find out what happens to it after reaching the boundaries of the domain $\mathcal{D}_j$. Considering various cases of the behaviour of the phase trajectories near the surfaces of discontinuity S_α one can be led to the following description of the motion of the phase point (fig.3)

I). If the phase point belongs to the domain $\mathcal{D}_j$, its motion is described by the equation (2.2.6).

2). If the phase point reaches the surface S_α *), it passes through it when $X_+(S_\alpha) X_-(S_\alpha) > 0$ and remains on it when $X_+(S_\alpha) < 0$, $X_-(S_\alpha) > 0$;

3). If the phase point as a result of sliding motion on the surface S_α comes on to the boundary $X_+(S_\alpha) = S_\alpha = 0$, it goes to the domain $\mathcal{D}^+$, when $X_+^2(S_\alpha) > 0$ and remains on the boundary under the condition $X_+^2(S_\alpha) < 0$. Similary, when it comes on to the boundary $X_-(S_\alpha) = S_\alpha = 0$, it goes to the domain $\mathcal{D}^-$ if $X_-^2(S_\alpha) < 0$ and it remains on the boudary if $X_-^2(S_\alpha) > 0$. Hence the description of the motion of the representative point has been reduced to the scheme discussed above, and the domains $\mathcal{D}_1$, $\mathcal{D}_2$, ..., $\mathcal{D}_K$, the regions of sliding motions, the boundaries of the region of sliding motion, which the phase point does not leave will be the phase subspaces etc.

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*) Here S_α is some of discontinuity surfaces whose equation is $S_\alpha(t, x_1, \ldots, x_n) = 0$, $\dot{x}_i = f_i^+(t, x_1, \ldots, x_n)$ is the equation of motion in the domain $\mathcal{D}^+ \{S_\alpha(t, x) \geq 0\}$ and X_+ is the differential operator $X_+ = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i^+ \frac{\partial}{\partial x_i}$.

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8. The construction of a point mapping in the neighbourhood of periodic motion. / 75.76 /

To periodic motion in the phase space Φ there corresponds a closed trajectory Γ, constituted from the segments $\Gamma_{j_1}, \ldots, \Gamma_{j_m}$ of phase trajectories of the system of equations (2.2.7). The segment Γ_{j_k} of the trajectories Γ situated in the phase subspace Φ_{j_k}, originates on the surface $S_{j_{k-1}j_k}$ and terminates on the surface $S_{j_k j_{k+1}}$. Let T_{pq} defined by the phase trajectories Φ_q denote the mapping of the surface S_{pq} into the surface S_{qz}, and let R_{pq} denote the transformation (2.2.4) corresponding to a change of variables from $x_1^p, \ldots, x_{n_p}^p$ to $x_1^q, x_2^y, \ldots, x_{n_q}^q$. Let $M_1, M_2, \ldots, M_m$ be consecutive points of the intersection of the closed trajectory Γ with the surfaces $S_{j_1 j_2}, \ldots, S_{j_m j_1}$. To the succession of the transition from the point M_1 to M_2, from M_2 to M_3 ..., from M_m to M_1, there corresponds the mapping:

$$T = R_{j_1 j_2} T_{j_m j_1 j_2} R_{j_m j_1} T_{j_{m-1} j_m j_1} \cdots R_{j_2 j_3} T_{j_1 j_2 j_3} \qquad (2.2.7)$$

of the surface $S_{j_1 j_2}$ into itself. To the fixed point of this mapping there corresponds a closed phase trajectory passing consecutively through the spaces $\Phi_{j_1}, \Phi_{j_2}, \ldots, \Phi_{j_m}$.

The surfaces $S_{j_1 j_2}, \ldots, S_{j_m j_1}$ can be of various dimension. If σ is the least dimension of these surfaces, the δ - neighbourhood of the point M_1 on the surface $S_{j_1 j_2}$ is taken to the ζ-"neighbourhood" of the point M_1 of dimension σ by means of the mapping (2.2.7), and therefore under the repeated application of the mapping (2.2.7) one may confine himself to the consideration of the mapping of the set $\bar{S}$ into itself. Thus searching and investigating periodic motions composed from segments of the phase trajectories $\Gamma_{j_1}, \Gamma_{j_2}, \ldots, \Gamma_{j_m}$ is expedient to reduce to the consideration of the mapping into itself of the $\sqrt{}$ least dimension σ. One of the surfaces $S_{j_1 j_2}, \ldots, S_{j_m j_1}$ having the least dimension σ, or a surface especially choosen in one of the phase spaces $\Phi_{j_1}, \ldots,$

$\sqrt{}$ surface of the

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..., Φ_{jm} of dimension $\sigma+1$ can be such a surface.

If $\bar{S}$ is another, different from S cutting surface of dimension $\bar{\sigma} \geqslant \sigma$, and $\bar{T}$ is the mapping $\bar{S}$ into itself corresponding to the considered type of periodic motion, the characteristic equation of the mapping $\bar{T}$, as it is immediately seen, has $\bar{\sigma}-\sigma$ zero roots, and the rest of roots coincide with the roots of the characteristic equation of the mapping T.

The method itself of the construction of a point mapping, generally speaking, depends on the structure of the phase space of the system under consideration, but in all well-known problems it consists of the repetition of the construction described below may be over and over. At some stages of this construction the right-hand sides of the differential equations describing the system and the functions $S(t; x_1, \ldots, x_n)$ and $\bar{S}(t; x_1, \ldots, x_n)$ (which will be discussed below) are analytical or differentiable sufficient number of times functions.

Assume that in a certain neighbourhood of the phase trajectory L_0 crossing the surfaces S and $\bar{S}$ at the points M_0 and $\bar{M}_0$. bet-

ween the surfaces S and $\bar{S}$ the motions of the system are described by the equations:

$$\frac{dx_i}{dt} = f_i(t; x_1, \ldots, x_n) \qquad (2.2.8)$$

and at the point $\bar{M}_0$ the condition

$$\frac{\partial \bar{S}}{\partial t} + \sum_{i=1}^{n} \frac{\partial \bar{S}}{\partial x_i} f_i \neq 0 \qquad (2.2.9)$$

is fulfilled. Then the phase curve L starting from any sufficiently close to M_0 the point M will cut the surface $\bar{S}$ in some point $\bar{M}$ near $\bar{M}_0$. Thus a point mapping of the surface S into $\bar{S}$ in the neighbourhood of the points M_0 and $\bar{M}_0$ respectively is defined by means of the phase trajectories of the system of equations (2.2.8).

The finding of the point mapping which was described is not difficult if a general solution of the system of equations (2.2.8) is known. In case when it is impossible to obtain a general solution of the system (2.2.8), one should resort to one or another approximate method of integration. The most suitable method for our purposes is a method, based on finding the solution in the form of a power series in

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the so-called small parameter.

Let $x_i = \varphi_i(t)$ $(t_0 \le t \le t_1)$ be an equation of the given phase trajectory L_0. Introduce new variables $\xi_1, \xi_2, \ldots, \xi_n$ and a small parameter μ, letting $x_i = \varphi_i(t) + \mu\xi_i$ we shall search for the solution of the system of equations (2.2.I6) satisfying the initial conditions $x_i = \varphi_i(t_0) + \mu\xi_{i0}$ when $t = t_0$. The equations (2.2.8) in variables $\xi_1, \xi_2, \ldots, \xi_n$ will be written in the form:

$$\frac{d\xi_i}{dt} = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)_{x_s=\varphi_s} \xi_j + \frac{1}{2}\mu \sum_{j,k} \left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right)_{x_s=\varphi_s} \xi_j \xi_k \qquad (2.2.I0)$$

It is necessary to find the solution $\xi_i(t)$ of this equation which is mapped into ξ_{i0} when $t = t_0$. This solution can be found in the form of a power series in μ.

$$\xi_i(t) = \xi_i^0(t) + \mu\xi_i'(t) + \mu^2\xi_i^2(t) + \ldots$$

where $\xi_i^0(t)$ satisfies the same initial conditions as $\xi(t)$ and ξ_i', ξ_i^2 vanish when $t = t_0$.

Thus the point mapping of the plane $t = t_0$ into the plane $t = \bar{t}_0$ in the neighbourhood of the points M_0 and $\bar{M}_0$ of the intersection of these planes with the known phase trajectory L_0 can be found in the form of a series relative

to the initiel deviations of the coordinates of the point M from the point M_0. In particular, a linear approximation of this mapping is obtained by integrating an equation in variations, that is the equation (2.2.I0) when $\mu = 0$. The finding of the subsequent approximations (quadratic relative to the differences $x_j - x_{j_0}$, cubic etc.) is carried out in quadratures by the formula (2.2.9) provided the general solution of the equation in variations (2.2.I0) is known.

The mapping of S and $\bar{S}$ in the neighbourhood of the points M and $\bar{M}$ can be obtained by successive mapping the surface S into the plane $t = t_0$, the plane $t = t_0$ into the plane $t = \bar{t}_0$ the plane $t = \bar{t}_0$ into the surface $\bar{S}$.

The mappings of the planes $t = t_0$ and $t = \bar{t}_0$ into the surfaces S and $\bar{S}$ in a linear approximation are written respectively in the form:

$$x_i = x_i' + (f_i)_{M_0} \left\{X(S)\right\}_{M_0}^{-1} \sum \left(\frac{\partial S}{\partial x_j}\right)_{M_0} (x_j' - x_{j_0}) \qquad (2.2.II)$$

and in the form:

$$\bar{x}_i = \bar{x}_i' + (f_i)_{\bar{M}_0} \left\{X(\bar{S})\right\}_{\bar{M}_0}^{-1} \sum \left(\frac{\partial \bar{S}}{\partial \bar{x}_j}\right)_{\bar{M}_0} (\bar{x}_j' - \bar{x}_{j_0}) \qquad (2.2.I2)$$

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and the linearized mapping of the plane $t = t_0$ into $t = \bar{t}_0$ coincides with the mapping given by the linearized equation (2.2.8) that is by the equation in variations.

C. Theorems on periodic motions stability

There exists correspondence not only between periodic motions and the stable points of the mapping (2.2.7) but correspondence between their stability [75-76].

Theorem 2.8 . For the stability (asymptotic stability) in the sense of Liapunov of the periodic motion of a non-autonomous system it is necessary and sufficient that the fixed point of the point mapping T be stable (asymptotically stable).

Theorem 2.9. For the orbital stability (asymptotic stability) of the periodic motion of an autonomous system it is necessary and sufficient that the fixed point of the mapping T be stable (asymptotically stable).

Let T_τ be the mapping whose corresponding points lying on the same phase trajectory and being an interval of time equal to the period of the periodic motion τ apart.

Theorem 2.10. The characteristic polynomial T_τ in case of an autonomous system differs only by a factor $(1-\lambda)$ from the characteristic polynomial of the mapping T .

Theorem 2.11. For the stability in the sense of Liapunov of the periodic solution of an autonomous system it is sufficient that all the roots of the characteristic polynomial of the mapping T lie inside a unite circle.

In particular, in case of an autonomous system with smooth right-hand sides the relation

$$\rho_\kappa = \frac{1}{\tau} \ln \lambda_\kappa$$

between the characteristic exponent ρ_κ of periodic motion and the roots λ_κ of the characteristic equation of a point mapping, and the theorems 2.10 and 2.11 lead the well-known theorem of Andronov and Witt [95] .

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3. On dependence of fixed points on parameters

A. Stability domains and existence domains of the fixed point of mapping

Let a point mapping depend on the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ so that the formulas of the mapping T have the form :

$$\bar{x}_i = f_i(x_1, x_2, \ldots, x_n; \sigma_1, \sigma_2, \ldots, \sigma_m)_{(i=1,\ldots,n)}, \quad (2.3.1)$$

where the functions f_i and their derivatives $\partial f_i / \partial x_j$ are continuous with respect to the variables $x_1, x_2, \ldots, x_n$ and to the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ in the considered domains of their variation. Let the mapping T admit the fixed point M^* under certain particular values of the parameters $\sigma_1 = \sigma_1^*, \ldots, \ldots, \sigma_m = \sigma_m^*$. Introduce a space of the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ into our consideration for geometric clearness. To every point of this space there corresponds a definite mapping T.

The fixed points of the mapping T are determined from the system of n equations:

$$x_i^* = f_i(x_1^*, x_2^*, \ldots, x_n^*; \sigma_1, \sigma_2, \ldots, \sigma_m). \quad (2.3.2)$$

It follows from (2.3.2) that the coordinates of the fixed point are the functions of the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$. Therefore during continuous changing of parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ the point M^* in the space of the variable $x_1, x_2, \ldots, x_n$, generally speaking, will continuously move. It will hold till the system (2.3.2) is solvable relative to $x_1^*, x_2^*, \ldots, x_n^*$, for this is necessary that the Jacobian of the system (2.3.2) equal to

$$J(\sigma_1, \sigma_2, \ldots, \sigma_m) = \left\| \left(\frac{\partial f_i}{\partial x_j}\right)_* - \delta_{ij} \right\| \quad (2.3.3)$$

does not vanish. It follows, in particular, that the fixed point under consideration, when the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ are continuously changing, can vanish only when the Jacobian vanishes.

The characteristic polynolial $\chi(z)$ of the mapping (2.3.1) has the form :

$$\chi(z; \sigma_1, \ldots, \sigma_m) = \left\| \left(\frac{\partial f_i}{\partial x_j}\right)_* - \delta_{ij} z \right\| \quad (2.3.4)$$

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If the fixed point M^* is stable, all the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ of the polynomial (2.3.4) lie inside a unite circle. We shall change the parameters $\sigma_1, \sigma_2, \dots, \sigma_m$ continuously. Then the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ will change continuously too, and the violation of stability can occur as a result of crossing the boundaries the unite circle by one of (or by a few) roots λ_i. The crossings of the unite circle are divided into three principal cases / / : a) one of the roots λ_i turns into +I, b) one of the roots λ_i turns into -I, and c) two complex-conjugate roots $e^{i\varphi}$ and $e^{-i\varphi}$ appear. It implies that the boundary of the stability domain of the fixed point M^* in this space of the parameters $\sigma_1, \sigma_2, \dots, \sigma_m$ consists of three different surfaces $\mathcal{N}_{+1}$, $\mathcal{N}_{-1}$ and $\mathcal{N}_\varphi$ whose equations can be written respectively in the form:

$$\chi(+1; \sigma_1, \dots, \sigma_m) = 0, \quad \chi(-1; \sigma_1, \dots, \sigma_m) = 0, \quad \chi(e^{i\varphi}, \sigma_1, \dots, \sigma_m) = 0. \qquad (2.35)$$

Comparing the equations (2.3.5) with the equation of the boundary of the existence domain

of the fixed point M^* we arrive at a conclusion: the part $\mathcal{N}_{+1}$ of the boundary of the stability domain corresponding to a unit root coincides with the boundary of the existence domain of the fixed point. Thus the boundary of the existence and stability domain of the fixed point consists of three surfaces $\mathcal{N}_{+1}$, $\mathcal{N}_{-1}$ and $\mathcal{N}_\varphi$, and the equations of these surfaces can be written if the characteristic polynominal (2.3.4) is known.

Not only the surfaces $\mathcal{N}_{+1}$, $\mathcal{N}_{-1}$ and $\mathcal{N}_\varphi$, but the surfaces S_1', S_2', ..., $S_{\mathcal{N}}'$ in the space of parameters $\sigma_1, \sigma_2, \dots, \sigma_m$ can be the boundary of the existence and stability domain of the fixed point. On these surfaces the continuity of the functions f_i or their derivatives $\frac{\partial f_i}{\partial x_i}$ $\left(\frac{\partial f_i}{\partial \sigma_j}\right)$ with respect to the variables x_1, $x_2, \dots, x_n$ or to the parameters σ_j is violated.

B. Bifurcations of fixed points in passing through the boundary surfaces $\mathcal{N}_{+1}$, $\mathcal{N}_{-1}$ and $\mathcal{N}_\varphi$

Let a mapping T depend on a parameter μ,

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and when $\mu = 0$, let it admit a fixed point M^* one of the characteristic roots of which is equal to unit. A problem is to investigate fixed points of the mapping T in the neighbourhood of the point M^* when μ are small. The mapping T in the neighbourhood of the fixed point M^* can be written in the form:

$$\bar{x}_1 = a_{10} + x_1 + a_{11} x_1 + \cdots + a_{1n} x_n + a x_1^2 + \cdots$$
$$\bar{x}_i = a_{i0} + \lambda_i x_i + a_{i1} x_1 + \cdots + a_{in} x_n + \cdots , \qquad (2.3.6)$$

where all the values $a_{s\kappa}$ vanish together with μ. Since all λ_i are different from unit, the system $n-1$ equations

$$x_i^* = a_{i0} + \lambda_i x_i^* + a_{i1} x_1^* + \cdots + a_{in} x_n^* + \cdots \qquad (2.3.8)$$

is solvable with respect to x_2^*, x_3^*, ..., x_n^* when μ and x_1 are sufficiently small. Upon substituting the found expressions x_2^*, ..., x_n^* into the first relation (2.3.6) we shall find that

$$\bar{x}_1 = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \cdots , \qquad (2.3.8)$$

where $\alpha_0 = 0$, $\alpha_1 = 1$ and $\alpha_2 = a$ when $\mu = 0$. There holds.

Theorem 2.I2./ 74/. If α_0 is of the first order of smallness concerning μ and changes its sign together with μ , there are not fixed points of the mapping T in the sufficiently small neighbourhood M^* if $a\alpha_0 > 0$. When μ is sufficiently small there are two and only two fixed points M_1^* and M_2^* (one is stable, the other is unstable) if $a\alpha_0 < 0$. When $\mu \to 0$ and $a\alpha_0 < 0$ the points M_1^* and M_2^* coalesce into the point M^* and then disappear when μ increases.

2. In case of passing through the boundary surface $\mathcal{N}_1$ one of the roots of the characteristic polynominal becomes equal to -I.

Then the mapping T can be written in the form :

$$\bar{x}_1 = \lambda_1(\mu) x_1 + \sum_{s,\kappa} a_{1s\kappa} x_s x_\kappa + \sum_{s,\kappa,j} b_{1s\kappa j} x_s x_\kappa x_j + \cdots$$
$$\bar{x}_i = \lambda_i(\mu) x_i + \sum_{p,q} a_{ipq} x_p x_q + \cdots , \qquad (2.3.9)$$

where $\lambda_1(0) = -1$, $|\lambda_i(0)| \neq 1$. Let us compose the mapping T^2

$$\bar{\bar{x}}_1 = \lambda_1^2 x_1 + (\lambda_1 + \lambda_1^2) a_{11} x_1^2 + \sum_{\kappa=2}^{n} \lambda_1 (\lambda_\kappa + 1) (a_{1\kappa} + a_{\kappa 1}) x_1 x_\kappa + \cdots$$
$$\bar{\bar{x}}_i = \lambda_i^2 x_i + \sum_{p,q} a_{ipq} (\lambda_i + \lambda_p \lambda_q) x_p x_q + \cdots \qquad (2.3.10)$$

the system of equations

$$\bar{\bar{x}}_i = x_i = x_i^* \qquad i = 2, 3, \ldots, n$$

is solvable and

$$x_i^* = (1 - \lambda_i^2)^{-1} (\lambda_i + \lambda_i^2) a_{i11} x_1^2 + \cdots . \qquad (2.3.11)$$

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Upon substituting (2.3.II) into the first formula of the mapping (2.3.I0) we shall find with precision of the third order of smallness with respect to x_i that

$$\bar{\bar{x}}_i = \lambda_i^2 x_i + \lambda_i (1+\lambda_i) a_{iii} x_i^2 + g x_i^3 + \cdots, \quad (2.3I2)$$

where

$$g = \sum_{k=1}^{n} (a_{iik} + a_{ikk}) \lambda_i (1-\lambda_k)^{-1} (\lambda_k + \lambda_i^2) a_{kii} +$$
$$+ (\lambda_i + \lambda_i^3) b_{iiii} + \lambda_i (a_{iik} a_{kii} + a_{ikk} a_{kii}) 2\lambda_i a_{iii}^2 . \quad (2.3.I3)$$

Assume that when $\mu = 0$, $g = g_0 \neq 0$. In this case we have the following theorem :

Theorem 2.I3 [95.747]. Let a point mapping T which depends on a parameter μ and is continuously differentiable a sufficient number of times admit a fixed point $M^*(\mu)$ which is stable when $\mu < 0$ and unstable when $\mu > 0$, the violation of stability when $\mu = 0$ occuring because of the appearance of a root equal to -I of the characteristic equation. Then depending on the sign of the quantity g_0 one of the following cases is possible :

a) if $g_0 > 0$ the first point $M^*(\mu)$ of the mapping T is unstable when $\mu < 0$, and when μ increases up to zero two-fold fixed points $M_1^*(\mu)$ and $M_2^*(\mu)$ coalesce with the point $M^*(\mu)$;

b) if $g_0 < 0$ the fixed point $M^*(\mu)$ of the mapping T is stable when $\mu = 0$, and when μ increases from zero two two-fold stable fixed points $M_1^*(\mu)$ and $M_2^*(\mu)$ are generated from the fixed point $M^*(\mu)$. In the case "b" there exists a fixed neighbourhood of the point $M^*(0)$ and such $\mu_0 > 0$ that when $|\mu| < \mu_0$, all the points of the neighbourhood except the points tending to $M^*(\mu)$ under a repeated application of the mappin T asymptotically approach either the points $M_1^*(\mu)$ and $M_2^*(\mu)$ if $\mu > 0$ or the point $M^*(\mu)$ if $\mu < 0$.

3. In the neighbourhood of the fixed point for the values the parameters close to the bifurcation surface $N\varphi$ the mapping T can be written in the form :

$$\bar{u} = \rho (u \cos\varphi - v \sin\varphi) + \cdots,$$
$$\bar{v} = \rho (u \sin\varphi + v \cos\varphi) + \cdots,$$
$$\bar{x}_i = \lambda_i x_i + \cdots, \quad (2.3.I4)$$

ρ being equal to unit on the surface $N\varphi$. The results obtained here can be formulated in the form of the following theorem :

Theorem 2.I4 [95.747]. Let a mapping which

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is dependent on a parameter μ and is continuously differentiable a sufficient number of times, when μ is near zero, admit a fixed point $M^*(\mu)$ which is stable when $\mu<0$ an unstable when $\mu>0$, the violation of stability occuring because of the appearance of two com - plex-conjugate roots $e^{i\varphi}$ and $e^{-i\varphi}$ of the characteristic polynominal. Then depending on the sign of the quantity $\bar{g}_0$ found in values wi - thin the fourth order of smallness in the expansion of the point mapping T into a series in the neighbourhood of the fixed point when $\mu = 0$ one of the following cases is possible:

a) if $\bar{g}_0<0$, the fixed point $M^*(r)$ of the mapping T is unstable when $\mu = 0$, and when μ increases up to zero, an invariant two--dimensional unstable closed curve $\gamma^*(\mu)$ coalesce with the point $M^*(\mu)$.

b) if $\bar{g}_0>0$ the fixed point $M^*(\mu)$ of the mapping T is stable when $\mu = 0$, and when μ increases from zero, an invariant two-dimensional stable closed curve $\gamma^*(\mu)$ is generated from the fixed point $M^*(\mu)$.

The mapping (2.3.I4) determines a certain

point mapping on the invariant curve $z=\gamma(\theta,\rho,\mu)$.

$$\bar{\theta} = \varphi + \theta + \beta(\theta,\rho,\mu). \qquad (2.3.I5)$$

The mapping (2.3.I5) can be considered as the mapping of a circumference into itself, depending on two parameters φ and ρ.

The nature of the mapping (2.3.I5) is determined in a great extent by the so-called rotation number introduced by A. Poincare in connection with the study of the partition of a torus into trajectories. Let θ_0 be a point of a circumference. We shall find points θ_1, θ_2... as a result of the successive application of the transformation (2.3.I5) to the point θ_0. The rotation number ω is determined as

$$\omega = \lim_{n\to\infty} \frac{\theta_n - \theta}{2\pi n}. \qquad (2.3.I6)$$

This limit always exist (if $\bar{\theta}$ according to the transformation (2.3.I5) uniformly increases together with θ and the transformation is one - one; these conditions are fulfilled in the case under consideration) and does not depend on the initial point θ_0. The rotation number ω depends on the parameters continuously if the transformation itself depends on them continuously [97].

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Now a question of multiple fixed points of the transformation (2.3.I5) is solved by the theorem of Poincare-Denjen [98], stating that ω can be rational and equal to an irreducible fraction $\frac{p}{q}$ if and only if the considered transformation of a circumference into itself has q-multiple fixed points.

The rotation number ω is a continuous function of the parameters ρ and φ, and according to the form of the transformation (2.3.I5) since $z(\Theta, \rho, \varphi) \to 0$ when $\rho \to 1$,

$$\lim_{\rho \to 1} \omega(\varphi, \rho) = \frac{\varphi}{2\pi}. \qquad (2.3.I7)$$

Let now L be any continuous curve connecting two points of the boundary curve N_φ with various values of the parameter φ and situated, excluding the ends, in the domain $1+\delta > \rho > 1$, where δ is sufficiently small. The rotation number ω along the curve L changes continuously from some value φ_1 to φ_2, taking all the intermediate values. And to every intermediate rational value $\frac{p}{q}$, where p and q are relative primes, there corresponds the transformation (2.3.I5) having a q-multiple fixed point.

If there are stable points in linear approximation among these q-multiple points, that is a general case, there is a whole interval on the curve L corresponding to the q-multiple stable fixed point. Let L_1, L_2, ... be a sequence of curves which come together to some point of the boundary N_φ whose parameter $\varphi = \frac{2\pi p}{q}$, and for the curves L_1, L_2, ... all along $\varphi_1 < \varphi < \varphi_2$. Beginning from some number each of the curves L_i crosses "a domain" corresponding to the existence of the q-multiple fixed point. In this sense one may say that an existence domain of the q-multiple fixed points of the transformation (2.3.I5) adjoins every point N_φ corresponding to the value of the parameter φ of the form $\frac{2\pi p}{q}$. These facts show that the structure of the partition of the space of parameters in the neighbourhood N_φ possesses a complicated nature.

3. Generation of fixed points from a set of invariant points of dimension m

Let a mapping $T(\mu)$

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$$\bar{x}_i = f_i(x_1, x_2, \ldots, x_n; \mu) \quad i = 1, 2, \ldots, n \qquad (2.3.18)$$

admit a set J of invariant points of dimension m when $\mu = 0$. We shall say that the fixed point $M^*(\mu)$ of the mapping T is generated from the set of invariant points J if the point $M^*(\mu)$ approaches the set J when $\mu \to 0$.

If in a domain G the mapping T in the neighbourhood of $\mu = 0$ depends on the parameter μ and when $\mu = 0$ there are not fixed points of the mapping T different from the points J in any strictly internal towards G bounded domain G' when $|\mu| < \bar{\mu}$ where $\bar{\mu}$ is sufficiently small, here are not fixed points which would not be generated by the set J.

Therefore the investigation of the fixed points of the mapping T when μ are sufficiently small is reduced to the investigation of the generation of fixed points from the invariant set J .

Let the set invariant points in the do - main under consideration can be represented in

the form (2.I.I0).

If we assume that $n - m$ roots of the characteristic equation are different from unit , the rank of the matrix $\left(\frac{\partial f_i}{\partial x_j} - \delta_{ij}\right)$ is equal to $n - m$. The following theorems hold here .

Theorem 2.I5 [74.69]. For the correspondence of the fixed point $M^*(\mu)$ tending to $M^*(0)$ when $\mu \to 0$ to the fixed point $M^*(0)$ of the mapping T where $\mu = 0$ and $\Delta \neq 0$ it is necessary that the coordinates of this point satisfy the system of equations

$$\Phi_j(u_1, \ldots, u_m) = 0 \quad j = 1, 2, \ldots, m \qquad (2.3.19)$$

and sufficient that the Jacobian $\bar{D} = \frac{D(\Phi_1, \ldots, \Phi_m)}{D(u_1, \ldots, u_m)}$ when the parameters satisfy the system of equations (2.3.20) be not equal to zero.

When the Jacobian $\bar{D}$ is not equal to zero, the point $M^*(\mu)$ is the only fixed point tending to $M^*(0)$ when $\mu \to 0$.

The question of the fixed point stability of the mapping T when μ is sufficiently small is settled by the following theorem:

Theorem 2.I6 [74.69]. The fixed point $M^*(\mu)$ is stable when $\mu > 0$ if $n-m$ roots of the characteristic equation for the point $M^*(0)$ lie inside a

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unite circle, and if all the roots of some equation

$$\Delta(\sigma)=0 \qquad (2.3.20)$$

have negative real parts.

The fixed point $M^*(\mu)$ is unstable when sufficiently small $\mu \neq 0$ if the characteristic equation for the point $M^*(0)$ has at least one root whose modulus is more than unit, or if at least one root of the equation (2.3.20) has the positive real part. In a particular case, when the formulae of the transformation (2.3.I8) are written in the form:

$$\bar{u}_i = \lambda_i u_i + \mu f_i(u_1,\dots,u_n;\mu), \quad \lambda_1=\dots=\lambda_m=1, \quad i=1,2,\dots,n \qquad (2.3.2I)$$

the equations (2.3.I9) and (2.3.20) assume the following form:

$$\Phi_i(u_1,\dots,u_m)\equiv f_i(u_1,u_2,\dots,u_m,0,\dots,0)=0, \quad i=1,2,\dots,n$$
$$\Delta(\sigma)=\left|\left(\frac{\partial\Phi_i}{\partial u_j}\right)_*-\delta_{ij}\sigma\right|=0. \qquad (2.3.22)$$

4. A point mapping of a straight line into a straight line and a circumference into a circumference.

A. A continuous point mapping of a straight line into a straight line.

Let the coordinates x and $\bar{x}$ of the ori-

ginal and transformed points M and $\bar{M}$ of some straight line be in the relation

$$\bar{x}=f(x) \qquad (2.4.I)$$

where $f(x)$ is a continuous function x in the domain under consideration.

For the investigation of the transformation (2.4.I) one should resort to the theorem of Koenigs [79.80], to a parametric representation of point mapping, and to a geometric interpretation with the aid of Lameray's diagram.

B. A point mapping of a circumference into itself

For the first time a mapping of a circumference into itself was studied by Poincare in connection with the qualitative investigation of the trajectories of a torus [5]. These investigations were proceeded by Denjeu and by others [98]. In the work of A.G. Mayer [97] in I939 the conditions of structural stability of this transformation were found, and a theorem of continuous dependence of Poincare's rotation number was proved, and the structure of

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structural stability of a point mapping of a circumference into itself was examined.

C. A discontinuous point mapping of a straight line into itself

A whole series of concrete problems leading to the investigation of a discontinuous point mapping of a straight line into itself can be pointed out [48-50,31-33]. Moreover, it should be noted that a mapping of a circumference into itself is also reduced to a discontinuous point mapping of a straight line into itself.

Let the mapping T of a straight line into itself (2.4.I) satisfy the following conditions: I) $f(x)$ is a continuous nearly everywhere function having points of discontinuity of two types a_i and b_i, and either $f(a_i-0)<a_i$, $f(a_i+0)<a_i$ or $f(a_i-0)>a_i$, $f(a_i+0)>a_i$ and $f(b_i-0)<b_i<f(b_i+0)$. 2) the mapping (2.4.I) of a straight line into itself has ordinary fixed points α_j.

Let (σ_{k-1}, σ_k) denote intervals into which the interval (α, β) is devided by the points b_i, α_j. There hold the following theorems [99-101]:

Theorem 2.I7 [99]. Let the mapping T satisfy the above conditions and $\sigma_{k-1}<f(x)<\sigma_k$ when $x\in(\sigma_{k-1},\sigma_k)$. Then the mapping T can have only ordinary fixed points α_j. Every stable fixed point α_j has its own region of attraction bouded by the nearest b_i or unstable α_j.

Theorem 2.I8 [99]. If the mapping T is such that $f[f(x)]$ satisfies the conditions of theorem 2.I7, it can have fixed points of multiplicity not more than 2.

Suppose now that the mapping T depending on a parameter μ

$$T(\mu)x=\begin{cases}T_1(\mu)x=a(\mu)+f(x,\mu), & x<0\\ T_2(\mu)x=b(\mu)+\varphi(x,\mu), & x>0\end{cases}\qquad(2.4.2)$$

satisfies the following conditions:

I) $a(0)=b(0)=0$, 2) $f(0,\mu)=\varphi(0,\mu)=0$ 3) $a(\mu)$, $b(\mu)$, $a'(\mu)$, $b'(\mu)$ are continuous with regard to μ 4) $f(x,\mu)$, $\frac{\partial f(x,\mu)}{\partial x}$, $\frac{\partial^2 f(x,\mu)}{\partial x^2}$ are continuous with regard to μ and x when $x\le 0$, 5) $\varphi(x,\mu)$, $\frac{\partial\varphi(x,\mu)}{\partial x}$, $\frac{\partial^2\varphi(x,\mu)}{\partial x^2}$ are continuous with regard to μ and α when $x\ge 0$.

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When $\mu = 0$, the mapping T is continuous, and when $\mu > 0$ there appears one discontinuity of the first kind at the point $x = 0$.

It is found that the mapping T admits when $\mu > 0$ the existence stable fixed points of arbitrary large multiplicity but of quite definite types which correspond to various periodic motions of dynamic systems. Moreover, unperiodic, bounded, stable in the sense of Poissin motions can exist under certain values of the parameter μ .

The existence of the discontinuity of the mapping T causes the appearance of bifurcation surfaces from N_1 , N_2 . There exists, for example, a bifurcation surface on one side of which the domain of the existence of the only stable periodic motion is situated, and on the other side in an arbitrary thin layer there is a countable set of existence domains of more complicated periodic motions and uncountable set of existence domains of unperiodic, bounded, stable in the sense of Poisson motions.

CHAPTER III

APPLICATION OF THE METHOD OF POINT MAPPING TO SOME GENERAL PROBLEMS OF THE THEORY OF NON-LINEAR DYNAMIC SYSTEMS

I. On dependence of periodic motions on parameters

The study of the dependence of the structure of the phase space partition of a dynamic system into trajectories on its parameters and, related to it, the distinguishing in a space of parameters domains corresponding to the same structure of the partition of a phase space represents a basic problem of the qualitative investigation of dynamics of the system. For systems with one degree of freedom, having two-dimensional phase space, this problem under a certain restriction has been solved in a large measure in the works of A.A. Andronov , L.S. Pontrjagin, E.A. Leontovich and A.G.Mayer

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[102-106,3]. The restriction, imposed upon the systems under consideration, consists of the requirement of the so-called structural stability [102] and of the assumption that a change of a qualitative picture of the partition of the phase space occurs through systems with a finite degree of structural instability. These restrictions, which are natural from the physical point of view, permit to determine the whole qualitative picture of the partition of the phase space into trajectories through a finite number of elements : singular points, separatrices of saddle points, and limit cycles. Therefore a question of the dependence of a qualitative picture of the phase space partition into trajectories is reduced to the study of the dependence on the parameters of singular points, limit cycles and separatrices. The values of parameters under which there occurs a change of one of just enumerated elements leading to a change of the whole picture of the phase space partition into trajectories are called bifurcational. Under any bifurcation

there occurs either the generation, or the disappearance of singular points or limit cycles, or a change in the course of separatrices.

The realization of such program of the investigation for systems with a number of degrees of freedom more than unit involves a series of considerable difficulties connected first of all with the fact that there is not any qualitative theory for a system of differential equations of more than the second order as a whole. Thus one has to confine himself now to the local considerations without their immediate connection with the structure of the partition of the whole phase space into trajectories, i.e. to the investigation of the dependence on parameters of individual most important types of motions and first of all states of equilibrium and periodic motions.

Let a non-autonomous system of the type described in p.2 of Chapter 2 depend on parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$. To its stable or unstable motion Γ with a period τ there corresponds one-one the stable or unstable fixed point M^* of the mapping T_τ. Under continuous

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changing of the parameters $\sigma_1, \sigma_2, \ldots, \sigma_m$ the fixed point M^* can disappear, without leaving the domain of the definition of the point mapping T_τ, only under the condition of the appearance of a unite root of its characteristic equation and its coalescence with another fixed point. It follows from here, that the periodic motion Γ with the period τ remaining in the considered domain of the phase space under continuous changing of parameters of the system can disappear only by means of coalescence with another periodoc motion, and at the moment of coalescence one of the characteristic exponents Γ vanishes. Note that only the points for which a segment of the phase trajectory originating from them of the period τ either crosses one of the boundaries of the surface $S_{\rho\varphi}$, or is tangent to one of these surfaces can be boundary points of the definition domain of the mapping T_τ. The violation of fixed point stability occurs as a result of the appearance of a root of its characteristic equation with modulus equal to unit (i.e. a root $z = 1$ or $z = -1$,

or a couple of complex-conjugate roots of the form $z = e^{i\varphi}$ and $z = e^{-i\varphi}$. Denote by means of $\mathcal{N}_{+1}$, $\mathcal{N}_{-1}$ and $\mathcal{N}_\varphi$ the surfaces in the space of parameters corresponding to these cases.

Thus the existence domain of the periodic motion Γ is bounded by the surface $\mathcal{N}_{+1}$ and by the surfaces which correspond to the cases when Γ passes through the boundary points of the surfaces $S_{\rho\varphi}$ or when Γ is tangent to one of the surfaces. The surfaces $\mathcal{N}_{-1}$ and $\mathcal{N}_\varphi$ distinguish a subdomain in which Γ is stable from the existence domain of periodic motion Γ.

The bifurcation of the periodic motion Γ occuring in crossing the surface $\mathcal{N}_{-1}$ is described by the following theorem:

Theorem 3.1 [96.767]. If for the point mapping T_τ when $\mu = 0$ the quantity $g_0 > 0$, the periodic motion Γ when $\mu = 0$ is unstable, and when μ increases up to zero, the unstable periodic motion having at the moment of coalescence the period 2τ where τ is a period Γ when $\mu = 0$ coalesces with it; on the contrary, if $g_0 < 0$, then when $\mu = 0$ the periodic motion Γ

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is stable, and when μ increases from zero, the periodic motion Γ becomes unstable, but a stable periodic motion of a double period is generated from it.

A case of passing through the surface $\mathcal{N}_\varphi$ is more complicated.

The facts obtained here can be summed up in the form of the following theorem:

Theorem 3.2 [96,76]. If for the point mapping T_τ when $\mu = 0$ the quantity $\bar{g}_0 > 0$, the periodic motion Γ in a critical case of two characteristic roots $e^{i\varphi}$ and $e^{-i\varphi}$ is unstable, and when μ increases up to zero, an unstable two-dimensional torus composed from the phase trajectories with the rotation number of Poincare at the moment of coalescence equal to $\frac{\varphi}{2\pi}$ merges with the periodic motion Γ. If $\bar{g}_0 < 0$, then when $\mu = 0$ the periodic motion Γ is stable, and when μ increases from zero, a stable two-dimensional toroidal surface consisting of phase trajectories with the rotation number of Poincare equal to $\frac{\varphi}{2\pi}$ at the moment of generation is generated from the phase closed curve Γ.

Proceed now to the consideration of the bifurcation of periodic motions of an autonomous system. Let Γ be a closed phase trajectory and let π be some plate cutting orthogonally Γ at the point M^*. The phase trajectories close to Γ form a point mapping of the plate π into itself in some neighbourhood $\varepsilon(M^*)$ of the point M^*. The segments of the phase trajectories originating from the points $\varepsilon(M^*)$ untill their repeated intersection with the plate π form a certain $\varepsilon(\Gamma)$ neighbourhood of the closed curve Γ. Under a continuous change of the parameters of the system up to the moment of bifurcation of the periodic motion Γ two possibilities are thought:

a) the construction of the mapping T defined in some ε - neighbourhood of the point M^* is possible up to the moment of the bifurcation Γ, and ε all the time exceeds some positive number;

b) the construction of such a mapping is impossible, for in approaching the moment at

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which the bifurcation Γ occurs, the ε - neighbourhood of the point M^*, at which the continuously differentiable mapping T is defined a necessary number of times, infinitely decreases.

In the first case to the bifurcations Γ there correspond some bifurcations of the fixed point M^*, and therefore all the stated above regarding the bifurcation of periodic motions of non-autonomous systems, occuring under the intersection of the surfaces N_{+1}, N_{-1} and N_φ, can be repeated completely about them. In the second case, on the contrary, the study of the bifurcation Γ is not reduced immediately to the consideration of the bifurcation of the fixed point M^* of the mapping T. The disappearance the ε - neighbourhood of the point M^*, in which the point mapping T is defined under a continuous change of the parameters of the system, is possible only in one of the following cases : I) the closed phase curve Γ leaves any bounded domain of the phase space; 2) Γ infinitely increases its period τ ; 3) Γ cuts one of the boundaries of the surfaces S_{pq} ; 4) Γ is tangent to one of the surfaces S_{pq}; 5) Γ coalesces with one of the states of equilibrium.

Thus, summing up all the stated we arrive at a conclusion that only the surfaces N_{+1}, N_{-1}, N_φ and the surfaces corresponding to the realization of one of the just enumerated five cases can be the boundaries of the existence and stability domain of the periodic motion Γ.

The bifurcations corresponding to all enumerated five cases at present time have not been studied. Some of the well-known results are formulated below.

Theorems 3.3, 3.4 relate the case when a periodic motion passes to a trajectory which emerges from and again returns to a state of equilibrium. Theorem 3.5 relates the case when the periodic motion is drawn together to a state of equilibrium, or, that is the same, when under an inverse change of parameters a periodic motion is generated from a state of equilibrium.

Theorem 3.3 [107]. If the trajectory Γ_0 emerging from a complex state of equilibrium of the type saddle-node I) when $t \to \infty$, returns to it, 2) Γ_0 does not lie on the separatrix surface ,

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then under the disappearance of the saddle-node a single periodic motion stable when either $t \to +\infty$,or $t \to -\infty$ (fig.) is generated from Γ_0 .

Let all $\lambda_i(0)$ except $\lambda_n(0)>0$ of the system of differential equations

$$\frac{dx_i}{dt} = \lambda_i(\mu)x_i + P_i(x_1, x_2, \ldots, x_n; \mu) \quad (i = 1, 2, \ldots, n)$$

where P_i is double continuously differentiable with respect to $x_1, x_2, \ldots, x_n; \mu$ and is not less than the second order of smallness relative to $x_1, x_2, \ldots, x_n$ be different and have real parts of the same sign. In this case, as it is known, the state of equilibrium($0, 0, \ldots, 0$) is of a saddle type, and there exist one-dimensional M_1 and $n-1$ -dimensional manifolds M_{n-1} composed of O -curves . The manifold M_{n-1} is tangent to the surface $x_n = 0$ at the point ($0, \ldots, 0$) and divides the neighbourhood of the state of equilibrium into domains which we denote by D^+ and D^- .

Theorem 3.4. If I) the trajectory Γ_0 emerging from the saddle and passing through D^+ , when $\mu = 0$, comes into the saddle when $t \to +\infty$, and when $\mu > 0$ passes by the saddle, crossing D^+ , 2) $|Re\,\lambda_i(0)| > \lambda_n(0)$, then a single periodic motion which will be stable is generated from Γ_0 when sufficiently small $\mu > 0$.

Let the equations of motion in the neighbourhood a state of equilibrium can be written in the form :

$$\frac{dx_i}{dt} = \lambda_i(\mu)x_i + \sum a_{ijk}x_jx_k + \ldots, \qquad (3.I.I)$$

where $\lambda_1 = \sigma + i\omega$, $\lambda_2 = \sigma - i\omega$, $Re\,\lambda_k < 0$ $(k = 3, \ldots, n)$ and let when $\mu = 0$, $\sigma = 0$, $\omega \neq 0$ and σ increase together with μ . In this case the phase trajectories in the neighbourhood of the state of equilibrium generate a point mapping T , and the following conclusion holds:

Theorem 3.5 [76,96] . If a quantity g_0 for the mapping T when $\mu = 0$ is positive , the state of equilibrium of the system (3.I.I) when $\mu = 0$ is unstable, and with this state of equilibrium when μ increasing approaches zero there coalesces an unstable periodic motion with the period equal to $\frac{2\pi}{\omega_0}$ at the moment of coalescence; if $g_0 < 0$ when $\mu = 0$, the state of equilibrium is stable, and when μ increases

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from zero from the state of equilibrium there generates a stable periodic motion with the period equal to $\frac{2\pi}{\omega}$ at the moment of the appearance.

For the quantity g_0 one can obtain an explicit [108] expression through the coefficients of the original system (3.I.I)

$$g_0 = a_{1112} + a_{1211} + a_{1121} + a_{2221} + a_{2122} + a_{2212} - \frac{i}{\omega}\left[a_{222}(a_{212}+a_{221}) - a_{111}(a_{112}+a_{121})\right] - \sum_{\kappa}\left\{\frac{a_{\kappa 12}+a_{\kappa 21}}{\lambda_\kappa}\left(a_{22\kappa}+a_{2\kappa 2}+a_{11\kappa}+a_{1\kappa 1}\right) + \frac{a_{\kappa 11}(a_{12\kappa}+a_{1\kappa 2})(\lambda_\kappa+2i\omega)}{\lambda_\kappa^2}\right\} + \frac{a_{\kappa 11}(a_{22\kappa}+a_{2\kappa 1})(\lambda_\kappa-2i\omega)}{\lambda_\kappa^2}. \quad (3.2.2)$$

2. Unsafe and safe parts of the boundaries of the stability domains of periodic motions and states of equilibrium.

Together with the investigation of stability of states of equilibrium and periodic motions a considerable interest represents the study of the behaviour of the phase trajectories close to a state of equilibrium or to a periodic motion under small violations of their stability. It was in connection with such investigation that the notions of "unsafe"(опасное) and "safe"(безопасное) parts of the boundaries of the stability domain of a state of equilibrium were introduced by N.N. Bautin [109].

Let Γ be a certain element of the attraction of the phase trajectories (a stable state of equilibrium or a stable periodic motion) and let $\sigma(\Gamma)$ be a region of attraction Γ, i.e. a set of phase trajectories asymptotically approaching Γ when $t \to +\infty$. In changing the parameters of the system leading to the violation of stability Γ two cases are possible :

a) at the moment when the parameters of the system pass through the boundary of the stability domain, the region of attraction $\sigma(\Gamma)$ disappears constricting to Γ ;

b) in changing the parameters on the boundary of the stability domain $\sigma(\Gamma)$ does not disappear, i.e. it contains inside itself some neighbourhood Γ .

In this first case after an arbitrary small violation of stability and on the very boundary of the stability domain the phase points close to Γ when $t \to \infty$ approach another,

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different from Γ element of attraction which under an inverse change of the parameters does not coalesce with Γ. On the conrary, in the second case when the violations of stability are sufficiently small the trajectories beginning near Γ, when $t \to \infty$, remain in a small neighbourhood Γ, though Γ itself in this case becomes unstable (another stable element of attraction is "generated off" Γ under the violation of its stability, and the region of attraction $\sigma(\Gamma)$ passes to the region of attraction of this new element).

The violation of the stability of a periodic motion is possible in passing through the surfaces $\mathcal{N}_{+1}$, $\mathcal{N}_{-1}$ and $\mathcal{N}_{\varphi}$. In passing through the surface $\mathcal{N}_{+1}$ there occurs the disappearance of the periodic motion by means of its coalescence with another periodic motion; therefore the boundary $\mathcal{N}_{+1}$ is unsafe. In crossing the boundary $\mathcal{N}_{-1}$ two cases are possible : the periodic motion on the boundary itself is stable, and in passing through the boundary a stable periodic motion of a double period "is gegerated off" it ; the periodic motion on the boundary itself of the domain is stable, and in passing through the boundary an unstable periodic motion of a double period coalesces with it. The first case corresponds to a safe part of the boundary $\mathcal{N}_{-1}$, the second - to critical.

The violation of the stability of a state of equilibrium occurs when there appear either a zero root, or a couple of pure imaginary roots of the characteristic polynomial. In a general case when a zero roots appears there occurs a coalescence of stable and unstable states of equilibrium with their subsequent disappearance, so that this case corresponds to an unsafe boundary. When the roots are pure imaginary, according to theorem 3.5 the unsafe part of the boundary of the stability domain corresponds to the case $g_0 > 0$, and the safe one to the case $g_0 < 0$.

3. The method of small parameters and averaging from the point of view of the method of point mappings.

A. The method of small parameters.

Let there be given a system of n-differen-

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tial equations

$$\frac{dx_s}{dt} = F_s(t, x_1, x_2, \ldots, x_n; \mu) \qquad (3.3.I)$$

depending on a parameter μ. In connection with problems of celestial mechanics A Poincare showed a method permitting to find periodic solutions of the system (3.3.I) when $\mu = 0$ is sufficiently small, basing on known periodic solutions of the same system when $\mu = 0$. The greatest practical interest represents the case when if $\mu = 0$, the system of equations (3.3.I) admits a family of periodic solutions depending upon one or several parameters, a problem arising to find the so-called generating solutions, i.e. those periodic solutions of the family to which there tend the periodic solutions of the equation (3.3.I) when $\mu \to 0$.

This method of finding periodic motions in combination with the stability theory of A.M.Liapunov played a very important role in the discovery and investigation of a series of non-linear phenomena [110-118].

Here we shall state the method of small

parameters meaning by it a combination of the methods of A.Poincare and A.M. Liapunov from the point of view of the method of point mappings. Such statement is interesting not only due to the fact that it differs from the adopted one, but due to that it leads naturally to the practically important generalization of the method of small parameteres over systems of differential equations with discontinuous right-hand sides [69.70] (systems with dry friction, with relay, with piecewise linear characteristics of non-linearities, impacts etc.).

Let the system (3.3.I) when $\mu = 0$ be converted into a system of n-linear equations with constant coefficients which after an appropriate change of the variables can be reduced to the form :

$$\frac{dx_i}{dt} = \lambda_i x_i + \omega_i(t) + \mu f_i(t, x_1, \ldots, x_n; \mu) \quad (i = 1, 2, \ldots, n) \qquad (3.3.2)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of the characteristic equation of the system (3.3.2) with $\mu = 0$, and $\omega_i(t)$ and $f_i(t; x_1, \ldots, x_n; \mu)$ are periodic functions of t with the period 2π. We shall suppose that the function

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$\omega_i(t)$ are continuously differentiable, and the functions f_i are piecewise double continuous differentiable with respect to t; $x_1, x_2, \ldots, x_n$; μ. As for the surfaces of discontinuity $S_1 = 0, \ldots S_2 = 0$ of the functions $f_1, \ldots, f_n$ suppose that the conditions (2.2.9) are fulfilled.

In considering the periodic solutions of the non-autonomous system of equations (3.3.2) two possibilities should be distinguished.

a) non-resonance case, when among the roots $\lambda_1, \ldots, \lambda_n$ there are not roots close (with the order of nearness μ) to the number ip where p is an integer.

b) a resonance case, one of the roots λ_i is close to ip where p is an integer. In this case the system (3.3.2) is reduced to the form:

$$\frac{dx_1}{dt} = -px_2 + \mu f_1, \quad \frac{dx_2}{dt} = px_1 + \mu f_2,$$
$$\frac{dx_i}{dt} = \lambda_i x_i + \omega_i(t) + \mu f_i \quad (i = 1, 2, \ldots, n) \qquad (3.3.3)$$

The distinction of the non-resonance case from the resonance one lies in the fact that the corresponding to the equations (3.3.2) in the first case mapping T when $\mu = 0$ of the hyperplane $t = 0$ into the hyperplane $t = 2\pi$ identified with the hyperplane $t = 0$

$$\bar{x}_i = (x_i - \varphi_i(0))e^{2\pi\lambda_i} + \varphi_i(2\pi) \quad (i = 1, 2, \ldots, n) \qquad (3.4.4)$$

has a single fixed point $x_i^* = \varphi_i(0)$ $(i = 1, 2, \ldots, n)$ meanwhile in the second case this mapping

$$\bar{x}_1 = x_1, \bar{x}_2 = x_2, \bar{x}_i = (x_i - \varphi_i(0))e^{2\pi\lambda_i} + \varphi_i(2\pi) \quad (i = 3, \ldots, n) \qquad (3.3.5)$$

has a two-dimensional plane of the fixed points $x_1^* = \varphi_1(0)$ $(i = 3, \ldots, n)$. In the non-resonance case the following conclusion holds:

Theorem 3.6 [69]. When μ is sufficiently small, the periodic motions of a non-autonomous system are represented in the form:

$$x_i(t) = \varphi_i(t) + \mu(\ldots) \quad (i = 1, 2, \ldots, n). \qquad (3.3.6)$$

The periodic solution (3.3.6) will be stable if $Re\ \lambda_i < 0$ and unstable in an opposite case.

In a resonance case upon constructing a point mapping of the plane $t = 0$ into the plane $t = 2\pi$ by the method described in p.2.2 and upon using theorems 2.16 and 2.17 the following result can be established.

Theorem 3.7 [69]. When μ are sufficiently small, the periodic solutions of the sy-

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stem (3.2.2) are written in the form :

$$x_1(t) = x_1^* \cos\rho t - x_2^* \sin\rho t + \mu(\dots),$$
$$x_2(t) = x_2^* \sin\rho t + x_2^* \cos\rho t + \mu(\dots),$$
$$x_i(t) = \varphi_i(t) + \mu(\dots) \quad (i = 3, \dots, n) \qquad (3.3.7)$$

where x_1^* and x_2^* are defined from the equations

$$\Phi_1(x_1^*, x_2^*) = \sum_{j=0}^{2} \int_{t_0^j}^{t_0^{j+1}} ([f_1]\cos\rho\tau + [f_2]\sin\rho\tau)\,d\tau = 0,$$
$$\Phi_2(x_1^*, x_2^*) = \sum_{j=0}^{2} \int_{t_0^j}^{t_0^{j+1}} (-[f_1]\sin\rho\tau + [f_2]\cos\rho\tau)\,d\tau = 0 \qquad (3.3.8)$$

and $t_0', t_0^2, \dots, t_0^2$ $(t_0^{2+1} = 2\pi)$ are the moments of time of intersection of the trajectory with the surfaces $S_1 = 0$, ... , $S_2 = 0$ when $\mu = 0$ defined from the equations $[S_k] = 0$, k = 1, 2, ..., 2 (here and below brackets mean that the function is calculated when $x_1(\tau) = x_1^* \cos\rho\tau - x_2^* \sin\rho\tau$, $x_2(\tau) = x_1^* \sin\rho\tau + x_2^* \cos\rho\tau$, $x_i(\tau) = \varphi_i(\tau)$, $(i = 3, 4, \dots, n)$.

The periodic motion (3.3.7) will be stable when μ are sufficiently small, if the roots of the equations

$$\Delta(\sigma) = \sigma^2 - \left(\frac{\partial \Phi_1}{\partial x_1^*} + \frac{\partial \Phi_2}{\partial x_2^*}\right)\sigma + \frac{\partial \Phi_1}{\partial x_1^*}\frac{\partial \Phi_2}{\partial x_2^*} - \frac{\partial \Phi_1}{\partial x_2^*}\frac{\partial \Phi_2}{\partial x_1^*} = 0 \qquad (3.3.9)$$

have the sign of the real parts coinciding

with the sign μ and simultaneously

$$\mathrm{Re}\,\lambda_i < 0 \quad (i = 3, 4, \dots, n).$$

Let the system (3.2.I) be autonomous and after an appropriate change of the variables be represented in the form :

$$\frac{dx_1}{dt} = -x_2 + \mu f_1(x_1, x_2, \dots, x_n; \mu), \quad \frac{dx_2}{dt} = x_1 + \mu f_2(x_1, x_2, \dots, x_n; \mu), \quad \frac{dx_i}{dt} = \lambda_i x_i + \mu f_i(x_1, x_2, \dots, x_n; \mu), \quad (i = 3, 4, \dots, n) \qquad (3.3.10)$$

which corresponds to the fact that its characteristic equation has two pure imaginary conjugate roots. The mapping of the halfhyperplane $x_2 = 0$, $x_1 > 0$ into itself formed by the phase trajectories of the system (3.3.10) admits a straight line of the fixed points $x_3^* = \dots = x_n^* = 0$ (it is assumed that $e^{2\pi\lambda_i} \neq 1$ when $i = 3, 4, \dots, n$)

Theorem 3.8 [69]. When μ are sufficiently small the system (3.3.10) has the periodic motion of the form :

$$x_1(t) = x_1^* \cos t + \mu(\dots), \quad x_2(t) = x_1^* \sin t + \mu(\dots),$$
$$x_i(t) = \mu(\dots), \quad (i = 3, 4, \dots, n) \qquad (3.3.11)$$

where x_1^* is found from the system of equations

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$$\Phi_i(x_i^*)=\sum_{j=0}^{2}\int_{t_j^*}^{t_{j+1}^*}\left([f_1(t)]\cos t+[f_2(t)]\sin t\right)dt=0,$$
$$[S_k]=0, \quad k=1,2,\dots,2. \qquad (3.3.12)$$

The found periodic equation will be stable if the sign of the expression $\frac{\partial \Phi_i}{\partial x_i^*}$ coincides with the sign $-\mu$ and $Re\,\lambda_i<0$ $(i=3,\dots,n)$.

The method of small parameters admits generalization [70] over the system described in §.2, Chapter II. We shall confine ourselves to an example.

Consider oscillations with an impact of a ball falling on an almost elastic plane. The equation of the motion of the ball:

$$\ddot{x}_1=-1+\mu F(x,\dot{x}), \quad (x>0);$$
$$\dot{x}_1=-(1+\mu\lambda(\dot{x}_0))\dot{x}_0 \quad (x=0). \qquad (3.3.13)$$

(The second relation (3.3.13) determines the conditions of the impact).

From fig.4, which shows the phase trajectories of the system when $\mu=0$, it is clear that the finding of discontinuous periodic solutions when μ can be reduced to the finding of the fixed points $T=RT_1$, where T_1 is the mapping of the positive semi-axis Ox_2 into the negative semi-axis Ox_2, and R is the mapping $\bar{x}_2=-(1+\mu\lambda(x_2))x_2$ when $x_1=0$. We easily find that T has the following form:

$$x_2^2=x_2^0+\mu\left[\lambda(-x_2^0)x_2^0+\frac{1}{x_2^0}\int_0^{2x_2^*}\int_0^{\tau}F\left(-\frac{\tau^2}{2}+x_2^0\tau,-\tau+x_2^0\right)d\tau dt_1-\right.$$
$$\left.-\int_0^{2x_2^*}F\left(-\frac{\tau^2}{2}+x_2^0\tau,-\tau+x_2^0\right)d\tau\right]+\mu^2(\dots). \qquad (3.3.14)$$

The fixed points of the mapping T are defined from the equation

$$\Phi(x_2^*)=\lambda(-x_2^*)x_2^*+\frac{1}{x_2^*}\int_0^{2x_2^*}\int_0^{\tau}F\,d\tau dt_1-\int_0^{2x_2^*}F\,d\tau=0 \qquad (3.3.15)$$

In a particular case when

$$F(x,\dot{x})=-h\dot{x} \text{ and } \lambda(\dot{x})=\frac{|\dot{x}|}{1+\dot{x}^2} \qquad (3.3.15)$$

is reduced to the equation

$$\Phi(x_2^*)=\frac{(x_2^*)^2}{1+(x_2^*)^2}-\frac{2}{3}h(x_2^*)^2 \qquad (3.3.16)$$

From (3.3.16) we find the coordinate of the fixed point $x_2^*=\sqrt{\frac{3}{2}h^{-1}-1}$. When $\mu>0$, $\Phi(x_2^*)<0$ and therefore the periodic motion corresponding to this fixed point is stable. The corresponding bifurcation diagram is shown in fig.5.

B. The method of averaging.

The grounding of the method of averaging and therefore Van der Pol's method consist of

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the establishment under some assumptions and when μ are sufficiently small, of the relation between the solutions of the differen - tial equation

$$\frac{dx}{dt} = \mu X(x,t) \qquad (3.3.I8)$$

and the solution of the equation

$$\frac{dx}{dt} = \mu \bar{X}(x) \qquad (3.3.I9)$$

obtained from it by averaging the right-hand side with respect to time which is contained explicitly.

ε - nearness of solutions with equal initial conditions of the original and averaged equations on a finite interval of time of duration $\frac{T(\varepsilon)}{\mu}$ was established by L.I.Mandelstam and D.N. Papalexi [II9]. In this case the function $X(x, t)$ was supposed to be continuously differentiable and periodic with respect to t. In the works of N.N. Bogoljubov [I20], I.I.Gihman [I2I],M. L. Krasnocelsky and S.G. Krein [I22] was shown that this assertion holds for more general assumptions. For an infinite interval of time Bogoljubov's

results are fundamental [I23] according to which to stable (unstable) state of equilibrium or to stable (unstable) periodic motion of the averaged system (3.3.I9) when μ are sufficiently small there corresponds stable (unstable) motion or respectively stable (unstable) two-dimensional integral manifold of the original system of equations (3.3.I8). Besides, it is supposed that the characteristic exponents of periodic motion or state of equilibrium of the averaged system have real parts different from zero. The method of point mappings admits to simplify the proof of Bogoljubov's theorems and to establish some new results of the correspondence of motions of the original and averaged systems in the large. Being applied to systems of the second order such consideration helps to establish the identity of the topological structures of the partition of the phase space into the trajecto - ries of the averaged and original systems provided the averaged system is structural stable in the sense of Andronov-Pontrjagin and μ is

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sufficiently small.

I) The equations (3.3.I8) and (3.3.I9) represent a vectorial form of the notation of systems of ordinary differential equations. Let $x_1, x_2, \ldots, x_n$ and $X_1, X_2, \ldots, X_n$ be components of the vectors x and X respectively. We shall assume that the right-hand sides of the equations (3.3.I8) are the continuous differentiable functions (3.3.I8) and continuous and periodic functions of time t (of the period τ).

Let M be a certain point of the plane $t = 0$, a trajectory starting from the point M will cross again the plane $t = 0$ in a certain point $\bar{M}$ in time τ. To find the point mapping T_τ taking the point M into $\bar{M}$ it is necessary to integrate the equations (3.3.I8) at an interval of time τ. With any degree of accuracy the needed solution of the equations (3.3.I8) can be found by means of integrating the relation *)

*) One can note that the availability of discontinuities of the first order of the right-hand sides of the system (3.3.I8) on some manifolds does not represent essential changes.

$$x^m(t) = x^0(0) + \mu \int_0^t X(x^{m-1}(\sigma), \sigma)\, d\sigma \qquad (3.3.20)$$

connecting an $m-1$ approximation with an m approximation. Restricting ourselves to the first approximation we shall find that

$$x(t) = x(0) + \mu \int_0^t X(x(0), \sigma)\, d\sigma + \mu \Omega(x, \mu) \qquad (3.3.2I)$$

where in any bounded domain of the phase space when $\mu \to 0$ is uniform with respect to x

$$\Omega(x, \mu) \to 0, \quad \Omega_x'(x, \mu) \to 0. \qquad (3.3.22)$$

According to (3.3.2I) the point mapping T_τ taking a point x of the plane $t = 0$ into a point $\bar{x}$ of the same plane is written in the form:

$$\bar{x} = x + \mu \int_0^\tau X(x, t)\, dt + \mu \Omega(x, \mu) =$$

$$= x + \mu\tau \bar{X}(x) + \mu \Omega(x, \mu). \qquad (3.3.23)$$

Let us find now the point mapping T_τ for the averaged equation (3.3.I9). Similarly as in the previous case applying to the method of successive approximations and confining ourselves to the first approximation we shall find that

$$\bar{x} = x + \mu\tau \bar{X}(x) + \mu \bar{\Omega}(x, \mu) \qquad (3.3.24)$$

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where $\bar{\Omega}(x,\mu)$ as well as $\Omega(x,\mu)$ satisfies the conditions (3.3.22). Thus, from the point of view of the method of point mappings the transition from the consideration of the equation (3.3.I8) to the consideration of the averaged equation (3.3.I9) corresponds to the consideration of the point mapping (3.3.24) instead of the mapping (3.3.23). It is clear at once that all the assertions which can be done for the point mapping (3.3.24), irrespective of adding $\mu\Omega(x,\mu)$ will hold for the point mapping (3.3.23) too.

The fixed points x^* of the point mapping T_τ are found from the equation

$$\tau\bar{X}(x^*)+\Omega(x^*,\mu)=0 \qquad (3.3.25)$$

and the characteristic equation upon substituting $z=1$ by $\mu\lambda$ is written in the form

$$Det\left\|\tau\left(\frac{\partial\bar{X}_i}{\partial x_j}\right)_{x^*}+O(\mu)-\delta_{ij}\lambda\right\|=0 \qquad (3.3.26)$$

Here X_i and x_i are the components of the vectors $X(x,t)$ and x accordingly, and $O(\mu)$ defines here and further quantities approaching zero together with μ.

To every solution x_0^* of the equation (3.3.25) with $\mu=0$, for which the roots λ of the equation (3.3.26) when $\mu=0$ are different from zero there corresponds a single solution $x^*=x_0^*+O(\mu)$ of the equation (3.3.25) when μ are sufficiently small. As a result we obtain [I24] the following theorem:

Theorem 3.9 [I20]. To every state of equilibrium of the averaged system (3.3.I9) with non-zero characteristic roots there corresponds the periodic motion of the original system (3.3.I8) with the characteristic exponents $\rho=\frac{m\lambda}{\tau}+\mu O(\mu)$.

Suppose that the averaged system (3.3.I9) has a periodic solution Γ_0 with the characteristic exponents different from zero. Using the principle of constricted mappings one can prove the existence of the closed invariant curve Γ_μ of the mapping T_τ and hence prove [I23--I28] theorem.

Theorem 3.I0 [I20]. When μ are sufficiently small, to a structurally stable periodic motion of the averaged system there corresponds

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an integral toroidal surface of the original system. This toroidal surface is stable (unstable) if the periodic motion Γ_0 is stable (unstable).

Let G be any compact part of the region of attraction of a structural stable state of equilibrium or structural stable periodic motion of the averaged system. Here there holds the following

Theorem 3.II [124]. When μ are sufficiently small, all the trajectories of the original system (3.3.I8) passing through G asymptotically tend to the periodic motion or the toroidal integral surface when $t \to +\infty$.

A more complete result can be stated in application to non-autonomous systems of the second order.

Theorem 3.I2 [124]. If the averaged system is structurally stable in the sence of Andronov-Pontragin, in any part of the phase space finite with respect to x every cell [106,3] of the autonomous system interpreted in the space (x,t) there corresponds the same cell for the original non-autonomous system

with accuracy of the quantities approaching zero together with μ.

4. On stability of a state of equilibrium situated on the surface of discontinuity.

A problem of stability of a state of equilibrium situated on the surface of discontinuity, and, in particular, a problem of stability of a state of equilibrium of a relay system which lately attracted attention of a series of investigators [129-141] can serve a good illustration of the application of the theory of point mappings to the investigation of dynamic systems.

A relay system can have states of equilibrium of two types [132,135] : a state of equilibrium on a plate of "sliding" motions and a state of equilibrium similar to a focus, but with the distinction that the frequency of rotation of the phase point around the state of equilibrium, when asymptotically approaching it, tends not to a finite number but to infinity. Namely, if the transfer function $K(p)$ and the corresponding unit step response $\gamma(t)$, $\gamma(t)$ – is the

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response of the system to a unit step function, applied at $t = 0$ with the system at rest) of a relay system are such that

$$\lim_{\rho \to \infty} \rho K(\rho) = \varphi'(0) < 0,$$

there is the so-called a plate of "sliding" motions in the phase space on the plane of switching the relay (fig.6). The states of equilibrium, which is on it, will be stable if all the roots $K(\rho)$ lie on the left half-plane. When $\varphi'(0) \to 0$, the plate of "sliding" motions is drawn into a line. Besides, depending on a sign of the quantity $\varphi''(0)$ the phase trajectories either behave as shown in fig. 7, or turn round this line as shown in fig.8. In the first case the state of equilibrium of the relay system is unstable, in the second case it can be unstable depending on the fact whether the successive points of the intersection $M_0, M_1, M_2, \ldots$ of the phase trajectory (which is sufficiently near the state of equilibrium) with the plane of switching will be removed from or approach the state of equilibrium M^* when t increases. Thus the question of stability of the state of equilibrium in the latter case is reduced to the investigation of stability of the fixed point M^* of the mapping taking the point M_1 into M_2, the point M_2 into M_3 and so on.

The investigation of the stability of this point [illegible] cannot be carried out in a usual way, [illegible] all the roots of its characteristic equation are equal to unit. The question here is solved by theorem 2.5.

The similar consideration can be applied to a two-cascade relay system [illegible] and to a general problem of stability of the state of equilibrium situated on the surface [illegible] of discontinuity of the right-hand side of the system of differential equations of the form:

$$\frac{dx_i}{dt} \begin{cases} f_i^+(x_1, x_2, \ldots, x_n) & (\sigma_n > 0) \\ f_i^-(x_1, x_2, \ldots, x_n) & (\sigma_n < 0) \end{cases} \qquad (i = 1, 2, \ldots, n) \qquad (5.4.1)$$

THEOREM 5.15. In order that the point $M^*(0, 0, \ldots, 0)$ be a stable state of equilibrium of the system (5.4.1) it is sufficient that the inequalities $\sum_{j=1}^{n} a_{nj}^{+} a_j^{+} < 0$, $\sum_{j=1}^{n} a_{nj}^{-} a_j^{-} > 0$ *) and one of the fol-

*) a_j^{+}, a_{ij}^{+} and a_j^{-}, a_{ij}^{-} are constants in the decomposition of two functions f_i^{+} and f_i^{-} in the

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lowing three conditions (depending on the values

$$d = 4\sum_{j=1}^{n} a_{nj}^{-} a_j^{+} \sum_{j=1}^{n} a_{nj}^{+} a_j^{-} \Big(\sum_{j=1}^{n} a_{nj}^{+} a_j^{+} \sum_{j=1}^{n} a_{nj}^{-} a_j^{-}\Big)^{-1}$$

) be fulfilled :

I) $d < 4$ and the point M^* are stable state of equilibrium for the "sliding" notions on the surface $x_n = 0$;

2) $d = 4$, the conditions

$$a_i^{+} \sum_{j=1}^{n} a_{nj} a_j^{-} = a_i^{-} \sum_{j=1}^{n} a_{nj} a_j^{+} \quad (i = 1, 2, \ldots, n-1) \tag{3.4.2}$$

are fulfiled, and the roots of the equation

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} \beta_{ij} x_j + \cdots \quad (i = 1, 2, \ldots, n-1) \tag{3.4.3}$$

which is obtained by means of a limiting process from the corresponding point mapping (theorem 2.I5) have the negative real part;

3) $d > 4$, the preceding requirement I and the condition

$$a_i^{+} = 2\sum_{j=1}^{n} a_{nj}^{-} a_j^{+} \left(d - 2 + \sqrt{d(d-4)}\right)\left[\sum_{j=1}^{n} a_{nj}^{-} a_j^{-} \left(d + \sqrt{d(d-4)}\right)\right]^{-1} a_i^{-} \tag{3.4.4}$$

are fulfilled, and moreover the roots of the equation

$$\frac{dx_i}{dt} = W_i(x_1, x_2, \ldots, x_n) \quad (i = 1, 2, \ldots, n-1) \tag{3.4.5}$$

of the similar equation (3.4.3) lie on the left half-plane.

Theorem 3.I4. The point $M^*(0, 0, \ldots 0)$ will not be a stable state of equilibrium if the condition $\sum_{j=1}^{n} a_{nj}^{+} a_j^{+} < 0$ or $\sum_{j=1}^{n} a_{nj}^{-} a_j^{-} > 0$ are not fulfilled, or if depending on the value d , one of the following three conditions is fulfilled:

I) $d < 4$ and the point M^* is not a stable state of equilibrium for the "sliding" motions of the surface $x_n = 0$;

2) $d = 4$ the conditions (3.4.2) are not fulfilled, or at least one root of the equation (3.4.3) has the positive real part;

3) $d > 4$, the requirements I are fulfilled, or the condition (3.4.4) is not fulfilled, or at least one root of the equation (3.4.5) lies on the right half-plane.

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neighbourhood of the point M^*:

$$f_i^{\pm} = a_i^{\pm} + \sum_{j=1}^{n} a_{ij}^{\pm} x_j + \cdots .$$

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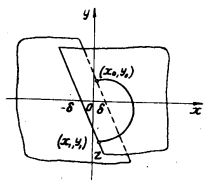
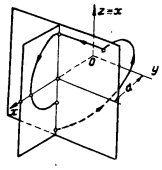


Fig. 1



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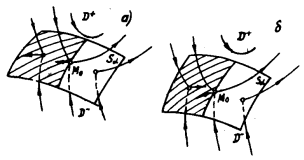

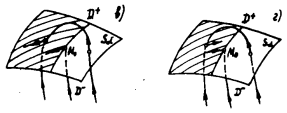

Fig. 3

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В. В. НЕМЫЦКИЙ

КОЛЕБАТЕЛЬНЫЕ РЕЖИМЫ МНОГОМЕРНЫХ ДИНАМИЧЕСКИХ СИСТЕМ

Si pour les sytèmes dynamiques définis par les systèmes linéaires d'équations différentielles a coefficients constants il est clair ~~quelsun~~ ce qu'il faut nommer régime oscillateur, pour les systèmes non linéaires cette question devient beaucoup plus compliquée puisqu'il surgissent des formes de mouvements qui ne peuvent pas se rencontrer dans les systémes linéaires. Il est maintenant trivial de parler des cycles limites comme d'un phénomene purement non linéaire. Cependant les régimes oscillateurs stables qui l'équilibre peuvent etre trés compliqués dans le cas différent de des systemes non linéaires et ne rappeler aucunement le cas linéaire.

Je me permets de proposer une certaine définition de régime oscillateur qui peut, si l'on veut, etre consideree comme une hypothese ouvriere.

Soit donnée une trajectoire $q = f(\rho, t)$; nous dirons qu'elle définit un regime oscillateur s'il existe un axe autour duquel s'effectuent les oscillations, c'est-a-dire s'il existe une droite L par rapport a laquelle la construction suivante est réalisable; soit C - le cercle unitaire ayant son centre sur L et situé dans un plan orthogonal a L. Soit $\bar{q} = f(\rho, t)$ un certain point de la trajectoire; projettons le sur le plan du cercle C. Soit q^* sa projection, menons le rayon-vecteur du point q^* et marquons sur le cercle unitaire le point d'intersection de ce rayon vecteur avec le cercle C, soit Z ce point. Considérons la representation

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$$Tq = z.$$

En faisant varier t on peut écrire

$$T(f(p,t)) = z(t)$$

Supposons maintenant que $t \to +\infty$. Nous dirons que $f(p,t)$ définit un regime oscillateur si le degré de T sur le cercle tend vers l'infini. pour $t \to +\infty$, c'est-a-dire si les points $z(t)$ decrivent le cercle une infinité de fois pour $t \to \infty$.

Pour representer analytiquement le régime oscillateur donné on peut proceder de la maniere suivante.

Supposant encore qu'une trajectoire $q = f(p,t)$ et un axe $\mathcal{L}$ sont donnés. faisons d'abord la construction precedemment décrite pour le point p et soit φ l'angle formé par le rayon-vecteur du point p^* et par l'axe Ox ; nous dirons que cet angle est la phase de l'oscillation par rapport a l'axe $\mathcal{L}$; projettons maintenant le point q^* sur l'axe Oy et designons par y^* sa projection; alors si l'on designe maintenant par z^* la distance du point q^* a l'origine des coordonnees et l'angle entre le rayon vecteur du point q^* et la direction φ par α, on a

$$y^* = z^* \sin(\alpha + \varphi),$$

ou bien en faisant varier t, nous avons

$$y^*(t) = z^*(t) \sin(\alpha(t) + \varphi)$$

Enfin pour que les formules possedent la forme habituelle, écrivons

$$y^*(t) = z^*(t) \sin\left(\left(\frac{\alpha(t)}{t}\right)t + \varphi_p\right),$$

$$y^*(t) = z^*(t) \sin(\omega(t)t + \varphi_p). \qquad \text{ou bien}$$

Dans ces notations $\max z(t)$, $0 < t < +\infty$ est l'amplitude de l'oscillation, $\omega(t) = \frac{\alpha(t)}{t}$ est la frequence variable, d'ailleurs il est commode de considerer la borne $\sup_{0 \le t < +\infty} \omega(t)$ la borne $\inf_{0 \le t < +\infty} \omega(t)$ - la borne inferieure gauche de la frequence. S'il est interessant d'etudier aussi le cas ou le temps est negatif, on peut parler de la frequence inferieur gauche. Il est evident que toutes ces quantites ne dependent pas de la phase mais elles dépendent evidemment du choix de l'axe $\mathcal{L}$, donc en etudiant les oscillations non lineaires il faut toujours dire par rapport a quel axe elles s'accomplissent.

Au lieu de considerer les amplitudes absolues $\sup z(t)$ pour tenir compte de certaines singulatites des oscillations non lineaires il est plus commode de considerer $\lim \sup_{t \to \infty} z(t) = z_0$.
Alors si $z_0 = 0$ les oscillations seront nommees amorties, si $z_0 = +\infty$ nous dirons qu'on se trouve dans le cas d'une resonance interieure, enfin si z_0 est fini et non nul nous dirons qu'on a des autooscillations non amorties ou bien simplement que $f(p,t)$ est un regime autooscillateur.

Pour le cas $n \geq 3$ les oscillations devront dependre du plan orthogonal a la droite $\mathcal{L}$. Par exemple, si l'on choisit κ plans mutuellement orthogonaux nous aurons un regime oscillateur par rapport a chacun de ces plans. Alors on peut dire que nous aurons des oscillations a κ -dimensions par rapport a l'axe L.
Passons maintenant aux criteres analytiques de l'existence des regimes oscillateurs. Nous utilisons ici une idée liee a la celebre seconde methode de Liapounoff.

Soit $\mathcal{L}$ un certain, et V un faisceau de surfaces dont

T la borne supérieure droite de la fréquence de l'oscillation et

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$\mathcal{L}$ est l'intersection et n'ayant d'autres points d'intersection. Supposons que ces plans remplissent un espace cylindrique au voisinage de L. Soit

$$U(x_1, x_2, \ldots, x_n) = \operatorname{arctg} \frac{F_1(x_1, \ldots, x_n)}{F_2(x_1, \ldots, x_n)} = \varphi$$

l'equation du faisceau, où φ varie de 0 à 2π.

Introduisons maintenant la definition suivante:

La fonction U est nommée fonction rotatoire de Liapounoff par rapport a l'axe L et pour le systeme $\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n)$ si

I° La famille $U = \varphi$ est formée de surface continues passant par l'axe L et n'ayant pas d'autre points d'intersection et il passe par chaque point du voisinage cylindrique U de l'axe L une et une seule surface de notre famille.

2° La derivée totale de U pour le systeme est continue et non nulle dans le voisinage U a savoir

$$\frac{dU}{dt} = \frac{d}{dt} \operatorname{arctg} \frac{F_1}{F_2} = \frac{\sum_{i=1}^{n} \left(\frac{\partial F_1}{\partial x_i} F_2 - \frac{\partial F_2}{\partial x_i} F_1 \right) f_i}{F_1^2 + F_2^2} > \alpha > 0.$$

Il est facile d'etablier par exemple le theoreme suivant:

Theoreme. Si dans un certain voisinage U de l'axe L il existe une fonction rotatoire de Liapounoff pour le systeme

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n)$$

on a dans ce voisinage un regime oscillatoire.

En effet on a l'identite evidente

$$U(t) = U_0 + \int_{t_0}^{t} \frac{dU}{dt} dt$$

donc, si le long de la solution on a $\frac{dU}{dt} > \alpha > 0$ on a

$U(t) = U_0 + \alpha(t - t_0)$ donc $U(t)$ croit infiniment le long de la solution, c'est-a-dire une infinite de fois le parametre angulaire φ prend toutes les valeurs de 0 à 2π. Il est certain qu'on doit tenir compte de ce que si la fonction rotatoire de Liapounoff existe, tous les regimes, dont les points initiaux se troivent dans le U-voisinage de l'axe L seront oscillatoires, donc il n'est pas exact.

Considerons un exemple. Considerons le systeme

$$\frac{dx}{dt} = y + \varepsilon P(x, y),$$
$$\frac{dy}{dt} = -x + \varepsilon Q(x, y),$$
$$\frac{dz}{dt} = R(x, y, z)$$

ou $P(x, y)$ et $Q(x, y)$ sont infiniment petits par rapport à $\sqrt{x^2 + y^2}$.

Considerons l'axe OZ comme l'axe L et comme fonction U la fonction $U = \operatorname{arctg} \frac{y}{x}$, alors nous avons

$$\frac{dU}{dt} = \frac{y^2 + \varepsilon x P + x^2 + \varepsilon y Q}{x^2 + y^2} = 1 + \varepsilon \frac{xP + yQ}{x^2 + y^2} \leq 1 + \varepsilon \frac{\sqrt{x^2 + y^2}\sqrt{P^2 + Q^2}}{x^2 + y^2};$$

$$\frac{dU}{dt} > 1 - \varepsilon \sqrt{\frac{P^2 + Q^2}{x^2 + y^2}}$$

pour une voisinage cylindrique assez petit $\frac{dU}{dt} > \frac{1}{2}$ donc le systeme donné possede un regime oscillateur par rapport a OZ. Pour les systemes non autonomes $\frac{dx_i}{dt} = f(x_1, \ldots, x_n, t)$ s'il l'on les donne au moyen de $n+1$ equations

$$\frac{dx_i}{dt} = f_i, \quad \frac{dt}{d\tau} = 1,$$

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on peut introduire les memes definitions. Le cas fondamental des régimes oscillatoires pour les systemes non autonomes c'est les regimes oscillatoires authour de l'axe t. Tâchons de distinguer les oscillations amorties et le cas d'une resonnance interieure du cas des autooscillations; a cet effect on peut profiter d'une nouvelle fonction de Liapunoff. A savoir, considerons la fonction cylindrique de Liapounoff, c'est-a-dire la famille des surfaces cylindriques avec un axe coincide avec l'axe L. Supposons que l'equation de telles surfaces est $W(x_1, x_2, \dots, x_n) = C$ Donc si pour l'une au moins de ces surfaces situées dans le voisinage cylindrique U on a $\frac{dW}{dt} > 0$, alors les cas des oscillations amorties est exclu, et si $\frac{dW}{dt} < 0$, le cas de resonnance interieure est impossible. Le long des autoiscillations ainsi trouvées les phase peuvent croitre indefiniment en module. Si nous voulons que le long des trajectoires les coordinnées restaint bornées, on peut considerer la fonction parabolique de Liapounoff $W_2(x_1, \dots, x_n) = L$ dont les surface de niveau coupent l'axe L en seul point et sont homeomorphes a un plan. Si parmi les surfaces de niveau W_2 nous trouvons deux surfaces $W_2 = C_1$ et $W_2 = C_2$ de cette nature et si $C_2 > C_1$ et

$$\left(\frac{dW_2}{dt}\right)_{C_2} < 0 \qquad \text{et} \qquad \left(\frac{dW}{dt}\right)_{C_1} > 0$$

alors dans le domaine ou toutes les trois fonctions sont definies il doit exister des autooscillations a coordonnées bornées pour $t \to +\infty$ et pour $t \to -\infty$.

Remarquons maintenat que si l'on a établi l'existence des autooscillations par la méthode de construction de

trois fonctions de Liapounoff, alors parmi les regimes oscillateurs obtenus il a aura necessairement des regimes periodiques.

Les methodes d'etude au moyen des fonctions de Liapunoff laissent toujours un sentiment d'insuffisance parcequon n'indique pas des methodes de la construction de ces fonctions. Cependant l'histoire de l'application de la seconde méthode de Liapunoff a l'etude de la stabilite montre que cette difficulté peut etre vaincue. J'espére qu'il sera ainsi dans le développement ultérieur de ma théorie. Cependant il est naturel d'indiquer des méthodes particulieres pour la recherche des regimes oscillatoirs. La théorie des sytemes non autonomes dont au second membre periode que nous fait connaitre que dans un certain nombre de cas l'existence d'une solution bornée implique l'existence des solutions periodiques. En 1960 Borg a étendu ce principe au cas des systemes autonomes. Il a etabli le theoreme suivant:

Soit donné systeme

$$\frac{dy}{dt} = f(y) \qquad \text{et} \qquad U(y) = \left\{ \frac{\partial f_i}{\partial y_j} \right\}.$$

Si dans un certain domaine borné D:

1° $\|f(y)\| \leq K$ et $\|f(y)\| \geq \delta > 0$,

2° il existe dans le domaine D une solutions située dans D pour $t_0 \leq t < +\infty$ et si 3° la matrice de Jacobi vérifie les conditions $\|U(y)\| \leq K$;

$(U(y)(y-z), y-z) \leq -\delta_1 \|y-z\|^2$ pour tous les y et z tels que $(f(y), y-z) = 0$ et $\|U(y) - U(z)\| = o(1)$ $\|y-z\| \to 0$

lors on a un cycle limite dans D.

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La seconde méthode consiste en ce qu'on établit d'abord l'existence d'une multiplicite invariante contenue dans le domaine borné donné, et on demontre ensuite qu'il se trouve sur cette multiplicite des regimes oscillateurs de nature differente /periodiques presque periodiques etc./. Pour les systemes autonomes la demonstration de l'existence d'une multiplicite invariante se trouve dans un article de V.S.Blinczevski, cependant il faut remarquer que dans un article precedent de L. Amerio l'hypothese que le systeme est non autonome est nonessentielle. Dans l'article d'une multiplicite invariante est garantie par l'existence d'un corps T du type d'un tore /produot topologique d'une sphere et d'un cercle/ tel que les trajectoires ne peuvent qu entrer ou sortir par la frontiere de ce corps. En utilisant l'analycite du systeme etudie et quelque conditions supplementaires quant a la conduite des solutions dans le domaine complexe, mais il est difficile de les verifier Amerio a etabli que toutes les solutions formant cette multiplicite invariante sont presque periodiques. Quant a Blinczevski, il suppose qu'il existe un nombre de sections (N_s) du corps toroidal par lesquelles les trajectoires passent dans meme direction. D'ailleurs pour chaque composante g_i du domaine $g=[T]-\Sigma N_s$ il se trouve vecteur α_i qui n'est pas orthogonal a aucun vecteur du champ defini par la composante g_i Dans ces conditions Blinczevski demontre qu'il existe dans le corps toroidal une solution periodique qui torne autour du tore une fois.

Enfin considerons la methode des transformations ponctuelles L'idee de cette methode appartient a Poincare et a été developpée par Birkhoff.

La methode consiste en ce que au lieu d'etudier les trajectoires elles mêmes on étude les traces de ces trajectoires sur certaine hypersurface S d'un certain nombre de dimensions. L'ensemble des trajectoires definit dans certaines hypotheses la transformations de S en elle-meme et l'on cherche un point fixe se cette transformation ou bien de sa puissance. Si l'on reussit de definir au moyen de formules analytiques la transformation étudiée, on peut non seulement démontrer l'existence d'une solution periodique, mais encore trouver quelques parametres caracterisant le mouvement periodique et parfois l'equation du mouvement lui-meme.

C'est A.A.Andronoff et ses eleves qui ont les premiers applique cette methode a des questions concretes. La technique de l'application de cette methode a été exposée dans des articles récents de G. Neymark.

Il est aussi important de remarquer que la methode des transformations ponctuelles permet quelquefois de mettre en evidence l'allure qualitative de la disposition des trajectoires dans l'espace des phase. Plusieurs auteurs ont applique cette methode a des problemes particuliers; citons, par exemple, V.Pliss et Rauch. E. Weisbrot a appliqué la methode des transformations ponctuelles a des classes tres generales d'equations du troisieme et du quatrieme ordre.

En terminant ma communication je voudrais dire que je suis sur que l'attention des mathematiciens sera attirée au probleme important de la recherches des methodes generales pour l'etude des regimes oscillateurs des systemes non lineaires.

T d'Amerio ainsi que dans celui de Blinczevski l'existance

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О ПЕРИОДИЧЕСКИХ ДВИЖЕНИЯХ ОДНОГО КЛАССА КОЛЕБАТЕЛЬНЫХ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

The periodic movements of the system described by the differential equation

$$x'(t)+2\mu x'(t)+\nu x(t)=T(t)x(t-\Delta(t,x))$$

are examined.

Here μ and ν are the parameters of the system $T(t)\geq 0$ continuous periodic function, delay function $\Delta(t,x)\geq 0$ is a continuous function ≥ 0 and in some way depends on the zeros of the unknown function.

I. Characteristics of the delay function. In some cases delay may depend on the unknown function $x(t)$. Often, for example in relay systems, the values of the $x(t)$ influence on the moment of engaging of some mechanism only and the transitive curve causing the rising of the delay has the strictly definite form which does not depend on t. Further we shall assume the "engaging" of the delay in those moments when $x(t)$ crosses the axis and if t is the zero of the solution $\bar{x}(t)$, then

$$t-\Delta(t,\bar{x})\geq \bar{t} \quad (t\geq \bar{t}). \qquad /I.1/$$

Now let an arbitrary point $t=A$ be the initial point and examine the solution $x_A(t)$ of the equation /C.I/ which satisfies the conditions

$$x_A(A)=0, \quad x_A'(A)=1. \qquad /I.2/$$

By /I.I/ these conditions determine the solution $x_A(t)$ when $t\geq A$.

Let $[A,t_1]$ be the first semicycle of the solution $x_A(t)$ and let the positive numbers k and ε be such, that $A+k+\varepsilon<t_1$ holds.

Then assume

$$0\leq\Delta(t,x_A)=\tilde{\Delta}\begin{cases} t-A, & (A\leq t\leq A+k),\\ k_\varepsilon(A+k+\varepsilon-t), & (A+k\leq t\leq A+k+\varepsilon),\\ 0, & (A+k+\varepsilon\leq t\leq t_1). \end{cases} \qquad /I.3/$$

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At last suppose that if t_i, t_{i+1} are the neighbour zeros of the arbitrary solution $\bar{x}(t)$ of the equation /0.1/ then

$$\Delta(t,\bar{x}) \equiv \begin{cases} \Delta(t-\bar{t}_i+A, x_A), & (\bar{t}_i+k\varepsilon \le t \le \bar{t}_i+k+\varepsilon), \\ 0, & (\bar{t}_i+k+\varepsilon < t < \bar{t}_{i+1}). \end{cases} \quad /1.4/$$

2. Nonautonomous case. Let $T(t+\frac{\pi}{2\omega}) \equiv T(t)$ holds in the equation /0.1/, ω being a given constant. We shall give the conditions, when $x_A(t)$ determined by /1.2/ will have a period $\frac{\pi}{\omega}$.

The substitution

$$x(t) = e^{-\mu t} y(t) \quad /2.1/$$

leads to

$$y''(t) + (\vartheta^2 - \mu^2) y(t) + T(t) e^{\mu \Delta(t,x)} y(t - \Delta(t,x)) = 0. \quad /2.2/$$

Let μ and $\lambda = \vartheta^2 - \mu^2$ be independent parameters of the equation /2.2/ and provisionally take the interval $[A, A+\frac{\pi}{2\omega}]$. Let the boundary problem of the equation /2.2/ with a fixed μ and with boundary conditions

$$y(A) = y(A+\tfrac{\pi}{2\omega}) = 0 \quad /2.3/$$

be examined.

If on the interval $[A, A+\frac{\pi}{2\omega}]$ we have

$$0 \le T(t) e^{\mu \Delta(t,x)} < \frac{\omega^2}{\pi^2} \left(\frac{\pi}{2} + \sqrt{2}\right)^2, \quad /2.4/$$

then all the characteristic values of the boundary problem are positive /1/. By /1.3/ and /1.4/ to the least of these characteristic values λ_o there corresponds the characteristic function $y_o(t)$ which has no zeros on $(A, A+\frac{\pi}{2\omega})$. Suppose $y_o'(A) = e^{\mu A}$.

LEMMA I. Let /2.4/ holds and in equation /2.2/ $\vartheta^2 - \mu^2 = \lambda_o$, λ_o being the least characteristic value of the boundary problem /2.2-3/. Then

$$\Delta(t+\tfrac{\pi}{2\omega}, x_A) \equiv \Delta(t, x_A) \quad /2.5/$$

and

$$x_A(t+\tfrac{\pi}{\omega}) \equiv x_A'(A+\tfrac{\pi}{\omega})\, x_A(t) \quad (x_A'(A+\tfrac{\pi}{\omega}) > 0). \quad /2.6/$$

Proof. By /2.1/ $x_A(t) = e^{-\mu t} y_o(t)$. Therefore in /1.3/ $t_1 = A + \frac{\pi}{2\omega}$. Then by /1.3/ the semicycle $[A, A+\frac{\pi}{2\omega}]$ is large, whence $x_A'(A+\frac{\pi}{2\omega}) < 0$. Let the initial point be transfered to the point $t = A + \frac{\pi}{2\omega}$.

Then by /1.4/ the interval $[A+\frac{\pi}{2\omega}, A+\frac{\pi}{\omega}]$ is the next semicycle of the solution $x_A(t)$, so by /1.4/ /2.5/ holds. Now for establishing /2.6/ it is enough to transfer the initial point to $t = A + \frac{\pi}{\omega}$.

Remark. Formula /2.6/ permits to continue the solution to the left from the initial point.

Similarly to [2] and using theorem 5 from [3] the following theorem may be proved

THEOREM I. Let $|\mu| \le 1$ in /0.1/ and conditions of lemma I hold uniformly for μ and

$$\sqrt{\vartheta^2 - \mu^2} \ge 10{,}04 \frac{T_o}{\omega} e^k \pi,$$

where $T_o = \max_{[A, A+\frac{\pi}{2\omega}]} T(t)$. Then there exists one and the only value of the parameter $\mu = \mu_o$ ($|\mu_o| < 1$) and corresponding to it $\vartheta = \vartheta_o (\vartheta_o^2 = \lambda_o(\mu_o) + \mu_o^2)$ such that corresponding to them solution $x_A(t)$ of the equation /0.1/ with the initial conditions /1.2/ has the period $\frac{\pi}{\omega}$.

Using lemma I let the delay in /2.2/ be fixed. Then

$$y''(t) + (\vartheta^2 - \mu^2) y(t) + T(t) e^{\mu \Delta(t, x_A)} y(t - \Delta(t, x_A)) = 0. \quad /2.2'/$$

Equation /2.2'/ is a linear equation of the second order with delay argument. If here $|\mu| < 1$ and

$$\sqrt{\vartheta^2 - \mu^2} \ge \sqrt{2}\, \frac{\pi}{\omega} T_o e^k \quad /2.7/$$

then, as it was shown in /3/ there exists one and the only equation without delay

$$y''(t) + p(t) y'(t) + q(t) y(t) = 0, \quad /2.8/$$

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[illegible] solution of which coincides with the general solution of /2.2'/ on $[A, \infty)$, coefficients being continuous periodic functions for t with a period $\frac{\pi}{2\omega}$

In the equation /2.8/ corresponding to the /2.2'/ the term with the first derivative vanish only in exceptional cases [2], so then the delay resembles friction qualitatively. This "friction" is characterised by the sign of the integral $\int_0^{\frac{\pi}{\omega}} p(t)\,dt$. The following theorem gives the conditions for "negative friction" [2], [3].

THEOREM 2. Let the conditions of lemma 1 and /2.7/ hold, $|\mu| < 1$ and $\frac{\sqrt{2}}{\sqrt{\nu^2-\mu^2}\,\tau_0 e^k} - k = \varepsilon_0 > 0$ and $\varepsilon < \varepsilon_0$ in /I.3/. Then corresponding to /2.2'/ equation /2.8/ $\int_0^{\frac{\pi}{\omega}} p(t)\,dt < 0$.

Let the system of the solutions of /2.2'/ $u(t)$, $v(t)$ defined by initial conditions

$$u(A)=0,\ u'(A)=1;\ v(A)=1,\ v'(A)=0$$

be examined.

Similary to [2] the following theorem may be proved

THEOREM 3. Let the conditions of theorem I hold. The value of the parameter $\mu=\mu_0$ will be positive, or, just the same, periodical movements will take place, the friction being positive, if the conditions of theorem 2 and one of the following conditions holds

$$u'\left(A+\frac{\pi}{\omega}\right) \geq v\left(A+\frac{\pi}{\omega}\right) \qquad /2.9/$$

or

$$2v\left(A+\frac{\pi}{\omega}\right) \leq 1+e^{-2\int_0^{\frac{\pi}{\omega}} p(t)\,dt} \qquad /2.10/$$

3. Autonomous case. Suppose now that $T(t)\equiv T>0$ in /0.1/. The conditions for the existence of the periodical solution of the equation /0.1/ with a period $\frac{\pi}{\omega}$ and the dependence of the frequency ω on the parameters of the system will be given.

LEMMA 2. Let the coefficients of /0.1/ satisfy the following condition $\nu^2-\mu^2 \geq 0$,

$$0 \leq \Delta(t, x_A) \leq \widetilde{\Delta} = \begin{cases} t-A, & (A \leq t \leq A+k), \\ \frac{k}{\varepsilon}(A+k+\varepsilon-t) & (A+k \leq A \leq A+k+\varepsilon), \\ 0, & (t \geq A+k+\varepsilon) \end{cases} \qquad /3.1/$$

and $\Delta_0 = \max_{t \geq A} \Delta(t, x_A) < k+\varepsilon < \frac{\frac{\pi}{2}+\sqrt{2}}{\sqrt{\nu^2-\mu^2}+\tau e^k}$.

Then there exists such a number $t_1 > A+k+\varepsilon$ that the solution $x_A(t)$ of the equation /0.1/ defined by the initial conditions /1.2/ satisfies

$$x_A(A)=x_A(t_1)=0,\ x_A'(t_1)<0,\ x_A(t)\neq 0,\ (A<t<A+t_1).$$

Lemma 2 gives an opportunity to use conditions /1.3/. Let t_1 be defined by lemma 2. Assume $\omega = \frac{\pi}{2t_1}$.

Then the following theorem takes place

THEOREM 4. Let the coefficients of /0.1/ satisfy /3.1/ and delay $\Delta(t, x_A)$ satisfy /1.3/ on the interval $\left[A, A+\frac{\pi}{2\omega}\right]$. If $\Delta(t, x_A)$ is determined by the condition

$$\Delta\left(t+\frac{\pi}{2\omega}, x_A\right) \equiv \Delta(t, x_A) \quad \text{for } t > A+\frac{\pi}{2\omega}, \qquad /3.2/$$

then for the solution $y_0(t)$ of the equation /2.2/ ($y_0(A)=0$, $y_0'(A)=e^{\mu A}$) the following holds:

$$y_0\left(t+\frac{\pi}{\omega}\right) \equiv y_0'\left(A+\frac{\pi}{\omega}\right) y_0(t) \qquad \left(y_0'\left(A+\frac{\pi}{\omega}\right) > 0\right). \qquad /3.3/$$

Let $\{t_k\}$ be a consequence of the zeros of the solution $y_0(t)$ of /2.2/, or, just the same, of the solution $x_A(t)$ of /0.1/.

Then $t-\Delta(t, x_A) \geq t_k \quad (t \geq t_k)$

and by /3.2/ and /3.3/

$$t_k = A + \frac{k\pi}{2\omega}. \qquad /3.4/$$

THEOREM 5. Let $0<\mu<1$ additionally holds in the conditions of the theorem 4 and $\omega=\omega(\mu)$ is such that /3.4/ holds with every $\mu \in (0,1)$. Let then

$$\sqrt{\nu^2-\mu^2} \geq 10{,}04\,\frac{T}{\omega}\,e^k\pi, \quad \frac{\sqrt{2}}{\sqrt{\nu^2-\mu^2}+\tau e^k} - k = \varepsilon_0 > 0$$

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and $\varepsilon < \varepsilon_0$ holds in /I.3/. If one of the conditions /2.9/ or /2.I0/ holds here then there is one and the only value of the parameter $\mu = \mu_0$ $(0 < \mu_0 < 1)$ such that corresponding to it solution $x_A(t)$ of /0.I/ defined by /I.2/ has the period $\frac{T}{\omega}$.

Conditions /2.9/ and /2.I0/ in theorem 5 can be replaced by the conditions that the first derivative $\Delta'_t(t, x_A)$ be continuous for t and $\frac{T}{\omega^2}$ be small.

Let $t_1 = \omega(t - A)$

in /2.2/ and determine

$$x_1(t_1) = x\left(\frac{t_1 + A\omega}{\omega}\right), \quad \Delta_1(t_1, x_0) = \Delta\left(\frac{t_1 + A\omega}{\omega}, x_A\right).$$

Now turning to the integral equation we'll get

$$y_0(t) = \frac{\sin \beta t}{\beta} - \frac{T}{\beta\omega^2}\int_0^t e^{\mu A(\tau, x_0)} \sin \beta(t-\tau) y_0(\tau - \omega\Delta(\tau, x_0))\, d\tau \qquad /3.5/$$

$\beta\omega = \sqrt{\vartheta^2 - \mu^2}$, index I is ommitted/.

Equation /3.5/ permits to get the asymptotic formula for the characteristic frequency of the system. Let $\vartheta^2\mu^2 > 1$ and $\frac{T}{\sqrt{\vartheta^2 - \mu^2}}$ in the conditions of the theorem 5 be small enough. Then $\omega = \frac{1}{2}\sqrt{\vartheta^2\mu^2} + \frac{1}{2\pi} B(\pi, 2, \Delta(t, x_0))\frac{T}{\sqrt{\vartheta^2 - \mu^2}} + O\left(\frac{T^2}{(\vartheta^2\mu^2)^{3/2}}\right)$.

Here $B(t, \beta, \Delta(t, x_0)) = \int_0^t e^{\mu\Delta(\tau, x_0)} \sin \beta(\tau - \omega\Delta(\tau, x_0))\, d\tau$,

with $|B(\pi, 2, \Delta(t, x_0))| < 2e^k$.

Using theorem 4 we get

THEOREM 6. Let the conditions of the theorem 4 hold. Then the general solution of the equation /0.I/ can be given as $x(t) = C_1 x_A(t - C_2)$, where C_1 and C_2 are arbitrary constants.

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ПРИМЕНЕНИЕ МЕТОДОВ АСИМПТОТИЧЕСКИХ РАЗЛОЖЕНИЙ ПО СТЕПЕНЯМ МАЛОГО ПАРАМЕТРА ДЛЯ ИССЛЕДОВАНИЯ КОЛЕБАНИЙ МЕХАНИЧЕСКИХ СИСТЕМ С УЧЕТОМ РАССЕЯНИЯ ЭНЕРГИИ В МАТЕРИАЛЕ

In many cases, N.M. Krylov's and N.N. Bogoliubov's methods of asymptotic expansions in powers of a small parameter can be used with great success in investigating nonlinear oscillations.

These methods are used most effectively under investigating the oscillatory processes which are described by nonlinear differential equations. These are sufficiently similar to linear equations. To these weakly nonlinear equations belong such which, being nonlinear, contain some parameter ε for whose zero value they will degenerate into linear equations.

In this case it is supposed that the parameter ε is small, i.e. it can assume only sufficiently small value /in absolute magnitute/.

The presence of a small parameter in a differential equation permits to use expansions in powers of the small parameter and then solve the problem in the first, second etc. order of approximation, assuming this order to be small.

The practical applicability of the method determine not by the properties of convergence when the power of the small parameter tends to infinity $/m \to \infty$ /, but by their asymptotic properties for a given fixed m as $\varepsilon \to 0$

The series herein obtained are asymptotic in the sense that the error of the m-th approximation

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is proportional to the /m + I/-th power of the small parameter ε . So for fixed m = I, 2, 3 ... the error will be under sufficiently small value ε . Naturally, increasing unboundedly m we shall not obtain convergence under $m \to \infty$, but absence of convergence is not essential for practical computation; as practically computation complicates so quickly under following powers ε that in fact approximations of not a great order /the first or the second/ can be used and applicability of these approximations is determined of asymptotic properties.

Clearly that efficiancy of the methods of asymptotic expansions in powers of a small parameter will be determined by how "small" is the nonlinearity of differential equations, describing oscillatory process of solved concrete problem and how many approximations will be need to obtain the practically sufficient accuracy of solution.

In many cases we can intuitively from physical sense of the problem foresee efficiancy of the methods of asymptotic expansions in powers of a small parameter ε , these solution in the first approximation will gave practical necessary accuracy.

To this kind of problem in particular we can attribute the problem about oscillations of mechanical systems when dissipation of energy in the material of elastic element of oscillatory system is taking into consideration.

In fact, assuming that dependence between deformations and force is linear in elastic element of system /the material of an elastic element follows Hoock's law/, oscillations of many mechanical systems can be described by linear differential equations.

In the case when we take into account dissipation of

energy in material, always taking place in real structural materials, it is necessary to suppose about presence /under cyclic deformation of material of elastic element of the system/ of formation of hysteresis loop in co-ordinates stress σ - relative deformation ξ . It is clear the hysteresis loop is possible only in case of nonlinear dependence $\sigma = f(\xi)$.

In connection with this under description of oscillatory process of mechanical system when dissipation of energy in material is taking into consideration, at the expense of hysteresis losses, the linear equation, which described the oscillatory motion of the system when hysteresis losses in material are not taking into account, in accordance with nonlinear dependence $\sigma = f(\xi)$ must be replaced by nonlinear differential equation. It is clear that such nonlinear equation, which nonlinearity is caused by some a very small deviation of the relationship $\sigma = f(\xi)$ from linear one, expressing Hoock's law, differs very little from linear equation, and power of this deviation can be shown easyly by introducing a small parameter ε , having quite a definite physical sense, into relationship. Indeed, considering symmetrical cycle of oscillations and representing the dissipation of energy in the unite volume of material with the amplitude of stress σ as a hysteresis loop with area A /fig.I/, under the amplytudic meaning of strain energy is accumulated in the same unit volume W /the strain energy is shown on fig.I by triangle OAB/, a small parameter ε can be assumed in the form:

$$\varepsilon = \frac{A}{W}$$

which represent nothing else but truch relative, dissipation of energy in material ψ , i.e. $\varepsilon = \psi = 2\delta$

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/δ - logariphmic decrement of oscillations/. As it is known for real structural material magnitude ψ is equal to a few percent, i.e. sufficiently small value, and it is quite possible to assume it as a small parameter.

Taking into consideration essential smallness of the above parameter ε, having quite definite physical interpretation, we may expect, that under solution of the class of problem under consideration, sufficient accuracy of solution can be secured in the first approximation.

Bellow we shall show applicability of the methods of assymptotic expansions in powers of a small parameter under solution of number of problems about free and forced oscillations of elastic system both with concentrated and distributed parameters.

I. The System with One Degree of Freedom

Differential equation of steady-state oscillations of the system with one degree of freedom when dissipation of energy in material of spring is taking into consideration can be written as follows:

$$m\frac{d^2u}{dt^2} + c[u + \varepsilon\Phi(u)] = \varepsilon\bar{q}\cos\omega t , \qquad /1/$$

where m - mass, u - travel, t - time, c - regidity, ε - a small parameter.

$\varepsilon\Phi(u)$ -some nonlinear function which takes into account dissipation of energy on the material of spring, $\varepsilon\bar{q}$ - amplytude of external perturbing force, magnitude of the same order of smallness that term characterising describing the loss of energy [illegible]

[illegible] force.

[illegible] ation.

[illegible] as follows:

$$\frac{d^2u}{dt^2} + p^2u = [illegible] + p^2\varepsilon\Phi(u) \qquad /2/$$

Consider [illegible] will be usual linear [illegible] tion of motion of the system with one degree of freedom

$$\frac{d^2u}{dt^2} + p^2u = 0 . \qquad /3/$$

The natural frequency of oscillation of this system will be p.

Taking [illegible] nonlinear [illegible] equation [illegible] which is [illegible] containing [illegible] meter in it, whose [illegible] follows [illegible] hysteresis type of problem under consideration. For [illegible] it is well to apply assymptotic methods of non-linear mechanic.

In accordance with the common methods of construct[illegible] assymptotic solutions, the solution of equation [illegible] we [illegible] find in the form of following expansion in powers of a small parameter ε:

$$u = a\cos\tau + \varepsilon u_1(a,\tau) + \varepsilon^2 u_2(a,\tau) + \cdots , \qquad /4/$$

where

$$\tau = at + \psi$$

Herein the amplitude a and phase ψ [illegible] depend upon the time and must be defined from the differential equations (in common case)

$$\frac{da}{dt} = \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots \qquad /5/$$

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[illegible]

here symbol $\overleftrightarrow{\Phi}$ - points, that the function has different expressions under ascending ($\overleftrightarrow{\Phi}$ and descending $\overleftrightarrow{\Phi}$) motion, in accordance with the equations of the branches of hysteresis loop, whose area is [illegible] reversible loss of energy in the unit volume [illegible] cyclically deformed material of the spring.

Under steady-state forced oscillations $\frac{d\psi}{dt} = 0$, i.e., $\psi = const$, and consequently $d\tau = d\theta$, [illegible] $\tau = \theta + \psi$. From this such equations follow:

$$\int_0^{2\pi} \cos\theta \cos(\theta+\psi)\, d\theta = \pi \cos\psi,$$

$$\int_0^{2\pi} \cos\theta \sin(\theta+\psi)\, d\theta = \pi \sin\psi.$$

And taking into consideration /8/, obviously:

$$\int_0^{2\pi} \left(\frac{\partial^2 u_1}{\partial \tau^2} + u_1\right) \sin\tau\, d\tau = 0,$$

$$\int_0^{2\pi} \left(\frac{\partial^2 u_1}{\partial \tau^2} + u_1\right) \cos\tau\, d\tau = 0.$$

Then from equation /10/ and /11/ find:

$$A_1(a) = \frac{p^2 \int_0^{2\pi} \overleftrightarrow{\Phi}(a\cos\tau)\sin\tau\, d\tau - \pi q \sin\psi}{2p\pi}, \qquad /12/$$

$$B_1(a) = \frac{p^2 \int_0^{2\pi} \overleftrightarrow{\Phi}(a\cos\tau)\cos\tau\, d\tau - \pi q \cos\psi}{2pa\pi}. \qquad /13/$$

As we know expressions for A_1 and B_1 differential

equations /5/ and /6/ can be rewritten in the form:

$$\frac{da}{dt} = \frac{p}{2\pi} \int_0^{2\pi} \varepsilon \overleftrightarrow{\Phi}(a\cos\tau)\sin\tau\, d\tau - \frac{\varepsilon q}{2p}\sin\psi, \qquad /14/$$

$$\frac{d\psi}{dt} = p - \omega + \frac{p}{2\pi a} \int_0^{2\pi} \overleftrightarrow{\Phi}(a\cos\tau)\cos\tau\, d\tau - \frac{\varepsilon q}{2ap}\cos\psi. \qquad /15/$$

In the case of steady-state forced oscillations we must put :

$$\frac{da}{dt} = 0 \quad \text{and} \quad \frac{d\psi}{dt} = 0.$$

Then from equation /14/ we find expression for sine of the phase angle:

$$\sin\psi = \frac{p^2}{\pi \varepsilon q} \int_0^{2\pi} \varepsilon \overleftrightarrow{\Phi}(a\cos\tau)\sin\tau\, d\tau \qquad /16/$$

and from equation /15/ the expression from frequency:

$$\omega = p + \frac{p}{2\pi a} \int_0^{2\pi} \varepsilon \overleftrightarrow{\Phi}(a\cos\tau)\cos\tau\, d\tau - \frac{\varepsilon q}{2ap}\cos\psi. \qquad /17/$$

In the basis of /16/

$$\cos\psi = \pm \sqrt{\pi^2(\varepsilon q)^2 - \left(\frac{p^2}{\pi \varepsilon q}\int_0^{2\pi} \varepsilon \overleftrightarrow{\Phi}(a\cos\tau)\sin\tau\, d\tau\right)^2}$$

Then formula /17/ can be represented in the form:

$$\frac{\omega}{p} - 1 + \frac{p^2 \int_0^{2\pi} \varepsilon \overleftrightarrow{\Phi}(a\cos\tau)\cos\tau\, d\tau \pm \sqrt{\pi^2(\varepsilon q)^2 - p^2 \int_0^{2\pi} \varepsilon \overleftrightarrow{\Phi}(a\cos\tau)\sin\tau\, d\tau^2}}{2\pi a p^2} \qquad /18/$$

This is the final formula, using it a resonance curve of oscillations can be built.

In the case of free oscillations the design formulas for determination of the amplitude and phase angle as the function of time can be obtained too from differential equations /14/ and /15/, in which it is neces-

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[illegible] to put [illegible] . As a resu[illegible]:

[illegible] /21/

[illegible] /22/

[illegible]ating the above differential equations we obtain:

[illegible] /21/

[illegible] /22/

[illegible] C_1 and C_2 are constants of integration which are [illegible]termined from the initial conditions.

For practical using of the above compative formulas [illegible], /21/ and /22/ it is necessary to have expression of the function $\varepsilon \Phi_{(\omega)}$, containing in the pointed formulas [illegible] explicity.

This function can be find from the dependence between [illegible]ress and deformation showing deviation of behaviour of [illegible] material from linear Hoock'law.

In principle this nonlinear function $\sigma = f(\xi)$ can of any form, in so far as it represents in sufficient [illegible] tail the hystersis loop /fig.1/, by whose area must ex[illegible]ressed dissipation of energy in unit volume of material [illegible]ected to variety of stresses with given amplitude for [illegible] cycle of oscillations. However as numerals investi[illegible] show from physical point of view the most groun[illegible] [illegible]nce are nonlinear dependence between stress

and deformations which was proposed by N.N. Davidenkov, these as applied to symmetrical cycle can be represented in the form:

$$\vec{\sigma} = E\left\{\xi \mp \frac{\eta}{n}\left[\xi_0 \mp \xi^{n} - 2^{n-1}\xi_0^{n}\right]\right\}, \qquad /23/$$

where the arrow directed to the right-hand side / [illegible] /, the upper sings refer to ascending motion and the arrow directed to the left-hand side / [illegible] / and lower sings refer to descending motion, E - elastic modulus, ξ - unit strain at any moment of time, ξ_0 - amplitude value of unit strain, and n, η geometric parameter of hysteresis loop which values depend upon demped properties of material. Unit strain under cyclic loading of material being the function of time can be represented in the form:

$$\xi = \xi_0 \cos\tau . \qquad /24/$$

Then relationship /23/ with /24/ under cosideration can be rewritten:

$$\vec{\sigma} = E\left\{\xi_0 \cos\tau \mp \frac{\eta}{n}\left[(1 \pm \cos\tau)^{n} - 2^{n-1}\right]\xi_0^{n}\right\}. \qquad /25/$$

From the analysis of equations /25/ it is easy to notice that the term in the right-hand side, containing multiplyer $\frac{\eta}{n}$ characterizes energy losses in material and in the initial differential equation /I/ explains the presence of term $\varepsilon \Phi_{(\omega)}$, bringing about perturbation and nonlinearity in comparison with linear equation [illegible] with respect to which the dependence [illegible] have the form:

[illegible]

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As to expression $\Phi(u)$, it will depend upon the form of oscillations and type of elastic element of oscillatory system /spring/.

So for example, in the case, when there is spring representing itself an elastic bar with a regidity $C=\frac{EF}{\ell}$ on the end of which was hung the mass undergoing vertical oscillations in accordance with /I/ and /25/

$$\varepsilon\Phi(a\cos\tau) = \frac{2}{n}C\left[(1\pm\cos\tau)^n - 2^{n-1}\right]\xi_0^n,$$

where ℓ - the length of the bar, ξ_0 - unit linear extension of the bar.

In the case of torsional oscillations of the mass at the end of the bar with torsional regidity $C=\frac{GJ_p}{\ell}$ the expression for $\Phi(u)$ must be represented in the form:

$$\varepsilon\Phi(a\cos\tau) = \frac{4\nu\tau^{\varkappa-1}}{\varkappa(\varkappa-3)\ell^{\varkappa-1}}\left[(1\pm\cos\tau)^{\varkappa} - 2^{\varkappa-1}\right]\gamma_0^{\varkappa},$$

where γ_0 - amplitude value of the angle rotation of lower end of the bar /where the mass is attached/ about the upper fixed end, z - radius of the bar; ν and $\varkappa$ - geometrical parameter of hysteresis loop in nonlinear dependence $\tau = f(\gamma)$, τ - tangential stress, γ - unit shear /analogous to /25/ /.

In the case of a spring undergoing variable bending during oscillations, under oscillations of the weight of the end of the cantilever of uniform cross-section with flexural regidity $C=\frac{3EJ}{\ell^3}$, expression for $\varepsilon\Phi(a\cos\tau)$ in accordance with /25/ becomes:

$$\varepsilon\Phi(a\cos\tau) = \frac{2\delta^{n-1}a^n}{n\ell^{3n-3}}\left[(1\pm\cos\tau)^n - 2^{n-1}\right]\int_F z^{n-1}dz\left[\iint(\ldots)dx\right]_{x=0}$$

$$\frac{\delta^{n-1}2\lambda^{n-1}a^n}{2\ln(n-1)\ell^{3n-3}}\left[(1\pm\cos\tau)^n - 2^{n-1}\right],$$

where λ - the depth of the rectangular section of the bar, a - amplitude of deflection of the end of cantilever /where the weight is attached/.

2. The System with Distributed Parameters

To illustrate an applycation of asymptotic methods of nonlinear mechanic to analysis of oscillations of elastic system with distributed parameters, when dissipation of energy in material is taken into consideration, we shall of a uniform bar, whose steady-state motion can be described by the equation:

$$m\frac{\partial^2 u}{\partial t^2} + EJ\left[\frac{\partial^4 u}{\partial x^4} + \varepsilon\frac{\partial^2}{\partial x^2}\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)\right] = \varepsilon\bar{f}(\partial t), \qquad /26/$$

where m - mass of unit length of beam, u -deflection of the beam in cross-section on the distance x from the beginning of co-ordinates /co-ordinate is directed along aseis of beam/ at any moment of time t, ε - a small parameter, $\varepsilon\frac{\partial^2}{\partial x^2}\left[\Phi\left(\frac{\partial^2 u}{\partial x^2}\right)\right]$ -some functional, taking into account dissipation of energy in material of the bar under oscillation, ν - frequency perturbing force acting on the bar.

Presence of a small parameter ε as factor of some numbers of differential equations points at weak perturbation, brought to nonlinear equation /26/ in comparison with unperturbed equation /when $\varepsilon = 0$ /

$$EJ\frac{\partial^4 u}{\partial x^4} + m\frac{\partial^2 u}{\partial t^2} = 0 \qquad /27/$$

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Introducing the following notation for shortening:

$$\frac{\partial^{\nu} u}{\partial x^{\nu}} = \frac{\partial^{\nu} \varphi}{\partial x^{\nu}} + \varepsilon \frac{\partial^{\nu} u_1}{\partial x^{\nu}} + \varepsilon^2 \frac{\partial^{\nu} u_2}{\partial x^{\nu}} + \ldots \qquad /28/$$

$$\frac{\partial^{\nu} f(x, \ldots, \frac{\partial^{\nu} u}{\partial x^{\nu}})}{\partial x^{\nu}} = \frac{\partial^{\nu} f(x, \ldots, \frac{\partial^{\nu} \varphi}{\partial x^{\nu}})}{\partial x^{\nu}} + \varepsilon \frac{\partial^{\nu} f(x, \ldots, \frac{\partial^{\nu} u_1}{\partial x^{\nu}})}{\partial x^{\nu}} + \ldots \qquad /29/$$

and

$$\varepsilon^{\nu} f^{(\nu)}(x, \ldots, \frac{\partial^{\nu} u}{\partial x^{\nu}}) = \varepsilon^{\nu} f^{(\nu)}(x, \ldots, \frac{\partial^{\nu} \varphi}{\partial x^{\nu}}) + \ldots \qquad /30/$$

Equation /26/ can be rewritten in the form:

$$\frac{\partial^{\nu} u}{\partial x^{\nu}} + \alpha^2 \frac{\partial^2 u}{\partial t^2} = \varepsilon f_1(\nu t + \psi) + \varepsilon f^{(\nu)}(x). \qquad /31/$$

As in the case of a system with one degree of freedom we shall find the solution of differential equation /31/ as following expansion in powers of a small parameter ε

$$u = u(x, \omega, \nu t, \psi) = \varphi(x) + a\cos\vartheta + \varepsilon u_1(a, \omega, \nu t, \vartheta) + \varepsilon^2 u_2(a, \nu t, \vartheta) + \ldots \qquad /32/$$

Where $\varphi(x)$ - the function of deffection and represent itself the solution of unperturbed equation /27/. $\vartheta = \frac{z}{s}\nu t + \psi$ / z and s are some mutual prime numbers/, amplitude a and phase angle ψ must be determined from the following system of differential equations

$$\frac{da}{dt} = \varepsilon A_1(a, \psi) + \varepsilon^2 A_2(a, \psi) + \ldots \qquad /33/$$

$$\frac{d\psi}{dt} = \varepsilon B_1(a, \psi) + \varepsilon^2 B_2(a, \psi) + \ldots \qquad /34/$$

To determine the functions $u_1(a, \nu t, \vartheta)$, $A_1(a, \psi)$, $B_1(a, \psi)$ when $i = 1, 2, 3 \ldots$ we shall take derivatives of /32/ $\frac{\partial^{\nu} u}{\partial x^{\nu}}$ and $\frac{\partial^2 u}{\partial t^2}$ when /33/ and /34/ are taken into consideration:

$$\frac{\partial^{\nu} u}{\partial x^{\nu}} = \frac{\partial^{\nu} \varphi}{\partial x^{\nu}} + a\cos\vartheta + \varepsilon \frac{\partial^{\nu} u_1(a, \nu t, \vartheta)}{\partial x^{\nu}} + \varepsilon^2 \frac{\partial^{\nu} u_2(a, \nu t, \vartheta)}{\partial x^{\nu}} + \ldots \qquad /35/$$

$$\frac{\partial^2 u}{\partial t^2} = -\alpha^2 \varphi(x) - a\left(\frac{z}{s}\nu\right)^2 \cos\vartheta - \varepsilon\left(\frac{z}{s}\nu\right)^2 u_1 + \alpha^2 \varphi(x) + \varepsilon\left(\frac{\partial^2 u_1}{\partial t^2} + 2\left(\omega - \frac{z}{s}\nu\right)\frac{\partial^2 u_1}{\partial \psi \partial t} + \left(\omega - \frac{z}{s}\nu\right)^2 \frac{\partial^2 u_1}{\partial \psi^2}\right) + \left[\left(\omega - \frac{z}{s}\nu\right)\frac{\partial A_1}{\partial \psi} - 2a\omega B_1\right]\cos\left(\frac{z}{s}\nu t + \psi\right) - \left[a\left(\omega - \frac{z}{s}\nu\right)B_1 + 2A_1\right]\sin\left(\frac{z}{s}\nu t + \psi\right)\Big\} + \qquad /36/$$

$$+ \alpha^2 \varphi(x) + \varepsilon^2\Big[\left(\omega - \frac{z}{s}\nu\right)^2 \frac{\partial^2 u_2}{\partial \psi^2} + 2\left(\omega - \frac{z}{s}\nu\right)B_1 \frac{\partial^2 u_1}{\partial \psi^2} + \frac{\partial^2 u_2}{\partial t^2} + 2A_1 \frac{\partial^2 u_1}{\partial a \partial t} + 2\left(\omega - \frac{z}{s}\nu\right)A_1 \frac{\partial^2 u_1}{\partial a \partial \psi} + 2\left(\omega - \frac{z}{s}\nu\right)\frac{\partial^2 u_2}{\partial \psi \partial t} + 2B_1 \frac{\partial^2 u_1}{\partial \psi \partial t} + \left(\omega - \frac{z}{s}\nu\right)\frac{\partial A_1}{\partial \psi}\frac{\partial u_1}{\partial \psi} + \left(\omega - \frac{z}{s}\nu\right)\frac{\partial B_1}{\partial \psi}\frac{\partial u_1}{\partial \psi}\Big] + \Big[-aB_1^2 - 2a\omega B_2 + A_1 \frac{\partial A_1}{\partial a} + B_1 \frac{\partial A_1}{\partial \psi} + \left(\omega - \frac{z}{s}\nu\right)\frac{\partial A_2}{\partial \psi}\Big]\cos\left(\frac{z}{s}\nu t + \psi\right) - \Big[2\omega A_2 + 2A_1 B_1 + \left(A_1 \frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_1}{\partial \psi} + \left(\omega - \frac{z}{s}\nu\right)\frac{\partial B_2}{\partial \psi}\right)a\Big]\sin\left(\frac{z}{s}\nu t + \psi\right)\Big\} + \varepsilon^3 \ldots$$

Taking into account relationships /35/ and /36/, the left-hand side of equation /31/ can be represented in the form of the following expansion in powers of a small parameter ε

$$\frac{\partial^{\nu} u}{\partial x^{\nu}} + \alpha^2 u = \frac{\partial^{\nu} \varphi}{\partial x^{\nu}} + \alpha^2 \varphi(x) + \varepsilon\left(\frac{\partial^{\nu} u_1}{\partial x^{\nu}} + \alpha^2 u_1\right) + \varepsilon^2\left(\frac{\partial^{\nu} u_2}{\partial x^{\nu}} + \alpha^2 u_2\right) + \ldots$$

$$= \alpha^2 \varphi(x) + a\cos\vartheta - \varepsilon\left(\frac{z}{s}\nu\right)^2 u_1 - \varepsilon^2\left(\frac{z}{s}\nu\right)^2 u_2 + \ldots$$

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[illegible] in the right-hand side of equation [illegible] can be found [illegible] from the above [illegible] depending [illegible] between [illegible]

[illegible]

$$\varepsilon f''(u'') = \varepsilon \frac{\partial^2}{\partial x^2} \Phi\left(\frac{\partial^2 u}{\partial x^2}\right) \approx \frac{d^2}{dx^2}\left(\frac{d^2\varphi(x)}{dx^2}\right)^n \times \qquad /45/$$

$$\times \left\{ \mp \frac{\eta}{nJ} \int_{(F)} z^{n+1} dF \left[(1 \pm \cos\tau)^n - 2^{n-1} \right] a^n \right\}$$

where z - is co-ordinate of the point of transverse cross-section of a bar, F - area of cross-section, η and n - geometrical parameters of hysteresis loop.

Introducing following notation:

$$\mp \frac{\eta}{nJ} \int_{(F)} z^{n+1} dF \left[(1 \pm \cos\tau)^n - 2^{n-1} \right] a^n = \mathcal{F}(a,\tau) \qquad /46/$$

Formula /45/ can be rewritten in the forms:

$$\varepsilon f''(u'') = \frac{d^2}{dx^2}\left(\frac{d^2\varphi(x)}{dx^2}\right)^n \mathcal{F}(a,\tau)$$

the right-hand side of equation /40/ when /36/ and /4[illegible]/ is taken into account, we can write as follows:

$$\varepsilon f_0(a,x,\nu t,\tau) = \varepsilon \bar{f}(\nu t) + \mathcal{F}(a,\tau)\frac{d^2}{dx^2}\left(\frac{d^2\varphi(x)}{dx^2}\right)^n \qquad /47/$$

Substituting /47/ into /40/ when /39/ is taken into consideration we shall have:

$$\alpha^2\varphi(x)\left\{\frac{d^2u_1}{dt^2} + 2\left(\omega - \frac{z}{s}\nu\right)\frac{\partial^2 u_1}{\partial\psi\partial t} + \left(\omega - \frac{z}{s}\nu\right)^2\frac{\partial^2 u_1}{\partial\psi^2} + \right.$$

$$\left[\left(\omega - \frac{z}{s}\nu\right)\frac{\partial A_1}{\partial\psi} - 2a\omega B_1\right]\cos\left(\frac{z}{s}\nu t + \psi\right) - \left[\left(\omega - \frac{z}{s}\nu\right)\frac{\partial B_1}{\partial\psi}a + \right.$$

$$\left.\left. + 2\omega A_1\right]\sin\left(\frac{z}{s}\nu + \psi\right) + \nu^2 u_1\right\} = \bar{f}(\nu t) + \mathcal{F}(a,\tau)\frac{d^2}{dx^2}\left(\frac{d^2\varphi(x)}{dx^2}\right)^n$$

Multiplying the right-and left-hand side of the latter equation by dx and integrating along the whole length of the bar /from 0 to ℓ /, obtain

$$\frac{d^2u_1}{dt^2} + 2\left(\omega - \frac{z}{s}\nu\right)\frac{\partial^2 u}{\partial\psi\partial t} + \left(\omega - \frac{z}{s}\nu\right)^2\frac{\partial^2 u_1}{\partial\psi^2} + \left[\left(\omega - \right.\right.$$

$$\left.\left. - \frac{z}{s}\nu\right)\frac{\partial A}{\partial\psi} - 2a\omega B_1\right]\cos\left(\frac{z}{s}\nu t + \psi\right) - a\left[\left(\omega - \frac{z}{s}\nu\right)\frac{\partial B_1}{\partial\psi} + \right. \qquad /48/$$

$$\left. + 2\omega A_1\right]\sin\left(\frac{z}{s}\nu t + \psi\right) + \nu^2 u_1 = f(a,\nu t,\tau).$$

Where

$$f(\nu t, a, \tau) = k\bar{f}(\nu t) + L\mathcal{F}(a,\tau). \qquad /49/$$

$$k = \frac{\ell}{\alpha^2 \int_0^\ell \varphi(x)\,dx}, \qquad /50/$$

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$$L = \int_0^{\ell} \alpha^2 \varepsilon \int_0^{\ell} \varphi(x)\,dx \left[\frac{d^2}{dx^2} \left(\frac{d^2\varphi(x))^n}{dx^2} \right) \right] dx \qquad /51/$$

To determine $u_1(a, \nu t, \tau)$ we shall use equation /48/ which doesn't contain the function of x.

We shall represent $u_1(a, \nu t, \tau)$, and the right-hand side of equation /48/ too as finite sums of Furiye.

$$u_1(a, \nu t, \tau) = \sum_n \sum_m u_{nm}^{(1)}(a)\, e^{i\left\{\left(\frac{\tau}{s}\right)\nu t + m\left(\frac{\tau}{s}\nu t + \psi\right)\right\}} \qquad /52/$$

$$f(a, \nu t, \tau) = \sum_n \sum_m f_{nm}^{(0)}(a)\, e^{i\left\{\left(\frac{\tau}{s}\right)\nu t + m\left(\frac{\tau}{s}\nu t + \psi\right)\right\}} \qquad /53/$$

Substituting the right-hand sides of the latter expressions to equation /48/ we obtain:

$$\left[\left(\frac{\tau}{s}\right)^2 - \left(\frac{\tau}{s}\right)\nu n - \omega m\right)^2\right] u_{nm}^{(1)}(a)\, e^{i\left\{\left(\frac{\tau}{s}\right)\nu t + m\left(\frac{\tau}{s}\nu t + \psi\right)\right\}} +$$

$$+ \left[\left(\omega - \frac{\tau}{s}\nu\right)\frac{\partial A_1}{\partial \psi} - 2a\omega B_1\right]\cos\left(\frac{\tau}{s}\nu t + \psi\right) - \left[\left(\omega - \frac{\tau}{s}\nu\right)\frac{\partial B_1}{\partial \psi} a + \right.$$

$$\left. + 2\omega A_1\right]\sin\left(\frac{\tau}{s}\nu t + \psi\right) = f_{nm}^{(0)}(a)\, e^{i\left\{\left(\frac{\tau}{s}\right)\nu t + m\left(\frac{\tau}{s}\nu t + \psi\right)\right\}} \qquad /54/$$

Where

$$f_{nm}^{(0)}(a) = \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} f_0(a, \nu t, \frac{\tau}{s}t + \psi)\, e^{-i\left\{\left(\frac{\tau}{s}\right)n\nu t + m\left(\frac{\tau}{s}\nu t + \psi\right)\right\}} d\nu t\, d\left(\frac{\tau}{s}\nu t + \psi\right). \qquad /55/$$

Whence, equating coefficience at similar harmonics, we shall find :

$$u_{nm} = \frac{f_{nm}^{(0)}(a)}{\left(\frac{\tau}{s}\right)^2 - \left(\frac{\tau}{s}\right)\nu n - \omega m)^2} \qquad /56/$$

for all n, m, satisfying condition $\left(\frac{\tau}{s}\right)^2 - \left(\frac{\tau}{s}\right)\nu n - \omega m)^2 \neq 0$

Substituting $f_{nm}^{(0)}(a)$ to the right-hand side of formula /52/ we shall have:

$$u_1(a, \nu t, \tau) = \frac{1}{4\pi^2} \sum_n \sum_m \frac{e^{i\left\{\left(\frac{\tau}{s}\right)n\nu t + m\left(\frac{\tau}{s}\nu t + \psi\right)\right\}}}{\left(\frac{\tau}{s}\right)^2 - \left(\frac{\tau}{s}\right)\nu n - \omega m)^2} \qquad /57/$$

$$[ns + (m \pm 1)z \neq 0]$$

We shall obtain the relationships for determination $A_1(a, \psi)$ and $B_1(a, \psi)$ from equation /54/ under condition :

$$\left(\frac{\tau}{s}\right)^2 - \left(\frac{\tau}{s}\right)\nu n + \omega m)^2 = 0$$

$$-\left[\left(\omega - \frac{\tau}{s}\nu\right)\frac{\partial A_1}{\partial \psi} + 2a\omega B_1\right]\cos\left(\frac{\tau}{s}\nu t + \psi\right) + \left[\left(\omega - \frac{\tau}{s}\nu\right)a\frac{\partial B_1}{\partial \psi} + \right.$$

$$\left. + 2\omega A_1\right]\sin\left(\frac{\tau}{s}\nu t + \psi\right) + \sum_n \sum_m f_{nm}^{(0)}(a)\, e^{i\left\{n\frac{\tau}{s}t + m\left(\frac{\tau}{s}\nu t + \psi\right)\right\}} = 0 \qquad /58/$$

$$[ns + (m \pm 1)z = 0].$$

Summation in equation /57/ proceeds through all interral /positive, negative and zero/ for whose:

$$ns + (m \pm 1)z = 0. \qquad /59/$$

In connection with this in this case there are complex exponents in the form:

$$e^{i\left\{(n + \frac{\tau}{s}m)\nu t + m\psi\right\}} = e^{i\left\{(ns + mz)\frac{\nu t}{s} + m\psi\right\}} = e^{i\left\{\mp\frac{\tau}{s}\nu t + m\psi\right\}} =$$

$$= e^{i\left\{\mp\left(\frac{\tau}{s}\right)\nu t + \psi) + (m \pm 1)\psi\right\}} = \left\{\cos\left(\frac{\tau}{s}\nu t + \psi\right) \mp i\sin\left(\frac{\tau}{s}\nu t + \psi\right)\right\} e^{i(m \pm 1)}$$

It is necessary to know that accordance to /58/ $m \pm 1$ is devided by S, as it can be written this multiplyer as follows:

$$S\sigma \quad (-\infty < \sigma < +\infty)$$

equated coefficients under $\cos\left(\frac{\tau}{s}\nu t + \psi\right)$ and $\sin\left(\frac{\tau}{s}\nu t + \psi\right)$ in equation /58/, substituting expression for $f_{nm}^{(0)}(a)$ in the case according to /55/, we obtain:

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/60/

/61/

where

$$\sin\tau = \frac{e^{i\tau} - e^{-i\tau}}{2i}$$

Considering, that external periodical force varies following sinusoidal law, the right-hand side of equation /48/ according /49/ can be represented by following binomial expression:

$$f(\vartheta t, a, \tau) = k\varrho \sin\vartheta t + \lambda F(a, \tau) \qquad /62/$$

where $\lambda F(a,\tau)$ - term, which taking into account dissipation of energy in material of oscillating bar, ϱ - amplitude of external perturbing force.

Then for the first approximation when $\varepsilon = 2 = 1$

$$u = \varphi(x) a \cos(\vartheta t + \psi)$$

Notating then $\vartheta t = \theta$ and considering, that $\psi = \tau - \theta$ the sum of integrals of the right-hand side equation /60/ becomes:

/63/

/64/

when $i\pi$ will be under $\sigma = 1$, and $-i\pi$ when $\sigma = -1$.

Thus the second sum of expression /63/ will be sist of two terms /under $\sigma = 1$ and under $\sigma = -1$ comes:

$$\varrho e^{i\psi}\pi \cdot i\pi + \varrho e^{-i\psi}\pi(-i\pi) = \varrho i\pi^2(e^{i\psi} - e^{-i\psi})$$

Or considering, that

$$\frac{e^{i\psi} - e^{-i\psi}}{2i} = \sin\psi$$

the expression of the second sum in /63/ according to /65/ will take the form:

$$-2\varrho\pi^2 \sin\psi.$$

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Then finally equation /61/ can be rewritten:

$$(\omega-\nu)\frac{\partial A_1}{\partial\psi}+2\omega B_1=\frac{1}{2\pi^2}\left[\int_0^{2\pi}\mathcal{L}\mathcal{F}(a,\tau)\cos\tau\,d\tau-2kq\pi^2\sin\psi\right] \qquad /66/$$

Then we shall consider integrals are contained in the right-hand side of equation /63/

$$\sum_\sigma e^{i\sigma\psi}\int_0^{2\pi}\int_0^{2\pi}\left[\mathcal{L}\mathcal{F}(a,\tau)+kq\sin\theta\right]e^{i\sigma(\tau-\theta)}\sin\tau\,d\theta\,d\tau=$$

$$=\sum_\sigma e^{i\sigma\psi}\int_0^{2\pi}\mathcal{L}\mathcal{F}(a,\tau)\sin\tau\,e^{-i\sigma\tau}d\tau\int_0^{2\pi}e^{i\sigma\theta}d\theta+$$

$$\sum_\sigma e^{i\sigma\psi}kq\int_0^{2\pi}\sin\tau\,e^{-i\sigma\tau}d\tau\int_0^{2\pi}\sin\theta\,e^{i\sigma\theta}d\theta. \qquad /67/$$

We shall take into account particular integrals which are contained in the right-hand side of the latter expression:

$$\int_0^{2\pi}e^{i\sigma\theta}d\theta\neq 0 \quad \text{only when} \quad \sigma=0 \quad .$$

Therefore the first sum of expression /67/ will be represented by the term

$$2\pi\int_0^{2\pi}\mathcal{L}\mathcal{F}(a,\tau)\sin\tau\,d\tau. \qquad /68/$$

The integrals contained in the second sum /67/ can be represented as follows

$$\int_0^{2\pi}\sin\tau\,e^{i\sigma\tau}d\tau=\int_0^{2\pi}\sin\tau(\cos\sigma\tau-i\sin\sigma\tau)d\tau=\mp i\pi$$

/ $i\pi$ when $\sigma=1$, $-i\pi$ when $\sigma=-1$ /

$$\int_0^{2\pi}\sin\theta\,e^{i\sigma\theta}d\theta=\int_0^{2\pi}\sin\theta(\cos\sigma\theta+i\sin\sigma\theta)d\theta=\pm i\pi$$

/ $i\pi$ under $\sigma=1$, and $-i\pi$ under $\sigma=-1$ /.

The latter integral is equal to zero at any other value σ .

Thus the second sum of the right-hand side of expression /67/ becomes

$$kq\,e^{i\psi}(i\pi+i\pi)+\ldots=kq\pi^2(e^{i\psi}+e^{-i\psi})=$$

$$=2kq\pi^2\cos\psi.$$

So, finally equation /60/ can be represented in the form:

$$(\omega-\nu)a\frac{\partial B_1}{\partial\psi}+2\omega A_1=$$

$$=\frac{1}{2\pi}\left[2\pi\int_0^{2\pi}\mathcal{L}\mathcal{F}(a,\tau)\sin\tau\,d\tau+2kq\pi^2\cos\psi\right]. \qquad /69/$$

Equations /66/ and /69/ be rewritten as follows:

$$(\omega-\nu)\frac{\partial A_1}{\partial\psi}+2\omega B_1=\frac{1}{\pi}\int_0^{2\pi}\mathcal{L}\mathcal{F}(a,\tau)\cos\tau\,d\tau-kq\sin\psi, \qquad /70/$$

$$(\omega-\nu)a\frac{\partial B_1}{\partial\psi}+2\omega A_1=\frac{1}{\pi}\int_0^{2\pi}\mathcal{L}\mathcal{F}(a,\tau)\sin\tau\,d\tau-kq\cos\psi. \qquad /71/$$

Solving simultaneous equations /70/ and /71/ it can be determined required function $A_1(a,\psi)$ and $B_1(a,\psi)$ as follows:

As we differentiate only along ψ , a we can consider as parameter and instead of partial derivatives it can be written usual ones, i.e.,

$$\frac{\partial A_1}{\partial\psi}=\frac{dA_1}{d\psi}\quad\frac{\partial B_1}{\partial\psi}=\frac{dB_1}{d\psi}$$

To solve a system of equation /70/ and /71/ we shall differentiate the first equation along ψ , taking into account that $\int_0^{2\pi}\mathcal{L}\mathcal{F}(a,\tau)\cos\tau\,d\tau=\text{const}$,

$$(\omega-\nu)\frac{\partial A_1}{\partial\psi}+2\omega\frac{\partial B_1}{\partial\psi}=kq\ldots$$

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from which

$$\frac{dB_1}{d\psi} = \frac{1}{2a\omega}\left[(\omega-\nu)\frac{d^2A_1}{d\psi^2} + kq\cos\psi\right]$$

Substituted value $\frac{dB_1}{d\psi}$ to the second equation we get:

$$\frac{\omega-\nu}{2\omega}\left[(\omega-\nu)\frac{d^2A}{d\psi^2} + kq\cos\psi\right] + 2\omega A_1 =$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} L\,F(a,\tau)\sin\tau\,d\tau - kq\cos\psi$$

or

$$(\omega-\nu)^2\frac{d^2A_1}{d\psi^2} + 2\omega A_1 =$$

$$= -\frac{1}{\pi}\int_0^{2\pi} L\,F(a,\tau)\sin\tau\,d\tau - \left(\frac{\omega-\nu}{2\omega}+1\right)kq\cos\psi \qquad /72/$$

We obtain the partial solution of the above equation as follows:

$$A_1 = \alpha + \beta\cos\psi,$$

$$\frac{d^2A_1}{d\psi^2} = -\beta\cos\psi. \qquad /73/$$

Substituting these value into equation /72/ we get:

$$-\frac{(\omega-\nu)^2}{2\omega}\beta\cos\psi - 2\omega(\alpha+\beta\cos\psi) = -\frac{1}{\pi}\int_0^{2\pi} L\,F(a,\tau)\sin\tau\,d\tau -$$

$$-\frac{\beta(\omega-\nu)}{2\omega}kq\cos\psi.$$

Equating coefficients of $\cos\psi$ and free terms we shall obtain the system:

$$\frac{(\omega-\nu)^2}{2\omega}\beta + 2\omega\beta = \frac{\beta(\omega-\nu)}{2\omega}kq$$

$$2\omega\alpha = -\frac{1}{\pi}\int_0^{2\pi} L\,F(a,\tau)\sin\tau\,d\tau$$

from which determine the constants α and β

$$\alpha = -\frac{1}{2\omega\pi}\int_0^{2\pi} L\,F(a,\tau)\sin\tau\,d\tau.$$

$$\beta = -\frac{q}{\omega+\nu}.$$

Then in accordand with /73/ finally expression for function $A_1(a,\psi)$ becomes:

$$A_1 = -\frac{1}{2\omega\pi}\int_0^{2\pi} L\,F(a,\tau)\sin\tau\,d\tau - \frac{qk}{\omega+\nu}\cos\psi \qquad /74/$$

and it derivatives

$$\frac{dA_1}{d\psi} = \frac{qk}{\omega+\nu}\sin\psi.$$

Substituting expression $\frac{dA_1}{d\psi}$ into equation /70/ we get:

$$\frac{\omega-\nu}{\omega+\nu}kq\sin\psi - 2a\omega B_1 = \frac{1}{\pi}\int_0^{2\pi} L\,F(a,\tau)\cos\tau\,d\tau - kq\sin\psi$$

whence final expression for function $B(a,\psi)$ becomes:

$$B_1 = -\frac{1}{2a\omega\pi}\int_0^{2\pi} L\,F(a,\tau)\cos\tau\,d\tau + \frac{qk}{a(\omega+\nu)}\sin\psi. \qquad /75/$$

Substituting the above value $A_1(a,\psi)$ and $B_1(a,\psi)$ and /75/ into expansions /33/ and /34/ according to formulas /74/ for the case of principal resonance under consideration $\tau = S - 1$, differential equation of the first approximation to determine amplitude a and phase ψ can be written in the form:

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$$\frac{da}{dt} = -\frac{1}{2\omega\pi}\int_0^{2\pi} \varepsilon L F(a,\tau)\sin\tau\, d\tau - \frac{\varepsilon g \kappa}{\omega + \nu}\cos\psi, \qquad /76/$$

$$\frac{d\psi}{dt} = \omega\nu - \frac{1}{2a\pi}\int_0^{2\pi} \varepsilon L F(a,\tau)\cos\tau\, d\tau - \frac{\varepsilon g \kappa}{\omega + \nu a}\sin\psi. \qquad /77/$$

To illustrate the application of the above formulas to computation of the resonance curve, let us consider the case of the transverse oscillations of a uniform cantilever bar having the dimensions: $l = 200$ mm $b = 30$ mm and $h = 10$ mm.

By using solution /44/ in accordance with the boundary conditions of the case under consideration

$$\varphi(0)=0;\ \varphi'_x(0)=0,\ \varphi''_x(l)=0,\ \varphi'''_x(l)=0$$

we obtain $c_1 = c_2 = 0$, as the well known frequency equation:

$$\cos\kappa l \operatorname{ch}\kappa l = -1.$$

Whence, considering /28/, and also that $\kappa = \sqrt{\alpha\omega^2}$ the natural frequency of oscillations of the bar will be determined

$$\omega = 1{,}3\cdot 10^3 \frac{1}{\sec}$$

Using adding condition $\varphi(l) = 1$ it can be determined the constants of integration C_3 and C_4 and, consequenty it can be determined the expression for function of deflection:

$$\varphi(x) = \frac{1}{2\sin\kappa l \operatorname{sh}\kappa l}\left[(\cos\kappa l + \operatorname{ch}\kappa l)(\operatorname{ch}\kappa x - \cos\kappa x) + (\sin\kappa l - \operatorname{sh}\kappa l)(\operatorname{sh}\kappa x - \sin\kappa x)\right]. \qquad /78/$$

We shall represent other expressions which are in the equations /76/ and /77/, using following concrete division:

$n = 2 \qquad \eta = 3 \qquad \kappa l = 1{,}875$

$E = 2\cdot 10^{11}\frac{kg}{m} \qquad \rho = \frac{\gamma}{g} = 8{,}002\cdot 10^{-6}\frac{kg\,sec^2}{m^4}$

$J = \frac{6h^3}{12}, \qquad \alpha^2 = \frac{12\pi\rho}{E b h^3}$

$F = bh, \qquad \frac{\eta}{nJ} = \frac{12\eta}{h^3 b h}$

$$\int_{(F)} z^{n+1}\, dF = 2b\int_0^{h/2} z^3\, dz = \frac{bh^4}{32},$$

$$\int_0^l \varphi(x)\, dx = \frac{1}{\kappa}\left(\frac{1}{\sin\kappa l} - \frac{1}{\operatorname{sh}\kappa l}\right),$$

$$\int_0^l \left[\frac{d^2}{dx^2}\left(\frac{d^2\varphi(x)}{dx^2}\right)^n\right] dx = 2\kappa^5\left(\frac{1}{\sin\kappa l} - \frac{1}{\operatorname{sh}\kappa l}\right) \qquad /79/$$

$$L = \frac{2\kappa^6}{\alpha^2} = \frac{\kappa^6 E b h^3}{6F\rho} = \frac{\kappa^6 E h^2}{6\rho}$$

$$\int_0^{2\pi} L F(a,\tau)\sin\tau\, d\tau = \frac{\kappa^6 E h^2}{6\rho}\,\frac{12\eta}{\pi b h^3}\,\frac{bh^4}{32}\left\{\int_0^{\pi}\left[(1-\cos\tau)^n - 2^{n-1}\right]\sin\tau\, d\tau + \int_{\pi}^{2\pi}\left[(1+\cos\tau)^n - 2^{n-1}\right]\sin\tau\, d\tau\right\} =$$

$$= \frac{\kappa^6 E h^3}{8\pi\rho}\int_0^{\pi}\left[(1-\cos\tau)^n - 2^{n-1}\right]\sin\tau\, d\tau,$$

$$\int_0^{2\pi} L F(a,\tau)\cos\tau\, d\tau = \frac{\kappa^6 E h^3}{8\pi\rho}\int_0^{\pi}\left[(1-\cos\tau)^n - 2^{n-1}\right]\cos\tau\, d\tau$$

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and when $\eta = 0$

$$\int_0^{\tau} [illegible] \sin\tau \, [illegible] \frac{\eta}{\delta},$$

$$\int_0^{\tau} [illegible]$$

$$\int_0^{\tau} [illegible] \sin\tau \, d\tau = [illegible] a^n,$$

$$\int_0^{\tau} [illegible] \cos\tau \, d\tau = \frac{\kappa^6 E \eta h^3 \alpha}{8 n \rho} a^n.$$

Supposing $\frac{da}{dt} = 0$ we obtain:

$$K \frac{\delta\psi}{\omega + \nu} \cos\psi = \frac{\kappa^6 E \eta h^3 (n+1)}{2^{4-n} \omega \pi n (n+1) \rho} \qquad /80/$$

Considering $\frac{d\psi}{dt} = 0$ we get:

$$\nu = \omega - \frac{\kappa^6 E \eta h^3}{6 a \omega n \rho} a^n + K \frac{\delta\psi}{(\omega + \nu) a} \sin\psi \qquad /81/$$

Expressing $\sin\psi$ as $\cos\psi$ from /80/ we shall obtain the formula for constructing of resonance curve

$$\sin\psi = \pm \sqrt{1 - \cos^2\psi}$$

of

$$\cos\psi = \frac{A}{B} a^n$$

where $A = \frac{\kappa^6 E \eta h^3 (n+1)}{2^{4-n} \omega \pi n (n+1) \rho}$, $B = \frac{K \delta\psi}{\omega + \nu}$.

Then

$$\nu = \pm \frac{1}{B} \sqrt{\varepsilon^2 - [illegible] a^{2n}}$$

and formula /81/ becomes

[illegible]

Substituting the [illegible]

[illegible]

[illegible]

As we are studying resonance curve near the natural frequency, i.e., under $\nu \approx \omega$, the expression [illegible] we may take into consideration as a constant magnitude which a great degree of accuracy. Substituting the numerical values into the formula /82/ we shall get:

$$\frac{\nu}{\omega} = 1 - 4{,}7066 \cdot 10^{-3} a \pm \sqrt{\frac{Q^2}{a^2} - 3{,}9901 \cdot 10^{-6} a^2}$$

which is the final desing.

Considering $a_{max} = 0{,}05$ cm we shall determine Q^2 from the condition that the expression under the radical sing is equal to zero:

$$Q^2 = 3{,}9901 \cdot 10^{-6} a^4 = 5{,}232 \cdot 10^{-8}$$

Then we have:

$$\frac{\nu}{\omega} = 1 - 4{,}707 \cdot 10^{-3} a \pm \sqrt{\frac{5{,}232 \cdot 10^{-8}}{a^2} - 3{,}99 \cdot 10^{-6} a^2} \qquad /83/$$

Using the formula /83/ resonance curve was constructed. Curve was shown on the fig.2.

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To construct the envelope of the free damped oscillations of the bar under consideration it is necessary to put $\varepsilon Q = 0$ in equation /76/, and we obtain:

$$\frac{da}{dt} = -\frac{1}{2\omega\pi}\int_0^{2\pi} f_1 F(a,\varphi)\cos\varphi\, d\varphi$$

or

$$\frac{da}{dt} = -\frac{\kappa^6 \sum_n \rho h^3 (n\nu)^4 a^n}{2^{n-2}\pi(2+n)\omega\rho}$$

Introducing the numerical value to the last formula /74/ we shall have

$$\frac{da}{dt} = -2{,}597a^2$$

whence

$$\frac{da}{a^2} = -2{,}597\,dt$$

$$-a^{-1} = -2{,}597t + C$$

under

$$t = 0 \qquad a = a_0, \qquad C = -a_0^{-1}$$

$$a = \frac{1}{2{,}597t + a_0^{-1}}$$

putting $a_0^{-1} = 3{,}333$ and using the latter formula it was constructed the envelope curve free oscillations of the bar /fig.3/.

Flexusal-Torsional Oscillations of a System with Two Degrees of Freedom

To illustrate the degree of accuracy of the solution in the first approximation, as compared to the second approximation, consider the flexural-torsional oscillations of a cylindrical cantiliver bar having an cally attached mass on its free-end /fig.4/.

The equations of steady-state forced flexural-torsional oscillation of a system with two degrees of free-

[illegible] the coefficient [illegible] equation [illegible] torsional oscillations [illegible] accounting [illegible] tension [illegible]

$$m\ddot{x} + c_x\dot{x} + c_{x\vartheta}\vartheta + \varepsilon\Phi_1\left(\frac{\dot{x}}{\omega}\right) = \mathcal{P}\cos\Omega t,$$

$$J\ddot{\vartheta} + c_\vartheta\dot{\vartheta} + c_{\vartheta x}x + \varepsilon\Phi_2\left(\frac{\dot{\vartheta}}{\omega}\right) = \mathcal{P}x_c\cos\Omega t, \qquad /84/$$

Where m - mass of a weight, J - moment of inertia of the mass of the weight about axis z, x_c - eccentricity of the mass, $c_x = \frac{3EJ}{e^3}$ - flexural rigidity of the bar, [illegible] - torsional rigidity of the bar, $\varepsilon\Phi_1(\omega)$ and $\varepsilon\Phi_2(\vartheta)$ - functions of dissipation of energy in the material respectively at the expense of the normal and tangential stresses, $\varepsilon\mathcal{P}\cos\Omega t$ - external perturbing periodical force with the amplitude $\varepsilon\mathcal{P}$, ε - a small parameter.

Neglecting the mass of the elastic bar in comparison with the end mass, the solution of equation /84/ can be sought in the form:

$$x(t) = a\cos(\Omega t + \varphi) = a\cos\psi,$$

$$\vartheta(t) = b\cos(\Omega t + \varphi - \beta) = b\cos\psi,$$

where coefficient $\beta = \frac{\delta}{a}$ - determines the [illegible] oscillations from equation /84/, putting $\varepsilon = 0$, i.e. from the unperturbed equations of oscillation of the [illegible] under consideration.

[illegible] the unperturbed equations it [illegible] frequency ω, corresponding to the [illegible] oscillation.

[illegible] amplitude a and [illegible] of oscillations [illegible]

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the first approximation

$$\frac{da}{dt} = -\frac{A}{M(\omega+\nu)}\sin\psi,$$

$$\frac{d\psi}{dt} = \omega - \frac{f_1(a)}{2\pi M\omega} - \frac{A}{aM(\omega+\nu)}\cos\psi \qquad /86/$$

and for steady-state oscillations, i.e., for $\frac{da}{dt}=0$ and $\frac{d\psi}{dt}=0$ assume the form:

$$\omega^2 = 1 - \frac{A}{M\omega^2 a}\left[\cos\psi - \frac{f_1(a)}{f_2(a)}\sin\psi\right],$$

$$\sin\psi = -\frac{f_2(a)}{\varkappa A}. \qquad /87/$$

Where the following notation are introuuced:

$$f_1(a) = \int_0^{2\pi}[f_{10}(u) - \beta f_{20}(\theta)]\sin\tau\, d\tau, \qquad /88/$$

$$f_2(a) = \int_0^{2\pi}[f_{10}(u) - \beta f_{20}(\theta)]\cos\tau\, d\tau,$$

$$M = m - mx_c\beta - mx_c|\beta|/\beta + |\beta|/\beta J,$$

$$A = \mathcal{P} + \mathcal{P}x_c|\beta|; \qquad /89/$$

$$f_{10}(u) = \pm\sum \bar{\varkappa}\bar{e}\, a^n[(1\pm\cos\tau)^n - 2^{n-1}],$$

$$f_{20}(\theta) = \pm G\bar{\eta}\,\theta^\kappa\left[(1\pm\cos\tau)^n - 2^{n-1}\right], \qquad /90/$$

$$\bar{e} = \varkappa\int_0^e\left(\frac{d^2u}{dz^2}\right)^{n+1}dz\iint_{(F)}\xi^{n+1}dF,$$

$$\bar{\eta} = \eta\int_0^e\left(\frac{d\theta}{dz}\right)^{\kappa+1}dz\iint_{(F)}z^{\kappa+1}dF.$$

Assuming for steel $n = \kappa = 2$

and taking into account that for the oscillatory system under consideration:

$$a^2\omega\frac{d\omega}{dz} = \frac{1}{C^2},$$

$$\omega\Theta = \frac{\omega}{\omega_0} = 1,$$

we have equations:

$$\bar{e} = \frac{\omega^2}{C^2},$$

$$\bar{\eta} = \frac{\omega^2}{\omega_0^2}.$$

Then each [illegible] can be represented [illegible]

$$f_1(a) = a^2[\quad + \quad a^2 + \quad a^4 + \quad a^6 + \quad a^8 + \quad a^{10}],$$

$$f_2(a) = [\quad + \quad a^2 + \quad a^4 + \quad a^6 + \quad a^8 + \quad a^{10}].$$

The solution of the system of [illegible] in the second of approximation can be represented [illegible] the first:

$$a(t) = a_0\cos(\quad t + \varphi_0) + a_1\cos(\quad t + \varphi_1) + \quad a_3\cos(\quad t + \varphi_3) + \quad a_5\cos(\quad t + \varphi_5), \qquad /92/$$

where

$$a_1(a_0) = \frac{1}{\pi}\int_0^{2\pi}\frac{\sin\tau}{\quad}\int_0^{2\pi}\quad\cos\tau\, d\tau,$$

$$a_3(a_0) = \frac{1}{\pi}\int_0^{2\pi}\quad\cos 3\tau\, d\tau,$$

$$a_5(a_0) = \frac{1}{\pi}\int_0^{2\pi}\quad\cos 5\tau\, d\tau.$$

The [illegible] and ψ are

$$\frac{d\varphi}{dt} = \frac{1}{2\omega}\left[\quad a_1^2 + \quad a_3^2 + \quad a_5^2\right],$$

$$\frac{d\psi}{dt} = \omega - 1 + \frac{1}{2\omega}\left[\quad a_1^2 + \quad a_3^2 + \quad a_5^2\right].$$

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where

$$A_1(a,\tau) = \frac{f_2(a)}{2\pi M\omega} - \frac{A_1}{M(\omega+\nu)}\sin\psi,$$

$$B_1(a,\tau) = \frac{f_1(a)}{2\pi a M\omega} - \frac{A_1}{M(\omega+\nu)}\cos\psi,$$

$$A_2(a,\tau) = \frac{1}{2\omega}\left[a\frac{\partial B_1}{\partial a}A_1 + a\frac{\partial B_1}{\partial\psi}B_1 + 2A_1B_1\right] - \frac{1}{2\pi\omega}\int_0^{2\pi} f(a,\tau)\sin\tau\,d\tau,$$

$$B_2(a,\tau) = \frac{1}{2a\omega}\left[\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial\psi}B_1 - aB_1^2\right] - \frac{1}{2a\omega\pi}\int_0^{2\pi} f(a,\tau)\cos\tau\,d\tau,$$

$$f(a,\tau) = f_{10}(a,\tau) + \beta f_{20}(a,\tau) = f_{10}(a\cos\tau) + \beta f_{20}(a\cos\tau) \qquad /96/$$

In accordance with A_1, when $2n \neq 0$ $u_1(a,\tau) = 0$ as when $n = 2$

$$\int_0^{\pi}\left[(1-\cos\tau)^2 - 2\right]\cos m\tau\,d\tau \int_{\pi}^{2\pi}\left[(1+\cos\tau)^2 - 2\right]\cos m\tau\,d\tau = 0$$

and

$$\int_0^{\pi}\left[(1-\cos\tau)^2 - 2\right]\sin m\tau\,d\tau \int_{\pi}^{2\pi}\left[(1+\cos\tau)^2 - 2\right]\sin m\tau\,d\tau = 0.$$

In connection with that $u_1(a,\tau) = 0$

$$f_1(a,\tau) = \frac{\partial f(a,\tau)}{\partial u_1}u_1(a,\tau) + \frac{\partial f(a,\tau)}{\partial u_1}\left[A\cos\tau - aB_1\sin\tau + \frac{\partial u_1}{\partial\psi}(\omega-\nu)\right] = 0$$

and consequently the last integrals are equal to zero in [illegible] /96/. So the functions of deflection and the angle of rotation in the second approximation will be

$$u(t) = a\cos(\nu t + \psi),$$

$$\theta(t) = \beta a\cos(\nu t + \psi) \qquad /97/$$

and amplitude a and phase angle ψ are determined from the equations of the second approximation

$$\frac{da}{dt} = \frac{f_1(a)}{2\pi M\omega} - \frac{A_1}{M(\omega+\nu)}\sin\psi - \frac{1}{2\omega}\left[a\frac{\partial B_1}{\partial a}A_1 + a\frac{\partial B_1}{\partial\psi} + 2A_1B_1\right]$$

$$\frac{d\psi}{dt} = \omega - \nu - \frac{f_1(a)}{2\pi a M\omega} - \frac{1}{2a\omega}\left[\frac{\partial A_1}{\partial a}A_1 + \frac{\partial A_1}{\partial\psi}B_1 - aB_1^2\right] - \frac{A_1\cos\psi}{M(\omega+\nu)} \qquad /98/$$

or taking into account /95/

$$\frac{da}{dt} = -\frac{f_2(a)}{2\pi M\omega} - \frac{3f_1(a)f_2(a)}{2(2\pi M\omega)^2\omega a} - \left[\frac{A_1}{M(\omega+\nu)} + \frac{f_1(a)A_1}{2\pi M^2(\omega+\nu)\omega^2 a}\right]\sin\psi - \frac{f_2(a)A_1}{4\pi M^2(\omega+\nu)\omega^2 a}\cos\psi,$$

$$\frac{d\psi}{dt} = \omega - \nu - \frac{f_1(a)}{2\pi a M\omega} - \frac{f_1^2(a) - f_2^2(a)}{2(2\pi M\omega)^2 a^2\omega} + \frac{f_2(a)A_1}{4\pi M^2(\omega+\nu)\omega^2 a^2}\sin\psi - \left[\frac{A_1}{Ma(\omega-\nu)} + \frac{f_1(a)A_1}{4\pi M(\omega+\nu)\omega^2 a^2}\right]\cos\psi. \qquad /99/$$

In the case of steady-state oscillations, i.e. when $\frac{da}{dt} = 0$ and $\frac{d\psi}{dt} = 0$, the equations of the second approximation /87/ can be represented in the form:

$$\frac{\nu^2}{\omega^2} = 1 - \frac{A_1}{M\omega^2 a}\left\{\left[1 + \frac{f_1(a)}{4\pi M\omega^2 a}\right]\cos\psi - \left[\left(f_1(a) + \frac{f_1^2(a) - f_2^2(a)}{4\pi M\omega^2 a}\right)\frac{1}{f_2(a)}\frac{\left(1 + \frac{f_1(a)}{2\pi M\omega^2 a}\right)}{\left(1 + \frac{3f_1(a)}{4\pi M\omega^2 a^2}\right)} + \frac{f_2(a)}{4\pi M\omega^2 a}\right]\sin\psi\right\}.$$

$$\text{or} \quad \frac{f_2(a)\left(1 + \frac{3f_1(a)}{4\pi M\omega^2 a}\right)}{\left(1 + \frac{f_1(a)}{4\pi M\omega^2 a}\right)}$$

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As an example of computation we shall consider the oscillatory system with the following concrete parameter:

$m = 431\,10^{-6}\ kg\,sec^2/cm$, $J = 463\,10^{5}\ kg\,sec^2 cm$

$E = 2\,10^{6}\ kg/cm^2$, $G = 0{,}8\,10^{6}\ kg/cm^2$

$l = 10\ cm$, $a = 1\ cm$.

From unperturbed system we find $\omega = 1793\ 1/sec$

$\beta = 0{,}4\ rad/cm$ when $n = \kappa = 2$ and $\varkappa = 32{,}2$; $\zeta = 14{,}9$ using /7/ we find

$$f_1(a) = 1168{,}7\,a^2,\quad f_2(a) = 495{,}9\,a^2$$

When the weight is perturbing $\varepsilon P = 0{,}0366\ kg$, according to /8/,

$$A = 0{,}06543\ kg,\quad M = 173\,10^{-6}\ \frac{kg\,sec^2}{cm}$$

Then the equations of resonance curve /87/ become:

$$\left(\frac{\nu}{\omega}\right)^2 = 1 - \frac{1{,}1761\,10^{-4}}{a}(\cos\psi + 2{,}35\sin\psi),$$

$$\sin\psi = -2413{,}7\,a^2,$$

$$\beta = \frac{\varepsilon}{a} = -0{,}4 \qquad /101/$$

Using equations /101/ the resonance curves of flexural-torsional oscillations, which are represented of fig.4 were built.

The equations of resonance curve of the second approximation in accordance with /100/ for case under consideration are:

$$\left(\frac{\nu}{\omega}\right)^2 = 1 - \frac{1{,}1761\,10^{-4}}{a}\left\{(1 + 0{,}1671\,a)\cos\psi - [0{,}3218\,a + 2{,}35)\frac{1 + 0{,}3343\,a}{1 + 0{,}5015\,a} + 0{,}0709\,a]\sin\psi\right\},$$

$$\sin\psi = -2413{,}7\,a^2\left(\frac{1 + 0{,}5015\,a}{1 + 0{,}3343\,a}\right),$$

$$\beta = \frac{\varepsilon}{a} = -0{,}4.$$

Using there equations it was constructed a resonance curve in the second approximation, the curve is shown on fig.5 by dotted line.

In the second approximation the maximum amplitude $a_{max} = 0{,}02032$ under $\frac{\nu}{\omega} = 0{,}99316$ while in the first approximation $a_{max} = 0{,}0203\,59$ under $\frac{\nu}{\omega} = 0{,}99306$, i.e., in the second approximation the maximum amplitude turned out to be 0,17 per cent less.

Thus, from the analysis of results of the numerical example it follows that in solving of considered class of nonlinear problems under consideration the accuracy of the first approximation is quite sufficient therefore the application of the small parameter method doesn't result in cumbersome transformation which are inevitable when a solution in the second approximation becomes necessary.

This fact is a very convineing proof of the effectiveness of application of assymptotic methods of nonlinear mechanic to the analysis of oscillations of ellastic systems with consideration of dissipation of energy in material.

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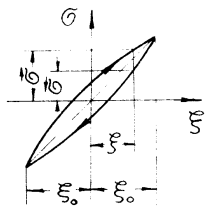

Fig. 1

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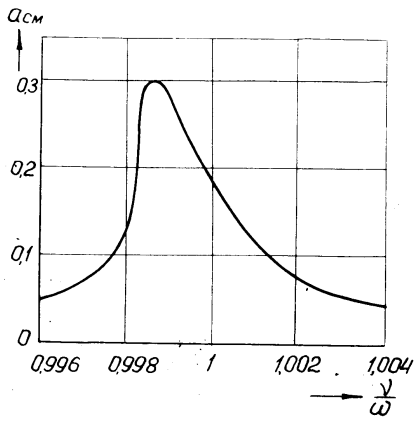

Fig.2

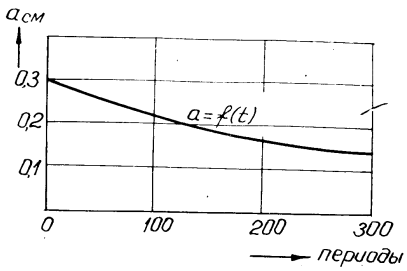

Fig.3

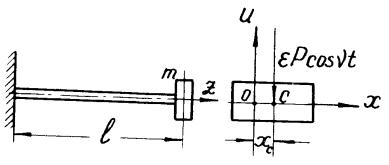

Fig.4

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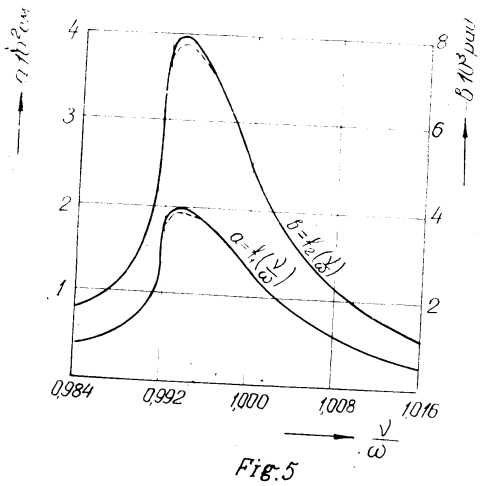

Fig. 5

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Е. П. ПОПОВ

ПРИБЛИЖЕННЫЕ МЕТОДЫ ИССЛЕДОВАНИЯ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ В АВТОМАТИЧЕСКИХ СИСТЕМАХ

APPROXIMATE [illegible]

IN AUTOM[illegible] CONTROL SY[illegible]

E.P. Popov

§ 1. Statement of the problem.

In the present investigation automatic control [illegible] are dynamic systems of any physical nature incorporating at least one closed-loop circuit.

For example, let the dynamics of a system with several degrees of freedom be described by a set of differential equations of the type:

$$\left.\begin{aligned} &F_1(\ddot{x}_1, \dot{x}_1, x_1) = f_1(x_2, \dots x_m; \dot{x}_2, \dots, \dot{x}_m; t), \\ &F_2(\ddot{x}_2, \dot{x}_2, x_2) = f_2(x_1, x_3, \dots, x_m; \dot{x}_1, \dot{x}_3, \dots, \dot{x}_m; t), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &F_\ell(\ddot{x}_\ell, \dot{x}_\ell, x_\ell) = f_\ell(x_1, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_m; \dot{x}_1, \dots, \dot{x}_{\ell-1}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \dot{x}_{\ell+1}, \dots, \dot{x}_m; t), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &F_m(\ddot{x}_m, \dot{x}_m, x_m) = f_m(x_1, \dots, x_{m-1}; \dot{x}_1, \dots, \dot{x}_{m-1}; t). \end{aligned}\right\} \quad (1.1)$$

It should be remembered that it is not necessary [illegible] all the variables given here for generality be included in each equation.

Such a system will have a closed-loop circuit [illegible] the right-hand part of every i-th equation to a certain [illegible] included (i = 2,3, 1)contains a variable with a pre[illegible] number (x_{i-1}) and the first equation cont[illegible]ns [illegible] x_1 .

[illegible] of equations (1.1) can be thought [illegible] ti[illegible] [illegible]ing a parti[illegible] element of [illegible]

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in the righ-hand part of the equation are referred to as input variables for the given element of the system, the left-hand part containing an output variable.

In our example the main input variables are the variables x_{i-1} which form the basic closed-loop circuit of the system containing l elements. (Fig. 1a) All other input variables in the first l equations of the system (1.1) represent redundant connections between these l elements. Fig. 1b illustrates a particular case when the righ-hand part of the second equation in the set (1.1) contains the variable x_3 and the right-hand part of the third equation - the variable x_1. The variable t in the right-hand parts of equations in an explicit form shows either that coefficients of the equations describing element of the system are time - varying or that the forcing function f(t) is applied (fig. 1b)

The set of equations (1.1) consists of m equations. If l = m, the schematic circuit of the system (i.e. the circuit of the type shown in fig. 1b, which is equivalent to a given set of equations) is described by m equations. If l m, it shows that there are some redundant connections beyond the basic closed-loop circuit.

They form compensating networks originating from any inner variable of the basic circuit (for instance, x_4 in fig. 1c) or from an external variable (for instance, f(t) in fig. 1d) The first of them can serve as redundant correcting devices of a controller redundant feedback and feedforward networks or they can indicate a complex nature of the controled plant. The other networks are introduced in feedforward control action. In our example m = l + [illegible]

The automatic control systems under consideration comprize control and regulating systems, servo systems, stabilizing and orientation systems, certain metering systems, etc. In all of these systems any single group of elements constitues a controlled plant and the other one - control equipment or a controller.

To simplify the analysis we shall consider control systems comprizing only one non-linearity of the general type

$$y = F(x, x) \qquad (1.2)$$

referring incidentally to the systems with several non-linearities as well as to the systems with multivariable [illegible] linear functions.

Thus, we assume that all equations describing the elements of the system (1.1) except any one are [illegible] the non-linear equation containing non-linearity of the type (1.2) with a corresponding index of the variable x. The coefficients of all equations are assumed constant. In this case the set of equations (1.1) can be reduced to one high-order differential equation with regard to one variable, namely, the variable which is involved in non-linearity. Omitting the index of the variable and designating for breaking the process of differentiation by the symbol p [illegible]

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one in the control theory) we can write the above differential equation describing a non-linear control system(with one non-linearity) as

$$Q(p)x + R(p)f(x, px) = S(p)f(t). \qquad (1.3)$$

where Q(p), R(p) and S(p) are transformed polynomials with constant coefficients. Any system of this structure can be reduced to this form, including the system of fig. 1 with any number of circuits provided there is only one non-linearity. Equation (1.3) corresponds to the system shown in fig. 2 the linear part of which is

$$Q(p)x = -R(p)y + S(p)f(t) \qquad (1.4)$$

and the non-linear part is

$$y = f(x, px), \qquad (1.5)$$

the equation

$$W(p) = \frac{R(p)}{Q(p)} \qquad (1.6)$$

being a linear part transfer function. This linear part is reduced and incorporates a number of elements; thus, it can have any complex inner structure, for example, a multiloop circuitry.

[illegible] automatic control system incorporates nonlinearity [illegible] nature, for instance:

$$y = f(x, x_2, px_2), \qquad (1.7)$$

[illegible] are reduced not to one equation as [illegible] pair of equations with two variables x_1 and x_2.

The same applies to a system with two non-lin[illegible]

$$y_1 = f_1(x_1, px_1), \quad y_2 = f_2(x_2, px_2).$$

as well as with several non-linearities. In all these the structural concepts can be also introduced but not one,as it is shown in Fig.2,but with several linear part and nonlinearities.

In this way,the problem is to analyze different dynam processes in control systems.The solution of this prob involves finding x(t) of nonlinear differential equation single high-order equation under different initial con tions and with different external disturbances (or them). It is necessary to determine the way in which change with varying parameters of a system(that is,when the coefficients of the equations vary) in sufficiently large space of these parameters. In particular, it is necessary to find in this space the areas of self-oscillations, equillibrium state stability, certain quality of transient processes, characteristics of forced oscillations, random processes and so on. This is essential for solving analysis and synthesis problems in designing automatic control systems. The techniques of obtaining the solution of these problems must be sufficiently simple and rapid, though rough approximation is permissible, and should be suitable for the preliminary calculation and assessment of the properties of a system being designed.(After the system is completely designed all processes can be studied more accurately using

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computers).

The analysis of oscillation processes in non-linear control systems is a problem of great importance. Among other things, the determination of oscillations is required when we want to get rid of them. In such cases the oscillation area whose boundary can be determined, will be a prohibited zone in choosing parameters. For instance, the stability boundary a non-linear system can often be defined as the boundary of self-oscillations or divergent oscillations and the operiodicity boundary - as the boundary of oscillation transient processes.

Besides, the investigation of oscillation processes is necessary for the analysis of non-linear systems subjected to different vibrations and random disturbances, etc.

A number of approximate methods are available for studying oscillations in non-linear systems. We shall consider the wellknown methods developed by N.M. Krylov, N.N.Bogolyubov(1) and B.V.Bulgakov( ). They are based on the principle of the introduction of the small parameter ε (or λ) in the equation of non-linearity, for instance,

$$F(x, px) = k_1 x + k_2 px + \varepsilon f(x, px), \quad (1.9)$$

as well as to the solutions of all variables of a non-linear system (insignificant deviation from the solution of an appropriate linear system). However, the utilizationof these methods in studying self-oscillations and stability of control systems was quite efficient in many cases when the

difference between the non-linear function $F(x, px)$ and the linear function $k_1 x + k_2 px$ could not be considered literally as small as in (1.9)

In fact, solutions of several variables in the same system (that is, variables solved simultaneously in different paints of the closed-loop circuit) may considerably differ.

Typical of many automatic control systems is the case when with a considerable difference of the solution of the variable $y = F(x, px)$ from the linear one $y = k_1 x + k_2 px$ ("at the non-linearity output") a really small difference is possible for the variable x ("at the non-linearity input") and various degrees of such difference for different variables ("in different points of the circuit"). This occurs mostly in periodic oscillation processes in non-linear control systems (self-oscillations and forced oscillations), and, consequently, in determining the equillibrium stability boundaries where periodic oscillations are formed. Here lies the explanation why these phenomena can be investigated in a most simple and reliable way using approximate quasi-linear techniques ("first harmonics analysis")

In a closed-loop system the most difficult thing is to determine the process form at least for one (any) varialbe. This involves solving a high-order nonlinear equation. The solution of all other variables on the basis of the first one is very easy for a single-loop system (fig. 1a) and does not present great difficulties for multiloop systems (fig. 1b,c) .

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[illegible]ore in the discussion to follow we shall require [illegible] solution of the equations of the system with one [illegible] of the type (1.5) only for one variable x should [illegible] small difference and we shall not require that [illegible] difference should exist between the nonlinear function itself and solutions for all other variables from the linear functions. The solution of the general equation of the system only for this variable x will be determined by quasi-linear techniques and then solutions for all other variables with full considering the nonlinearity will be defined.

[illegible] system with nonlinear function from two variables [illegible] type (1.7) or with two or more nonlinearities (1.8) [illegible] all require that the solution of it should have small difference from that of linear one for two or more variables respectively.

Modified in such a way quasi-linear method of Krylov and Bogoljubov applied to the above mentioned characteristics of non-linear automatic control systems will be called the describing function method /3/. Next paragraphs evolve the basis of this method considering all the peculiarities described of the systems under discussion and then the practical techniques of solving different problems will be briefly characterized.

[illegible] types of specific problems the results [illegible] ap[illegible] techniques coincide with the results obtained [illegible] solving these problems by frequency variants of [illegible] function method (or describing functions) of [illegible] /[illegible] and Kobenbürger /[illegible] as well as [illegible] and Magnus's /6/ techniques. This paper will discuss other methods of solving the same types of problems, as well as a lot of new problems, associated with investigation of non-linear oscillations, on the sole basis of harmonic linearization method.

§ 2. The basis of harmonic linearization method.

Consider at first periodic oscillation processes at free movement of the non-linear system /f(t) =0/, when its equation (1.3) has the form

$$Q(p)_x + R_{(p)} F(x_1 px) = 0 \qquad (2.1)$$

Assume that periodic solution for variable x is appraching the solution of some non-linear system(i.e.sinusoidal solution) namely

$$x = x_1 + \varepsilon Z(t), \qquad (2.2)$$

where

$$x_1 = A_1 \sin \Omega_1 t, \qquad (2.3)$$

whe n z(t) is an arbitrary limited time function, E - is a small parameter.

Note that one could introduce into equation (2.3) cosine or add any phase, but for periodical solution of homogenious equation it is not essential.

Let us find conditions, which must be satisfied by the transformed polynomials Q(p), R(p) included into the given equation (2.1) as well as by the nonlinear function F(x,px), which greatly differs from the linear one in order

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that a close approach of periodic solution (if it exists[1]) of non-linear system to the sinusoidal one expressed by the formula (2.2) takes place. Let us also find a respective linear differential equation, by which this sinusoidal solution is determined.

Let us present the given non-linear function $F(x_1, px)$ in the form

$$F(x, px) = F(x_1, px_1) + \left[F(x_1+\varepsilon z, px_1+\varepsilon pz) - F(x_1, px_1)\right]. \quad (2.4)$$

The first item i.e. $F(x_1 px_1)$ where x_1 has a form of (2.3) can be represented as harmonic expansion:

$$F(x_1, px_1) = F_0 + C\sin\Omega_1 t + B\cos\Omega_1 t + \sum_{k=2}^{n} F_k(t), \quad (2.5)$$

where

$$\left.\begin{aligned} F_0 &= \frac{1}{2\pi}\int_0^{2\pi} F(A_1\sin\psi,\ A_1\Omega_1\cos\psi)\,d\psi\ ,\ (\psi=\Omega_1 t)\\ C &= \frac{1}{\pi}\int_0^{2\pi} F(A_1\sin\psi,\ A_1\Omega_1\cos\psi)\,\sin\psi\,d\psi\ ,\\ B &= \frac{1}{\pi}\int_0^{2\pi} F(A_1\sin\psi,\ A_1\Omega_1\cos\psi)\,\cos\psi\,d\psi, \end{aligned}\right\} \quad (2.6)$$

but the higher harmonics $\sum_{k=2}^{n} F_k(t)$ are not small, since it has been given that $F(x_1 px)$ function greatly differs from the linear one. Let us write them in the form

$$F_k(t) = N_k \sin(k\Omega_1 t + \eta_k), \quad (k=2,3,\dots)$$

but we shall consider that expansion (2.5)-(2.7) has regular property, i.e. $E_k \longrightarrow 0$ at $K \longrightarrow \infty$.

Further, the expression in brackets of the formula (2.4)

1) Here it might be spoken not only about periodic solution in a strict sense, but also about any other solution, approaching periodic one (which in practice is regarded as periodic)

can be represented using Taylor series, in the following way[1]

$$\left[F(x_1+\varepsilon z, px_1+\varepsilon pz) - F(x_1, px_1)\right] = \varepsilon\left[\frac{\partial}{\partial x}F(x_1, px_1)\,z + \right.$$
$$\left. + \frac{\partial}{\partial px}F(x_1, px_1)\,pz\right] + \varepsilon^2[\dots] + \varepsilon^3[\dots] + \dots \quad (2.8)$$

Thus, this expression will be small if partial derivatives of F with respect to x and px are finite. Its small magnitude can also take place if we deal, in practice with discontinuous non-linear functions, such as relay functions when the above mentioned derivatives will be delta-functions (which can be seen from the example, explained below by Fig. 3.).

This is the first of conditions in question. Regarding it as fulfilled let us denote it according to (2.8)

$$F(x_1 + \varepsilon z,\ px_1 + \varepsilon pz) - F(x_1, px_1) = \varepsilon\Phi_1(t) + \varepsilon^2\Phi_2(t) + \dots$$

Then, according to (2.4) and (2.5) we have

$$F(x, px) = F(x_1, px_1) + \varepsilon\Phi(t). \quad (2.9)$$

where

$$\Phi(t) = \Phi_1(t) + \varepsilon\Phi_2(t) + \varepsilon^2\Phi_3(t) + \dots$$

and as a result

$$F(x, px) = F_0 + C\sin\Omega_1 t + B\cos\Omega_1 t + \sum_{k=2}^{n} F_k(t) + \varepsilon\Phi(t). \quad (2.10)$$

The expression (2.9) is an essential principal basis of the harmonic linearization method differring from the

1) If in (2.2) the expression εz were developed in the form of $\varepsilon z = \varepsilon z_1 + \varepsilon^2 z_2 + \dots$ the first term of the right-hand part of (2.8) would have included only x_1

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formula (1.9) ordinary for other quasi-linear methods .

Thus, it is in no way supposed that there exists in their case some small difference between the non-linear and linear functions but a less strict requirement is introduced (2.9), according to which the true periodic solution for variable $y = F(x_1 px)$ must differ little not from the linear solution, but from the true non-linear curve, obtained when substituting into the non-linear function $F(x_1 px)$ a sinusoid (2.3) i.e. the linear solution for the variable x.

Thus, small term $\varepsilon\Phi(t)$ in the formula (2.9) shows little effect of only small higher harmonic of the input variable x upon the form of the output non-linear oscillations but in no way on the small magnitude of the higher harmonics, resulting from the non-linearity itself in introducing into it a sinusoid, as it has been in (1.9)

For example, if a non-linearity has the form of an ideal relay (fig.3a) then the formula (2.9) which is a basis of the harmonic linearization method supposes that at the output of non-linearity y the true non-linear curve y takes place which is obtained in feeding to the input x the sinusoid 1 and that this curve andergoes little change 4 with the addition at the input a the small higher harmonic 2 . This accures in reality. The usual formula (1.9) supposes small difference of the non-linear curve 3 from the sinusoid 5, which is not justified in the case.

The formula (2.9) expresses the above mentioned condition which must be satisfied by the non-linearity $F(x_1 px)$.

Let us pass to obtaining other conditions and approximate equations for the definition of the periodic solution in question.

Substituting (2.10) and (2.2) into the given (i,j) equation of the system we shall have

$$Q(p)x_1 + Q(p)\varepsilon z(t) + R(p)\ell_0 + R(p)(C\sin\Omega_1 t + D\cos\Omega_1 t) + R(p)\sum_{k=2}^{\infty} F_k(t) + R(p)\varepsilon\Phi(t) = 0. \quad (2.11)$$

This equation must be satisfied identically. From this we shall obtain a number of conditions in the following way. If the periodic solution exists and if x_1 defined by the formula (2.3) represents its precise first harmonic, the functions $\varepsilon z(t)$ and $\varepsilon\Phi(t)$ can be expressed in the following way:

$$\varepsilon z(t) = \varepsilon\sum_{k} A_k \sin(k\Omega_1 t + \psi_k), \quad (2.12)$$

$$\varepsilon\Phi(t) = \sum_{i=1}^{\infty}\varepsilon^i\sum_{k=0}^{\infty}\Phi_{ik}(t) = \sum_{i=1}^{\infty}\varepsilon^i\Phi_{i0} + \sum_{i=1}^{\infty}\varepsilon^i\sum_{k=1}^{\infty}G_{ik}\sin(k\Omega_1 t + \vartheta_{ik}). \quad (2.13)$$

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Then from (2.11) and also (2.7) we obtain a set of equalities for respective harmonics

$$f_0 = -\varepsilon\Phi_{10} - \varepsilon^2\Phi_{20} - \dots, \qquad (2.14)$$

$$Q(p)x_1 + R(p)(C\sin\Omega_1 t + B\cos\Omega_1 t) = -R(p)\varepsilon\Phi_{11}(t) - R(p)\varepsilon^2\Phi_{21} - \dots \qquad (2.15)$$

$$Q(p)\varepsilon A_\kappa \sin(\kappa\Omega_1 t + \psi_\kappa) + R(p)N_\kappa \sin(\kappa\Omega_1 t + \eta_\kappa) = -R(p)\varepsilon\Phi_{1\kappa}(t) - R(p)\varepsilon^2\Phi_{2\kappa}(t) - \dots \quad (\kappa = 2,3,\dots) \qquad (2.16)$$

The equation (2.15) for the precise first harmonic $x_1 = A_1 \sin \Omega_1 t$ contains the right-hand small terms. Determining without these small terms the first approximation for the periodic solution in question which we denote

$$x^* = A\sin\Omega t, \qquad (2.17)$$

we obtain from (2.15) the equation

$$Q(p)x^* + R(p)(C\sin\Omega t + B\cos\Omega t) = 0 \qquad (2.18)$$

After substituting here according to (2.17)

$$\sin\Omega t = \frac{x^*}{A}, \quad \cos\Omega t = \frac{px^*}{A\Omega},$$

we shall write the equation in question for an approximate determination of the periodic solution in the form

$$\left[Q(p) + R(p)\left(q + \frac{q'}{\Omega}p\right)\right]x^* = 0, \qquad (2.19)$$

where from (2.6)

$$q = \frac{C}{A} = \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi, A\Omega\cos\psi)\sin\psi\, d\psi, \qquad (2.20)$$

$$q' = \frac{B}{A} = \frac{1}{\pi A}\int_0^{2\pi} F(A\sin\psi, A\Omega\cos\psi)\cos\psi\, d\psi. \qquad (2.21)$$

As we can see the coefficients q and q' depend in general case on the values in question A and Ω (and in many particular cases they depend only on A). Therefore the first approximation equation (2.19) defined for each periodic solution in question (A = const, Ω = const) for the variable x will be a linear equation with constant coefficients, the values of which, however, are changing with transition from one periodic solution to the other one.

The remaining equalities (2.14) and (2.16) give definite conditions which must be satisfied by Q(p), R(p) and F(x,px) and also the equations for defining higher harmonics and for more precise determining the first harmonic (2.17). From (2.14) to within ε and taking into account (2.6) we have

$$\int_0^{2\pi} F(A\sin\psi,\ A\Omega\cos\psi)\, d\psi = 0. \qquad (2.22)$$

This condition must be satisfied by the given nonlinearity F(x,px), that is it must be symmetrical (there should not appear a constant component at the nonlinearity output, in feeding to its input the symmetrical oscillations, in our case sinusoidal (x = AsinΩt). This is the second condition in question /after (2.9)/. Asymmetrical oscillations will be discussed below (§ 4) when the condition (2.22) is broken.

From (2.16) and (2.13) using complex method of determining sinusoidal oscillations and denoting $j = \sqrt{-1}$ we obtain

$$\varepsilon A_\kappa \sin(\kappa\Omega_1 t + \psi_\kappa) = -\left|\frac{R(j\kappa\Omega_1)}{Q(j\kappa\Omega_1)}\right| N_\kappa \sin(\kappa\Omega_1 t + \eta_\kappa + \beta_\kappa) - \qquad (2.23)$$

$$-\left|\frac{R(j\kappa\Omega_1)}{Q(j\kappa\Omega_1)}\right|\varepsilon G_{1\kappa}\sin(\kappa\Omega_1 t + \vartheta_{1\kappa} + \beta_\kappa) - \left|\frac{R(j\kappa\Omega_1)}{Q(j\kappa\Omega_1)}\right|\varepsilon^2 G_{2\kappa}\sin(\kappa\Omega_1 t + \vartheta_{2\kappa} + \beta_\kappa) - \dots, \quad (\kappa = 2,3,\dots)$$

where

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$$\beta_k = \arg \frac{R(jk\Omega_1)}{Q(jk\Omega_1)}. \qquad (2.24)$$

To carry out this set of equalities it is required that the first term of the right-hand part should be equal either to zero or should at least have the same order of ε as the rest of the equality terms. However, it can be seen from (2.7) and (2.5) that the values N_k (higher harmonic amplitudes at the nonlinearity output) in general case are not small as it has been given that the function F (x,px) greatly differs from the linear one. Therefore the values

$$\left|\frac{R(jk\Omega_1)}{Q(jk\Omega_1)}\right| \qquad (k=2,3,\ldots)$$

should have at least the order of ε.

It should be considered that from the equation of the first harmonic (2.19) it is obtained

$$\sin\Omega t = -\left|\frac{R(j\Omega)}{Q(j\Omega)}\right| \sqrt{q^2+(q')^2}\,\sin(\Omega t+\gamma+\beta), \qquad (2.25)$$

where

$$\gamma = \operatorname{arctg}\frac{q'}{q}, \qquad \beta = \arg\frac{R(j\Omega)}{Q(j\Omega)}, \qquad (2.26)$$

that is the value

$$\left|\frac{R(j\Omega)}{Q(j\Omega)}\right|$$

is the finite one.

Comparing these two results (and introducing into the first of them an approximate value of the frequency Ω) we come to a conclusion that the following condition should be carried out.

$$\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| = \varepsilon C_{1k}\left|\frac{R(j\Omega)}{Q(j\Omega)}\right| \ldots \left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| + \varepsilon^2 C_{2k}\left|\frac{R(j\Omega)}{Q(j\Omega)}\right|, \text{ или} \ldots (2.27)$$

where the values of C_{ik} are arbitrary finite positive numbers. Or in other words

$$\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right| \ll \left|\frac{R(j\Omega)}{Q(j\Omega)}\right|, \qquad (2.28)$$

where in general case K = 2,3,... or K = 3,5,..., if only uneven harmonics are changing as it is often the case in specific problems.

The inequality (2.28) is a condition in question, which should be satisfied by the transformed polynomials Q(p) and R(p) included in the given equation of the system dynamics (2.1).

If the power of the polynomial R(p) is lower than that of Q(p), which is often the case, we have

$$\left|\frac{R(jk\Omega)}{Q(jk\Omega)}\right|_{k\to\infty} \to 0. \qquad (2.29)$$

which is essential from the point of view of carrying out the condition (2.27) or (2.28).

The condition (2.28), in terms of automatic control theory, shows definite requirements to the type of gain-frequency characteristic of the linear part of the system. In fact according to (1.6) the gain-frequency characteristic of the linear part (transformed) will be

$$|W(j\omega)| = \left|\frac{R(j\omega)}{Q(j\omega)}\right|. \qquad (2.30)$$

Therefore the condition (2.28) will be satisfied by linear part of the system with gain-frequency characteristic for example, of one of the types shown in Fig. 4. Physically it means that the higher harmonics appearing at the

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linearity output (see Fig.2) will be filtered out by the linear part of the systems. In other words the linear part of the system should have the property of a filter and as a result of this the oscillations of the variable x at the output of the linear part of the system will, in fact, be close to sinusoidal ones (practically only the first harmonic remains) despite of the finite size of the amplitudes of the higher harmonics at its output y. This physical [illegible]ance /7/ now has a mathematical basis.

The calculations given in the paper have been done for the case when a periodic solution of the original nonlinear equation exists.

If the solution in strict sense will be not periodic but close to periodic solution (recognized as a periodic solution) then all the conditions obtained (2.9), (2.22) and (2.28) remain in force since additional terms causing the deviation of the solution from strict periodicity will be small and will not effect the relationship between the finite terms of the equations.

Also, in order the periodic solution of the nonlinear equation (2.1) should be close to sinusoidal one, defined by the linear* equation (2.1a) when the linear function

* This equation can be called a linear equation only conditionally and only along one periodic solution since its coefficients and will change when the amplitude A changes [illegible] they change when the frequency Ω [illegible]

greatly differs from nonlinear one three conditions should be carried out: (2.9), (2.22) and (2.28). These conditions are not too much strict and in practice (as it is for electromechanical servo systems, following object control systems, gyroscope stabilising systems, automatic control systems controlling processes with inertia, etc.) they can be nearly always carried out.

Besides, the carrying-out of the condition (2.28) can be strengthened, first, by the requirement (2.29) which is provided by the lower power of polynomial R(p) in comparison with Q(p).

Secondly - by the requirement that the polynomial Q(p) must not have roots with a positive material part or purely imaginary roots, that is in other words, by the requirement of having stability or neutrality (if such can exist) of the linear part of the system.

Generally speaking this requirement is not obligatory but it facilitates obtaining respective properties of the first approximation equation (2.19) in defining a periodic solution which will be mentioned below.

If a systems has several nonlinearities the conditions (2.9) and (2.22) must be carried out for all of them and the conditions (2.28) and (2.29) - for each linear part having variables at the output which then will be included under the signs of nonlinear functions.

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§ 3. The Approximate determination of symmetrical free and forced oscillations, more precise definition and calculation of the higher harmonics.

The above obtained equation (2.19) for finding the periodic solution approximately for the variable X in the sinusoidal form

$$x^* = A \sin \Omega t \qquad (3.1)$$

now can be obtained formally from the original nonlinear equation of the system

$$Q(\rho)x + R(\rho)F(x, px) = 0 \qquad (3.2)$$

by means of substitution X for x^* and the substitution

$$F(x, px) \longrightarrow (q + \frac{q'}{\Omega}\rho)x^*, \qquad (3.3)$$

when, as one can see from §2, the substitution (3.3) is in no way intailed with the assumption of an approximate equality of the left-hand and right-hand parts of the expression (3.3). The right-hand part of the expression (3.3) is only an approximate expression of the first harmonic of the left-hand part, the coefficients q and q' being calculated using formulas (2.20) and (2.21) as coefficients of the first harmonic of the Fourier series devided by the amplitude.

Thus for each desired nonlinearity F (x,px) the coefficients of harmonic linearization q and q' have quite definite expressions

$$q(A,\Omega),\ q'(A,\Omega) \quad \text{or} \quad q(A),\ q'(A). \qquad (3.4)$$

So as a result of the formal substitution (3.3) the original nonlinear equation (3.2) has the same form which has been obtained by a more strict method /see §2, (2,19)/:

$$\left[Q(\rho) + R(\rho)(q + \frac{q'}{\Omega}\rho)\right]x^* = 0. \qquad (3.5)$$

This simple operation incorporates the whole procedure of harmonic linearization of the desired nonlinear equation of the system in practical solution of specific problems.

Further, the harmonically linearized equation (3.5) should be solved by either this or that method in order to define the amplitude A and the frequency Ω of the required periodic solution (3.1) in the first approximation.

This is equivalent to finding such values of the coefficients q and q', which according to (3.4) depend at least on A, when the characteristic equation of the harmonically linearized set

$$Q(j\Omega) + R(j\Omega)(q + jq') = 0 \qquad (3.6)$$

has the pair of pure imaginary roots $p = \pm j\Omega$.

For example, with a direct substitution of $p = \pm j\Omega$ into the characteristic equation (3.6) we have

$$Q(j\Omega) + R(j\Omega)(q + jq') = 0. \qquad (3.7)$$

Substituting here q and q' for different values of A = const one can plot loci of the whole left-hand part, as vector, on the complex plane (similar to Mikhailov's curves providing their passing through the origine, and by means of

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this the required A and Ω are determined.

Another way allows obtaining the same solution by direct usage of the expression (2.25) equivalent to the equation (3.5), that is by writing of two equations

$$\left|\frac{R(j\Omega)}{Q(j\Omega)}\right|\sqrt{q^2+(q')^2}=1\,,\quad \gamma+\beta=\pi \qquad (3.8)$$

where γ and β are determined from the formulas (2.26). In this case the required A is a part of q and q'. This method can also be translated into the language of logarithmic frequency characteristics.

Instead of (3.7) one can write

$$\frac{R(j\Omega)}{Q(j\Omega)}=-\frac{}{q+jq'} \qquad (3.9)$$

and define the solution as a junction point of two loci with different parameters (Ω and A) on the com[illegible] plane as it has been done by Goldfarb /4/ and Kochenbu[illegible]

One can act in another way, that is tak[illegible] the expression (3.7) material and imaginary parts which given two scalar equations with real variables

$$X(A,\Omega)=0\,,\quad Y(A,\Omega)=0 \qquad (3.10)$$

equivalent to one complex equation (3.7). This leads either to analytical expressions for A and Ω through the parameters of the system (which is more desirable) or to the simple graph[illegible] [illegible]tions but without plotting the loci (see/3/)

Finally, one can demand that the last but one Hurwitz's determinant for the characteristic equation (3.6), all other Hurwitz's determinants for it being positive, should be equal to zero. This procedure is most likely to be [illegible] the solution of the problem by Bulgakov's method./2/

Afeter a periodic solution for the variable X in an approximate form $X = A\sin\Omega t$ has been somehow determined, periodic solution at the output of nonlinearity y should be determined in the form of

$$y=F(x^*,\,px^*), \qquad (3.11)$$

that is in no way in the sinusoidal form.

This coincides with the condition (2.9) which is the base for deriving the equations of the method, the real first harmonic X_1 of the input value being substituted by the approximatedly calculated X.

Similarly it is possible to define the approximate periodic solution if the system contains several nonlinearities with the only difference that instead [illegible] (3.5) there will be several equations which [illegible] together on the same basis.

Using the equations for the higher harm[illegible] it is possible to more precisely define [illegible] solution obtained.

According to (2-27) in equations [illegible] the [illegible] harmonics (2.23) the second term [illegible] right-hand part [illegible] magnitude order Σ^2 whereas [illegible] terms [illegible] magnitude order F. Therefore, [illegible] [illegible].23).

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$$\varepsilon A_k = \left| \frac{R(jk\Omega)}{Q(jk\Omega)} \right| N_k , \qquad (k = 2,3,\ldots) \quad (3.12)$$

$$\varphi_k = \pi + \eta_k + \arg \frac{R(jk\Omega)}{Q(jk\Omega)} . \qquad (k = 2,3,\ldots) \quad (3.13)$$

Introducing the relative amplitudes of the higher harmonics of the periodic solution for the variable X in the form

$$\delta_k = \frac{\varepsilon A_k}{A} \qquad (3.14)$$

and taking into account that the values N_k and η_k have appeared as a result of harmonic expansion of the nonlinear function F (x_1 , Px') which is now substituted approximately by F (x^*, px^*) we obtain from (3.12) and (3.13) the final formulae for the approximate definition of the relative amplitudes and phases of the small higher harmonics of the variable x:

$$\left.\begin{aligned} \delta_k &= \left| \frac{R(jk\Omega)}{Q(jk\Omega)} \right| \sqrt{\tau_k^2 + \delta_k^2} , \\ \varphi_k &= \pi + \operatorname{arctg} \frac{\delta_k}{\tau_k} + \arg \frac{R(jk\Omega)}{Q(jk\Omega)} , \end{aligned}\right\} \quad (k = 2,3,\ldots) \quad (3.15)$$

where

$$\left.\begin{aligned} \tau_k &= \frac{1}{\pi A} \int_0^{2\pi} F(A\sin\psi,\ A\Omega\cos\psi) \sin k\psi \, d\psi , \\ \delta_k &= \frac{1}{\pi A} \int_0^{2\pi} F(A\sin\psi,\ A\Omega\cos\psi) \cos k\psi \, d\psi . \end{aligned}\right\} \quad (3.16)$$

Thus each of the higher harmonics of the variable x is determined separately in the form

$$x_k = \delta_k A \sin(k\Omega t + \varphi_k) \qquad (k = 2,3,\ldots) \quad (3.17)$$

After such definition of the higher harmonics one can more precisely define the first harmonic defined above in the first approximation (3.1) A more precise definition is given considering the first term p of previously omitted small right-hand part R (p)$\varepsilon \Phi_n(t)$ in the equation for the first harmonic (2.15). From the definition the value $\varepsilon \Phi_n$ is the first harmonic of the first term of the right-hand part of the equality (2.8).

Considering that in (3.12) $N_k = 0$ for $k \to \infty$ we shall take into account for $\varepsilon\xi$ some finite number n of the higher harmonics (i.e. $\varepsilon\xi = \sum_{k=2}^{n} x_k$).

We obtain

$$\varepsilon \Phi_n(t) = \left(\Delta q + \frac{\Delta q'}{\Omega} p \right) x_1 , \qquad (3.18)$$

where

$$\left.\begin{aligned} \Delta q &= \frac{1}{\pi A} \int_0^{2\pi} \left[\frac{\partial}{\partial x} F(x^*, px^*) \sum_{k=2}^{n} x_k + \frac{\partial}{\partial px} F(x^*, px^*) \sum_{k=2}^{n} p x_k \right] \sin\psi \, d\psi \\ \Delta q' &= \frac{1}{\pi A} \int_0^{2\pi} \left[\frac{\partial}{\partial x} F(x^*, px^*) \sum_{k=2}^{n} x_k + \frac{\partial}{\partial px} F(x^*, px^*) \sum_{k=2}^{n} p x_k \right] \cos\psi \, d\psi \end{aligned}\right\} \quad (3.19)$$

Having calculated these extra coefficients we obtain according to (2.15) instead of (3.5) a new equation for defining the more precise first harmonic

$$x_1 = A_1 \sin \Omega_1 t \qquad (3.20)$$

in the form

$$\left[Q_1(p) + R(p) \left(q_1 + \frac{q_1'}{\Omega_1} p \right) \right] x_1 = 0 , \qquad (3.21)$$

where

$$Q_1(p) = Q(p) + R(p) \left(\Delta q + \frac{\Delta q'}{\Omega} p \right) , \qquad (3.22)$$

$$\left.\begin{aligned} q_1 &= \frac{1}{\pi A_1} \int_0^{2\pi} F(A_1\sin\psi,\ A_1\Omega_1\cos\psi) \sin\psi \, d\psi , \\ q_1' &= \frac{1}{\pi A_1} \int_0^{2\pi} F(A_1\sin\psi,\ A_1\Omega_1\cos\psi) \cos\psi \, d\psi . \end{aligned}\right\} \quad (3.23)$$

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Therefore the formal substitution is used instead of (3.3)

$$F(x, px) \longrightarrow (q + q'\frac{p-\xi}{\omega})x^*, \qquad (3.28)$$

where

$$\left.\begin{aligned} q &= \frac{1}{\pi A}\int_0^{2\pi} F(a\sin\psi,\ a\omega\cos\psi + a\xi\sin\psi)\sin\psi\, d\psi, \\ q &= \frac{1}{\pi A}\int_0^{2\pi} F(a\sin\psi,\ a\omega\cos\psi + a\xi\sin\psi)\cos\psi\, d\psi \end{aligned}\right\} \qquad (3.29)$$

(when the values of ξ are small, i.e. when the amplitude is slow-varying the value of ξ in the formulae (3.28) and (3.29) becomes negligible).

As a result of this we obtain instead of the equation (3.7) with the substitution of $p = \xi + tjw$ (in place of former $j\Omega$) the complex equation

$$Q(\xi + j\omega) + R(\xi + j\omega)(q + jq') = 0, \qquad (3.30)$$

which enables us to define $\xi(a)$ and $\omega(a)$. Design methods developed on this basis have been described in /3/.

Finally, using the similar methods one can determine single-frequency forced oscillations. Assume that we have the equation of the system dinamics

$$Q(p)x + R(p)F(x, px) = S(p)f(t) \qquad (3.31)$$

with a periodic external action

$$f(t) = B\sin\Omega t. \qquad (3.32)$$

The periodic solution can be found approximately in the form

$$x^* = A\sin(\Omega t + \varphi). \qquad (3.33)$$

Having realized the same harmonic linearization of nonlinearity (2.20) and (2.21) and formal substitution (3.3) and also having considered the relationships between the external action (3.32) and the solution (3.33) in complex form

$$f(t) = \frac{B}{A}e^{-j\varphi}x^*, \qquad (3.34)$$

we obtain instead of (3.3) the complex equation

$$Q(j\Omega) + R(j\Omega)(q + jq') = S(j\Omega)\frac{B}{A}e^{-j\varphi}, \qquad (3.35)$$

by means of which we determine two values required (A and φ) of the periodic solution (3.33) and also we determine the conditions of existance of such type of solution.

Similarly we determine the higher harmonics of the forced oscillations considering the higher harmonics of external periodic action f(t), if it is given in a non-sinusoidal form (see/3/).

§ 4. Slow-Changing Processes in Nonlinear Oscillatory Systems.

For nonlinear automatic control systems typical are those cases when self-oscillations and forced oscillations are vibratory processes which are superimposed on the main slow-changing regulation processes. Due to the system nonlinearity of both kinds of processes considerably effect each other,

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in particular, the changing of amplitude and frequency of vibrations can to a large extent change dynamic quality of a nonlinear system according to a slow-changing process (including boundary of stability, amplification ratio, frequency response and the quality of transient process under typical action). The process vibratory component may be a noise or specifically introduced for improvement of the quality and accuracy of the system. All this is extended to random processes in nonlinear systems.

Let for the nonlinear equation of the form

$$Q(p)x + R(p)F(x,px) = S(p)f(t) \qquad (4.1)$$

an approximate solution is being found in the form [1]

$$x = x^0 + x^*, \qquad x^* = A\sin\Omega t', \qquad (4.2)$$

where x^* is a vibratory (self-oscillating) component, $x^0(t)$ is a slow-changing component, that is x^0 little changes within the oscillation period x^* and the desired external action $f(t)$ also changes slowly.

In other words $f(t)$ and $x^0(t)$ are arbitrary in form time functions but their possible frequency changes are much less than the frequency Ω. The amplitude A and the frequency Ω of the vibratory component are also changing slowly in time together with changing of x^0. This is the reason why the time t^1 read out separately within each period is placed in the sine argument.

1) To be more exact $x = x_0 + x_1 + \xi(t)$

Note that due to the absence of x^* oscillation symmetry the carrying-out of the condition (2.22) is not required. Other conditions of §2 remain in force.

The harmonic linearization of nonlinearity in the given problem consists of the formal substitution (the reasoning of which is similar to the former one)

$$F(x,px) \longrightarrow F^0 + \left(q + \frac{q'}{\Omega}p\right)x^*, \qquad (4.3)$$

where

$$\left.\begin{aligned} F^0 &= \frac{1}{2\pi}\int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\,d\psi, \quad (\psi = \Omega t') \\ q &= \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\sin\psi\,d\psi, \\ q' &= \frac{1}{\pi A}\int_0^{2\pi} F(x^0 + A\sin\psi,\ A\Omega\cos\psi)\cos\psi\,d\psi. \end{aligned}\right\} \qquad (4.4)$$

In general case we have the definite expressions for each nonlinearity

$$F^0(x^0, A, \Omega),\ q(x^0, A, \Omega),\ q'(x^0, A, \Omega), \qquad (4.5)$$

and for many particular cases

$$F^0(x^0, A),\ q(x^0, A),\ q'(x^0, A). \qquad (4.6)$$

The substitution of the equation (4.3) allows us to consider(instead of desired equation (4.1)) the pair of interconnected equations

$$Q(p)x^0 + R(p)F^0 = S(p)f(t), \qquad (4.7)$$

$$\left[Q(p) + R(p)\left(q + \frac{q'}{\Omega}p\right)\right]x^* = 0, \qquad (4.8)$$

respectively for the slow-changing component and for the self-oscillating vibratory component.

With this division of the equation essential nonlinear properties and the absence of the superposition of the solution are preserved since their nonlinear interconnection remains

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due to incorporating the values (4.5) or (4.6) into these equations.

The equation (4.8) is similar to the equation (3.5) but with the difference that now the coefficients q and depend not only on the amplitude A and the frequency but also on the displacement x^0 of the vibratory component. Therefore having written the characteristic equation (3.6) one can define by any of the methods discribed in it the amplitude and frequency depending on x^0, that is

$$A(x^0) \quad u \quad Q(x^0). \tag{4.9}$$

Then from the formule (4.5) and (4.6) we define the displacement function.

$$F^0 = \Phi(x^0). \tag{4.10}$$

With this substitution the equation (4.7) takes the following form

$$Q(\rho)\,x^0 + R(\rho)\,\Phi(x^0) = S(\rho)\,f(t). \tag{4.11}$$

Consequently, the displacement function $\Phi(x^0)$ serves as a new nonlinearity with respect to slow-changing signal in the presence of imposed vibrations.

Usually the graph of the function $\Phi(x^0)$ is a smooth curve (even for nonsmooth and hysteresis type nonlinearities) which can be linearized in an ordinary way.

$$F^0 = k_H\,x^0, \tag{4.12}$$

where

$$k_H = \left(\frac{d\Phi}{dx^0}\right)_{x^0=0} \tag{4.13}$$

for the case shown in fig. 5-a or

$$F^0 = F_c^0 + k_H\,(x^0 - x_c^0), \quad F_c^0 = F^0(x_c^0), \quad k_H = \left(\frac{d\Phi}{dx^0}\right)_{x^0=x_c^0} \tag{4.14}$$

for the case shown in fig. 5-b

Then the equation (4.11) for a slow-changing component, that is for the main control process becomes a linear equation.

In case of (4.12) it will be

$$\left[Q(\rho) + k_H R(\rho)\right] x^0 = S(\rho) f(t) \tag{4.15}$$

This is an ordinary linear equation but the results of studying it will be not ordinary ones. The fact is that the value of amplification ratio of the slow-changing signal in nonlinear element of the system k_H depends on the amplitude and frequency of self-oscillating vibrations, which, in its turn, depend on the structure and relationship of all the parameters of automatic control system. This feature should be considered in analysing the effect of the structure and parameters of the system on its quality and also in synthesizing the system. This results in [illegible] nonlinear system behaviour when [illegible] was discussed at the beginning of [illegible]

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It is essential to note that slow-varying signal and vibrations pass through nonlinearity with different amplification ratios K_H and q respectively (or $q + jq'$).

Note the following property of the formula (4.13) which is essential from the point of view of simplifying coefficient k_H. In those cases when unevenly-symmetrical nonlinearities F(x) are investigated the value F^0 does not depend on Ω and, in addition, according to the first of the equation (4.4) we have

$$\left(\frac{\partial F^*}{\partial A}\right)_{x^0=0} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial f}{\partial x}\right)_{x=A\sin\psi}\sin\psi\, d\psi = 0,$$

since the derivative will be an even function. Consequently, for the unevenly-symmetrical nonlinearities F(x) both, simple and hysteresis type nonlinearities, the equation (4.13) for calculating k_H takes the following form

$$k_H = \left(\frac{\partial F^0}{\partial x^0}\right)_{x^0=0} \qquad (4.16)$$

This is an essential simplification because in this case the solution of the equation (4.8) for determining the relations (4.9) is not required but it is necessary after particular differentiation of the first of the expressions (4.5) or (4.6) by x^0 to substitute into it the values of A and Ω, defined at $x^0 = 0$, that is by solving a simpler problem - the equation (3.5).

If the displacement function $\Phi(x^0)$ does not yield to the above mentioned ordinary linearization and it is desirable to investigate nonlinear oscillations according to the slow-

changing component x^0 on the basis of nonlinear equation(4.11) one can realize harmonic linearization of the nonlinear function $\Phi(x^0)$ using the method described in paragraph 3. This will be a repeated harmonic linearization on a new lower frequency. In fact in such manner double-frequency nonlinear free oscillations (if f = 0) or complex oscillations (if f is periodic with low frequency) will be investigated.

So far we have considered self-oscillatory vibrations.

Similarly one can investigate the processes in nonlinear systems with forced vibrations. If we have the automatic control system

equation

$$Q(p)x + R(p)F(x, px) = S_1(p)f_1(t) + S_2(p)f_2(t) \qquad (4.17)$$

where

$$f_2(t) = B\sin\Omega t, \qquad (4.18)$$

and $f_1(t)$ is a slow-changing external action as compared with frequency.

Obtaining approximate solution in the form

$$x = x^0 + x^*, \qquad x^* = A\sin(\Omega t + \psi) \qquad (4.19)$$

and substituting (4.3) - (4.4) where $\psi = \Omega t + \psi$, we shall obtain instead of (4.7) - (4.8) a couple of equations

$$Q(p)x^0 + R(p)F^0 = S_1(p)f_1(t), \qquad (4.20)$$

$$\left[Q(p) + R(p)\left(q + \frac{q'}{\Omega}p\right)\right]x^* = S_2(p)f_2(t), \qquad (4.21)$$

respectively for slow-changing and vibratory components.

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We use from (4.21) the equation similar to (3.35) and determine the amplitude A and the phase ψ of forced vibrations as displecement function x^0. Substituting A (x^0) in the first of expression of (4.5) or (4.6) we determine displacement function (4.10) after which the further investigation involves the solution of the previous problem of § 4.

Essential difference is that the amplitude A effecting the value k_H in the linearized equation (4.15) depends not only on the structure and parameter of the system but also on the parameters of the external periodic action of B and Ω.

It is possible in this way to determine the influence, for instance, of external vibratory actions on the quality of operation of nonlinear automatic control systems on the basis of the main (slow-changing) signal and even on the stability region of this system.

Using the repeated harmonic linearization of the displacement function $\Phi(x^0)$ one can investigate double-frequency nonlinear forced oscillations (if $f_1(t)$ is a periodic low frequency function) complex ones (which are high frequency forced and low frequency free functions if $f_1 = 0$).

Note that from the equation (4.15) it is possible to investigate slow-changing random processes accompanied by self-oscillating or forced vibrations, probability characteristics for vibration amplitude being determined.

If in the nonlinear system (4.17) instead of periodic action (4.18) high-frequency random noises $f_2(t)$ given by its spectral density $S_f(\omega)$ an approximated solution can be found in the form $x = x^0 + x^*$ but x^* will be a random (centered) value, and x^0 - a slow-changing regular function (mathematical expectation). In this case statistical linearisation of nonlinearity instead of harmonic one suggested by I.E. Kazakov /8/ is realized. Here the same approach to the approximate solution of the problem is possible with the division of the equation (4.17) into two equations of the type (4.20) and (4.21), the second one having some different, statistical, sense.

The solution of such problem has been described in monograph /3/ which also contains a great number of particular investigations for particular nonlinear automatic control systems by all the above given methods developed on the basis of harmonic linearisation. Examples and various particular cases of the above presented general theory, as it is usually the case, particularly in the field of nonlinear features, reveal many interesting and specific properties of nonlinear automatic control systems.

This paper presents only general principal statements of the approximate methods of investigating the oscillatory processes in nonlinear automatic control systems.

Practical efficiency of the methods described provides further development in the field of application and more strict reasoning with defining the limiting conditions.

April 4, 1961.

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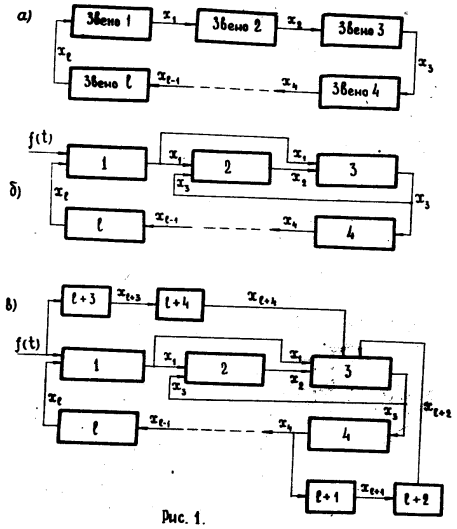

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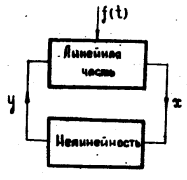

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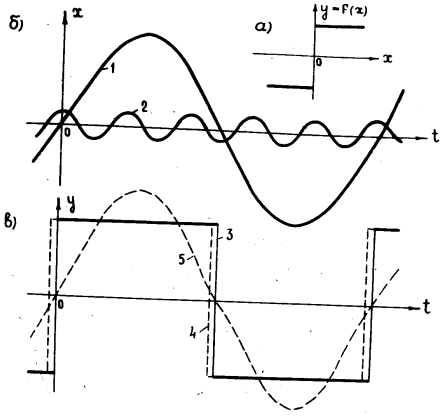

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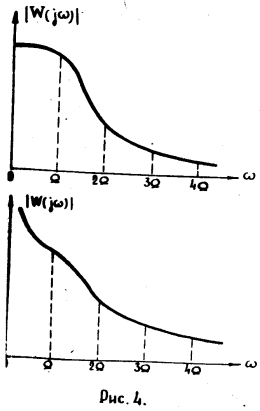

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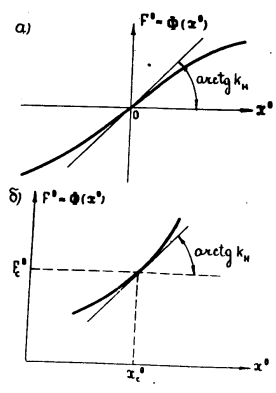

Рис. 5.

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А. П. ПРОСКУРЯКОВ

ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ КВАЗИЛИНЕЙНЫХ АВТОНОМНЫХ СИСТЕМ С ОДНОЙ СТЕПЕНЬЮ СВОБОДЫ В ВИДЕ РЯДОВ ПО ЦЕЛЫМ И ДРОБНЫМ СТЕПЕНЯМ ПАРАМЕТРА

Periodic solutions of quasi-linear systems usually are expressed by series in integer powers of small parameter [1, 2]. In this paper the solutions are considered in the form of series in fractional powers of this parameter.

1. In what follows we consider an quasi-linear oscillating system of the form

$$\frac{d^2x}{dt^2} + k^2 x = \mu f(x, \dot{x}, \mu). \tag{1.1}$$

We shall assume that $f(x, \dot{x}, \mu)$ is an analytic function of the variables in certain domain under consideration and μ is the small parameter.

Since our system is autonomous, we can choose the origin of the time reading so, that the following initial condition holds:

$$\dot{x}(0) = 0 \tag{1.2}$$

The general form of periodic solutions of d.e. (1.1) for $\mu = 0$ is then

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$$x_0(t) = A_0 \cos \kappa t \,. \qquad (1.3)$$

As to the other initial condition we can write it in the form

$$\dot{x}(0) = A_0 + \beta \,, \qquad (1.4)$$

where β is a function of μ ; if $\mu = 0$, one has also $\beta = 0$.

We shall assume that the domain, where function $f(x, \dot{x}, \mu)$ is analytic function of the variables, contains periodic solution (1.3) for $\mu = 0$. In accordance with the well known theorems [3] solution $x(t, \beta, \mu)$ of d.e. (1.1) is an analytic function of t . This solution is also analytic function of β and μ for sufficiently small value of parameter μ .

Let us set this solution in the form

$$x(t, \beta, \mu) = A_0 \cos \kappa t + \beta C(t) + \sum_{n=1}^{\infty} \left[C_n(t) + \frac{\partial C_n(t)}{\partial A_0} \beta + \frac{1}{2} \frac{\partial^2 C_n(t)}{\partial A_0^2} \beta^2 + \ldots \right] \mu^n . \qquad (1.5)$$

It is to be noted, that all the remaining terms of this series depending only on β are zero, as the coefficients at these terms satisfy linear homogeneous differential equations subjected to zero initial conditions.

Let us denote:

$$H_n(t) = \frac{1}{(n-1)!} \left(\frac{\partial^{n-1} f}{\partial \mu^{n-1}} \right)_{\beta = \mu = 0} .$$

We shall obtain that the coefficients $C_n(t)$ satisfy the following equation

$$\frac{d^2 C_n}{dt^2} + \kappa^2 C_n = H_n(t) \,.$$

The initial conditions are: $C_n(0) = \dot{C}_n(0) = 0$. In this connection we have

$$C_n(t) = \int_0^t H_n(t_1) \frac{\sin \kappa (t - t_1)}{\kappa} dt_1 \,. \qquad (1.6)$$

The first, second and third functions $H_n(t)$ are

$$H_1(t) = f(x_0, \dot{x}_0, 0) \,,$$

$$H_2(t) = \left(\frac{\partial f}{\partial x} \right)_0 C_1 + \left(\frac{\partial f}{\partial \dot{x}} \right)_0 \dot{C}_1 + \left(\frac{\partial f}{\partial \mu} \right)_0 ,$$

$$H_3(t) = \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2} \right)_0 C_1^2 + \frac{1}{2} \left(\frac{\partial^2 f}{\partial \dot{x}^2} \right)_0 \dot{C}_1^2 + \frac{1}{2} \left(\frac{\partial^2 f}{\partial \mu^2} \right)_0 + \left(\frac{\partial^2 f}{\partial x \partial \dot{x}} \right)_0 C_1 \dot{C}_1 + \left(\frac{\partial^2 f}{\partial x \partial \mu} \right)_0 C_1 + \left(\frac{\partial^2 f}{\partial \dot{x} \partial \mu} \right)_0 \dot{C}_1 + \left(\frac{\partial f}{\partial x} \right)_0 C_2 + \frac{\partial f}{\partial \dot{x}} \,.$$

Here the index "o" at the brackets denotes that it is necessary to substitute $x_0, \dot{x}_0, 0$ for $x, \dot{x}, \mu$ in the partial derivatives of f with respect to the variables $x, \dot{x}, \mu$.

It is known that period of solutions of autonomous systems depends on parameter μ and can be written in the form $T = T_0 + \alpha$, where $T_0 = 2\pi/\kappa$; α is a certain function of μ which approaches zero as μ approaches zero.

The conditions of periodicity for the function $x(t, \beta, \mu)$ and its first derivative are clearly

$$x(T_0 + \alpha, \beta, \mu) = A_0 + \beta \,, \qquad (1.7)$$

$$\dot{x}(T_0 + \alpha, \beta, \mu) = 0 \qquad (1.8)$$

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The equality (1.8) can be considered as equation for determination α as implicit function of β and μ. One-valued analytic function $\alpha(\beta,\mu)$ exists for $A_0 \neq 0$, since

$$\chi(T_0, 0, 0) = -\kappa^2 A_0 .$$

As all the partial derivatives of α with respect to β are vanishing for $\mu = 0$, we can set the function $\alpha(\beta,\mu)$ in the form

$$\alpha(\beta,\mu) = \sum_{n=1}^{\infty} \left(N_n + \frac{\partial N_n}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 N_n}{\partial A_0^2}\beta^2 + \dots \right) \mu^n . \quad (1.9)$$

Calculating we find

$$N_1 = \frac{1}{\kappa^2 A_0} C_1(T_0)$$

$$N_2 = \frac{1}{\kappa^2 A_0}\left[\dot{C}_2(T_0) + N_1 \ddot{C}_1(T_0)\right],$$

$$N_3 = \frac{1}{\kappa^2 A_0}\left\{\dot{C}_3(T_0) + N_2 \ddot{C}_1(T_0) + N_1 \ddot{C}_2(T_0) + \frac{1}{2} N_1^2 \left[\dddot{C}_1(T_0) + \frac{1}{3}\kappa^2 \dot{C}_1(T_0)\right]\right\}$$

and so on.

Substituting variable α as given by (1.9) into (1.7), one can obtain this equation in the form

$$\sum_{n=1}^{\infty} \left(M_n + \frac{\partial M_n}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 M_n}{\partial A_0^2}\beta^2 + \dots \right) \mu^n = 0 . \quad (1.10)$$

Values M_n are then

$$M_1 = C_1(T_0), \quad M_2 = C_2(T_0) + \frac{1}{2} N_1 \dot{C}_1(T_0),$$

$$M_3 = C_3(T_0) + N_2 \dot{C}_1(T_0) - \frac{1}{2} N_1^2 \ddot{C}_1(T_0),$$

$$M_4 = C_4(T_0) + N_3 \dot{C}_1(T_0) + \frac{1}{2}\kappa^2 A_0 N_2^2 - N_1\left[N_2 \ddot{C}_1(T_0) + \frac{1}{3} N_1^2 \dddot{C}_1(T_0) + \frac{1}{2} N_1 \ddot{C}_2(T_0)\right]$$

and so on.

2. One can consider equality (1.10) as equation for determination the implicit function $\beta = \beta(\mu)$. Dividing eq. (1.10) by μ and taking into account that $\beta(0) = 0$, we obtain as follows:

$$M_1 = C_1(T_0) = 0 . \quad (2.1)$$

Let $C_1(T_0)$ not equal zero identically. Then equation (2.10) determines the amplitude A_0 of the generating solution. If $C_1(T_0) \equiv 0$, all the derivatives of $C_1(T_0)$ with respect to A_0 vanish too. If this were the case, we should consider the equation

$$M_2 = 0$$

as the amplitude equation provided M_2 does not vanish identically.

Let us write equation (1.10) in considerable detail gethering the terms in the form of homogeneous polynomials of β and μ:

$$\Phi(\beta,\mu) = \frac{\partial C_1}{\partial A_0}\beta + M_2\mu + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\beta^2 + \frac{\partial M_2}{\partial A_0}\beta\mu + M_3\mu^2 + \frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}\beta^3 + \frac{1}{2}\frac{\partial^2 M_2}{\partial A_0^2}\beta^2\mu + \frac{\partial M_3}{\partial A_0}\beta\mu^2 + M_4\mu^3 + \dots = 0 . \quad (2.2)$$

We have then for $\mu = 0$:

$$\Phi(\beta,0) = \frac{\partial C_1}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\beta^2 + \frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}\beta^3 + \dots$$

In accordance with the theorem of Weierstrass about implicit functions [3, 4] the number of roots of equation (2.2), i.e. the number of implicit functions $\beta(\mu)$ determined from this equa-

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reduces to the consideration of the roots of the quadratic equation for $A_{1/3}$ and so on.

If some terms in the eq. (2.2) equal zero, it is possible to simplify the formulae. For instance, for $M_3=0$ one of the roots of eq. (2.6) equals zero. For $M_3=0$ and $\frac{\partial M_2}{\partial A_0}=0$ eq. (2.6) has 2-fold root $A_1=0$ and so on.

3) Value A_0 is the 3-fold root of eq.(2.1) $\frac{\partial C_1}{\partial A_0}=\frac{\partial^2 C_1}{\partial A_0^2}=0$; $\frac{\partial^3 C_1}{\partial A_0^3}\neq 0$.

In this case (2.2) has three roots $\beta=\beta(\mu)$. There are only three forms of expansions: in μ, $\mu^{1/2}$ and $\mu^{1/3}$. Since the sum of denominators in powers of μ does not exceed three in every considered case, we can have only two forms of expansions simultaneously: in μ and $\mu^{1/2}$.

3.1) If $M_2\neq 0$, the expansion for β is assumed in the form

$$\beta=\sum_{n=1}^{\infty} A_{n/3}\mu^{n/3} . \qquad \text{(III)}$$

Let us denote by:

$$U_n(A_{1/3})=\frac{1}{n!}\frac{\partial^n C_1}{\partial A_0^n}A_{1/3}^n+\frac{1}{(n-3)!}\frac{\partial^{n-3}M_2}{\partial A_0^{n-3}}A_{1/3}^{n-3}+\dots . \qquad (2.11)$$

We have the following equations for coefficients $A_{n/3}$:

$$U_3(A_{1/3})=\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{1/3}^3+M_2=0 ,$$

$$\frac{\partial U_3}{\partial A_{1/3}}A_{1/3}+U_4=0 ,$$

$$\frac{\partial U_3}{\partial A_{1/3}}A_{2/3}+\frac{\partial U_4}{\partial A_{1/3}}A_1+\frac{1}{2}\frac{\partial^2 U_3}{\partial A_{1/3}^2}A_{1/3}^2+U_5=0 ,$$

$$\frac{\partial U_3}{\partial A_{1/3}}A_{4/3}+\frac{\partial U_4}{\partial A_{1/3}}A_1+\frac{1}{6}\frac{\partial^3 U_3}{\partial A_{1/3}^3}A_{1/3}^3+\frac{1}{2}\frac{\partial^2 U_4}{\partial A_{1/3}^2}A_{2/3}^2+\frac{\partial U_5}{\partial A_{1/3}}A_{2/3}+U_6=0$$

and so on.

Since the first equation determines only one real value for $A_{1/3}$ and equations for the succesive coefficients are linear, we have only one expansion for β with the real coefficients. In the following this fact will always take place when one of the coefficients $A_{n/3}$ is determined from the binomial cubic equation.

3.2) Let M_2 equals zero, but $\frac{\partial M_2}{\partial A_0}\neq 0$. The expansion for β is determined in the form II.

We obtain the following system of equations for coefficients $A_{n/2}$

$$S_3(A_{1/2})=\left(\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{1/2}^2+\frac{\partial M_2}{\partial A_0}\right)A_{1/2}=0 ,$$

$$\frac{\partial S_3}{\partial A_{1/2}}A_1+S_4=0 ,$$

$$\frac{\partial S_4}{\partial A_{1/2}}A_{1/2}+\frac{1}{2}\frac{\partial^2 S_3}{\partial A_{1/2}^2}A_1^2+\frac{\partial S_4}{\partial A_{1/2}}A_1+S_5=0$$

and so on.

The first equation has two roots which do not equal zero and one zero-root. For the first and for the second roots we have expansion in the form II. If $A_{1/2}=0$, we have expansion for β in the form I. Coefficients A_n can be obtain-

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ed from the system (2.4).

3.3) Let $M_0=0$, $\partial M_1/\partial A_0=0$, but $M_3\neq 0$. In this case the expansion for β is obtained in the form III, but it starts from the term which contains $\mu^{1/3}$. Equations for coefficient $A_{1/3}$ are

$$\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{1/3}^3+M_3=0,$$
$$\left(\frac{\partial^3 C_1}{\partial A_0^3}A_1+\frac{\partial^2 M_2}{\partial A_0^2}\right)A_{1/3}^2=0,$$
$$\left[\frac{\partial^3 C_1}{\partial A_0^3}A_{1/3}A_{2/3}+\frac{\partial P_3}{\partial A_1}+\frac{1}{24}\frac{\partial^4 C_1}{\partial A_0^4}A_{1/3}^3\right]A_{1/3}=0$$

and so on.

3.4) Let $M_0=0$, $\partial M_1/\partial A_0=0$, $M_3=0$. In this case the coefficient A_1 is determined from the cubic equation

$$P_3(A_1)=\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_1^3+\frac{1}{2}\frac{\partial^2 M_2}{\partial A_0^2}A_1^2+\frac{\partial M_3}{\partial A_0}A_1+M_4=0. \tag{2.12}$$

Let us transform equation (2.2) using the substitution (2.7). Dividing it by μ^3 we obtain as follows:

$$\frac{\partial P_3}{\partial A_1}\gamma+P_4\mu+\frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}\gamma^2+\frac{\partial P_4}{\partial A_1}\gamma\mu+P_5\mu^2+$$
$$\frac{1}{6}\frac{\partial^3 P_3}{\partial A_1^3}\gamma^3+\frac{1}{2}\frac{\partial^2 P_4}{\partial A_1^2}\gamma^2\mu+\frac{\partial P_5}{\partial A_1}\gamma\mu^2+P_6\mu^3+\ldots=0. \tag{2.13}$$

If the roots of equation (2.12) are single, the expansion of β has the form I. Equations for the successive coefficients A_n are

$$\frac{\partial P_3}{\partial A_1}A_2+P_4=0,$$
$$\frac{\partial P_3}{\partial A_1}A_3+\frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}A_2^2+\frac{\partial P_4}{\partial A_1}A_2+P_5=0$$

and so on. In this case we have one or three forms of expansions for β with real coefficients depending on the number of real roots of equation (2.12).

3.5) Let us assume that there is 2-fold root among the roots of equation (2.12).

$$\frac{\partial P_3}{\partial A_1}=0\ ;\quad P_4\neq 0.$$

For this root the expansion of $\beta(\mu)$ is written in the form II. Equations for the determination of coefficients $A_{n/2}$ are

$$\frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}A_{1/2}^2+P_4=0,$$
$$\left(\frac{\partial^2 P_3}{\partial A_1^2}A_2+\frac{\partial P_4}{\partial A_1}+\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{1/2}^2\right)A_{1/2}=0$$

and so on. If the first equation has the real roots, we have three expansions for β; one expansion is in the form I and two expansions in the form II.

b) If in previous case $P_4=0$, coefficient A_2 can be determined from the following quadratic equation:

$$Q_3(A_2)=\frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2}A_2^2+\frac{\partial P_4}{\partial A_1}A_2+P_5=0. \tag{2.14}$$

The form of expansion for β depends on multiplicity of roots of this equation. Analysis of possible cases is analogous to the analysis in the case 2).

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3.6) a) Finally, let the roots of equation (2.12) be 3-fold roots

$$\frac{\partial P_3}{\partial A_1} = \frac{\partial^2 P_3}{\partial A_1^2} = 0 \quad , \quad P_4 \neq 0 \quad .$$

Expansions for β has the form III. Equations for coefficients $A_{n/3}$ are

$$\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3} A_{4/3}^3 + P_4 = 0 \quad ,$$

$$\left(\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^3} A_{4/3} A_{5/3} + \frac{\partial P_4}{\partial A_1}\right) A_{4/3} = 0 \quad ,$$

$$\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^3}\left(A_{4/3}^2 A_0 + A_{5/3}^2 A_{4/3}\right) + \frac{\partial P_4}{\partial A_1} A_{5/3} + \frac{1}{2}\frac{\partial^2 P_4}{\partial A_1^2} A_{4/3}^2 = 0$$

and so on.

b) If in previous case $P_4 = 0$, but $\frac{\partial P_4}{\partial A_1} \neq 0$, we shall determine expansion for β in the form II. Equation for coefficients $A_{3/2}$ is

$$\left(\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3} A_{3/2}^2 + \frac{\partial P_4}{\partial A_1}\right) A_{3/2} = 0 \quad .$$

For the real roots of this equation not being zero, we have the following

$$\left(\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^3} A_{3/2}^2 + \frac{\partial P_4}{\partial A_1}\right) A_0 + \frac{1}{2}\frac{\partial^2 P_4}{\partial A_1^2} A_{3/2}^2 + P_5 = 0$$

and so on. For the zero-root $A_{3/2} = 0$ we obtain expansion in the form I. Equations for coefficients A_n are

$$\frac{\partial P_4}{\partial A_1} A_2 + P_5 = 0 \quad ,$$

$$\frac{\partial P_4}{\partial A_1} A_3 + Q_4 = 0 \quad ,$$

$$\frac{\partial P_4}{\partial A_1} A_4 + \frac{\partial Q_4}{\partial A_2} A_3 + Q_5 = 0$$

and so on.

c) The roots are 3-fold roots, but

$$P_4 = 0 \quad , \quad \frac{\partial P_4}{\partial A_1} = 0 \quad , \quad P_5 \neq 0 \quad .$$

We have expansion for β in the form III. Here $A_{4/3} = 0$. The successive equations for coefficients $A_{n/3}$ are

$$\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3} A_{5/3}^3 + P_5 = 0 \quad ,$$

$$\left(\frac{\partial^3 C_1}{\partial A_0^3} A_2 + \frac{\partial^2 P}{\partial A_1^2}\right) A_{5/3}^2 = 0$$

and so on.

d) The roots are 3-fold roots, but $P_4 = 0$, $\frac{\partial P_4}{\partial A_1} = 0$, $P_5 = 0$. In this case coefficient A_2 is determined from the following cubic equation

$$\Phi_4(A_2) = \frac{1}{6}\frac{\partial^3 P_3}{\partial A_1^3} A_2^3 + \frac{1}{2}\frac{\partial^2 P_3}{\partial A_1^2} A_2^2 + \frac{\partial P_3}{\partial A_1} A_2 + P_6 = 0 \quad . \tag{2.15}$$

The following analysis is connected with the multiplicity of the roots of this equation and has analogy to above.

The cases, when value A_0 is m -fold root of equation (2.1) for $m > 3$, is not considered in this paper.

From the foregoing it follows that the **multiple root** of amplitude equation (2.1) corresponds to bifurcation of generating solution. Bifurcation is not in fact when only one root $\beta = \beta(\mu)$ is real or the all roots are equal. In this case expansion for β has the form I.

3. It is obvious that the form of expansion for period and for solution corresponds to the form of expansion for β .

Let us consider the case when expansion for β

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has the form II.
We have then

$$\alpha = T_0 \sum_{n=1}^{\infty} h_{n/2}\, \mu^{n/2} \quad . \tag{3.1}$$

For the coefficients we have the following formulas

$$h_{1/2} = 0 \quad , \quad h_1 = \frac{1}{T_0} N_1 \quad , \quad h_{3/2} = \frac{1}{T_0} \frac{\partial N_1}{\partial A_0} A_{1/2}$$

$$h_2 = \frac{1}{T_0}\left(\frac{1}{2}\frac{\partial^2 N_1}{\partial A_0^2} A_{1/2}^2 + \frac{\partial N_1}{\partial A_0} A_1 + N_2\right) \tag{3.2}$$

and so on.
Let us make the following introduction of new variables for the construction of periodic solution of eq. (1.1) with the constant period

$$t = \frac{\tau}{\kappa}\left(1 + h_1\mu + h_{3/2}\mu^{3/2} + h_2\mu^2 + \dots\right) .$$

We obtain then solution in the form of series in powers of $\mu^{1/2}$.

$$x(\tau) = x_0(\tau) + \mu^{1/2} x_{1/2}(\tau) + \mu x_1(\tau) + \mu^{3/2} x_{3/2}(\tau) + \dots \tag{3.3}$$

Coefficients of this solution have the constant period being equal 2π .
These coefficients are

$$x_0(\tau) = A_0 \cos\tau \quad , \quad x_{1/2}(\tau) = A_{1/2}\cos\tau \quad ,$$
$$x_1(\tau) = A_1\cos\tau + C_1(\tau) - h_1 A_0\, \tau \sin\tau \quad ,$$
$$x_{3/2}(\tau) = A_{3/2}\cos\tau + A_{1/2}\frac{\partial C_1(\tau)}{\partial A_0} - \left(h_{3/2}A_0 + h_1 A_{1/2}\right)\tau\sin\tau \quad ,$$
$$x_2(\tau) = A_0\cos\tau + C_2(\tau) + A_1\frac{\partial C_1(\tau)}{\partial A_0} + \frac{1}{2}A_{1/2}^2\frac{\partial^2 C_1(\tau)}{\partial A_0^2} +$$
$$+ h_1\tau\frac{\partial C_1(\tau)}{\partial\tau} - \left(h_2 A_0 + h_{3/2}A_{1/2} + h_1 A_1\right)\tau\sin\tau - \frac{1}{2}h_1^2 A_0\,\tau^2\cos\tau \tag{3.4}$$

and so on. Here we assumed that

$$C_n\left(\frac{\tau}{\kappa}\right) = C_n^*(\tau)$$

and then we dropped the index " * ".
If expansion of β is in the form III, we shall obtain

$$h_{1/2} = 0 \quad , \quad h_{2/2} = 0 \quad , \quad h_1 = \frac{1}{T_0} N_1 \quad ,$$
$$h_{4/2} = \frac{1}{T_0}\frac{\partial N_1}{\partial A_0} A_{1/2} \quad , \quad h_{3/2} = \frac{1}{T_0}\left(\frac{1}{2}\frac{\partial^2 N_1}{\partial A_0^2} A_{1/2}^2 + \frac{\partial N_1}{\partial A_0} A_{2/2}\right) ,$$
$$h_2 = \frac{1}{T_0}\left(\frac{1}{6}\frac{\partial^3 N_1}{\partial A_0^3} A_{1/2}^3 + \frac{\partial^2 N_1}{\partial A_0^2} A_{1/2} A_{2/2} + \frac{\partial N_1}{\partial A_0} A_1 + N_2\right) \tag{3.5}$$

and so on. In this case expansion of solution (1.1) has the following form after mentioned above introduction of new variables having been done

$$x(\tau) = x_0(\tau) + \mu^{1/2} x_{1/2}(\tau) + \mu^{2/2} x_{2/2}(\tau) + \mu x_1(\tau) + \dots \tag{3.6}$$

The first coefficients of this expansion are equal to

$$x_0(\tau) = A_0\cos\tau \quad , \quad x_{1/2}(\tau) = A_{1/2}\cos\tau \quad , \quad x_{2/2}(\tau) = A_{2/2}\cos\tau \quad ,$$
$$x_1(\tau) = A_1\cos\tau + C_1(\tau) - h_1 A_0\,\tau\sin\tau \quad ,$$
$$x_{3/2}(\tau) = A_{3/2}\cos\tau + A_{1/2}\frac{\partial C_1(\tau)}{\partial A_0} - \left(h_{3/2}A_0 + h_1 A_{1/2}\right)\tau\sin\tau$$
$$x_{2/2}(\tau) = A_{2/2}\cos\tau + A_{2/2}\frac{\partial C_1(\tau)}{\partial A_0} + \frac{1}{2}A_{1/2}^2\frac{\partial^2 C_1(\tau)}{\partial A_0^2} -$$
$$- \left(h_{2/2}A_0 + h_{3/2}A_{1/2} + h_1 A_{2/2}\right)\tau\sin\tau \quad ,$$

$$x_2(\tau) = A_0\cos\tau + C_2(\tau) + A_1\frac{\partial C_1(\tau)}{\partial A_0} + A_{1/2}A_{3/2}\frac{\partial^2 C_1(\tau)}{\partial A_0^2} + \tag{3.7}$$

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$$+ h_1 \tau \frac{\partial C_1(\tau)}{\partial \tau} - (h_2 A_0 + h_{3/5} A_{1/3} + h_{4/5} A_{2/5} + h_1 A_1 \tau \sin \tau -$$

$$- \frac{1}{2} h_1^2 A_0 \tau^2 \cos \tau \ .$$

In the case when β has expansion in integer powers of parameter μ, we must equate to zero some coefficients $A_{n/k}$ and $h_{n/k}$ in the formulae of one of the previous cases. The indexes in these coefficients are not equal to integer.

There is not discussed the question about the radius of convergence of the series considered in the paper.

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НИЗШИЕ ГАРМОНИЧЕСКИЕ И КОМБИНАЦИОННЫЕ КОЛЕБАНИЯ В ТРЕХФАЗНЫХ НЕЛИНЕЙНЫХ СИСТЕМАХ С ФЕРРОМАГНИТНЫМИ ЭЛЕМЕНТАМИ

Excitation and maintenance of autoparametric oscillations in three-phase electroferro-magnetic oscillation circuits have some specific peculiarities. As distinction from linear three-phase circuits three-phase nonlinear systems may not always be considered (even if there is full symmetry) to be a superposition of three one-phase circuits.

1. Subharmonics in three-phase systems.

Let us consider a multiphase system with symmetric elements (with identical characteristics) in every phase. The periodic function with the period T in m-phase system for λ-th phase may be represented by:

$$f_\lambda(x) = f\left[x + \frac{\lambda - 1}{m} \cdot T\right]. \qquad (1)$$

In general form it may be represented by series:

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$$f_\lambda(x) = A_{1\lambda}\sin\left[x + \frac{\lambda-1}{m}\pi\right] + A_{3\lambda}\sin\left[3\left(x + \frac{\lambda-1}{m}\pi\right) + \psi_3\right] +$$

$$+ A_{\frac{1}{2}\lambda}\sin\left[\frac{1}{2}\left(x + \frac{\lambda-1}{m}\pi\right) + \psi_{\frac{1}{2}}\right] + A_{1\lambda}\sin\left[x + \frac{\lambda-1}{m}\pi + \psi_1\right] +$$

$$+ A_{2\lambda}\sin\left[2\left(x + \frac{\lambda-1}{m}\pi\right) + \psi_2\right] + A_{3\lambda}\sin\left[3\left(x + \frac{\lambda-1}{m}\pi\right) + \psi_3\right] +$$

$$+ \ldots = \sum_{\frac{\kappa}{\varkappa},\kappa=0}^{\frac{\kappa}{\varkappa},\kappa=\infty} A_{\frac{\kappa}{\varkappa}\lambda}\sin\left[\frac{\kappa}{\varkappa}\left(x + \frac{\lambda-1}{m}\pi\right) + \psi_{\frac{\kappa}{\varkappa}}\right], \qquad (2)$$

where λ is the number of phase

m is the quantity of phases

$\frac{\kappa}{\varkappa}$ is the order of harmonic.

The order of harmonic $\frac{\kappa}{\varkappa}$ may take on any value: when $\kappa = 0$ there is a constant component, $\frac{\kappa}{\varkappa} = 1$ gives the component of the first harmonic, when κ is an integer, $\varkappa = 1$ we have higher harmonic components, $\kappa = 1$, $\varkappa$ - integer we have subharmonic components, $\kappa \neq 1$, $\varkappa \neq 1$ we have nonmultiple nigh- and subnarmonic components.

Let us use the operator of three phase system:

$$\alpha_m^\gamma = e^{i\gamma\frac{2\pi}{m}}, \qquad (3)$$

where γ are integers, which determine the order of sequence of phases.

Applying a matrix method of description of non-sinusoidal periodic quantities [1] for the whole three-phase system we have:

$$\|\dot{F}_{\gamma\kappa}\| = \|\alpha_m^{\gamma\kappa}\| \, \|\dot{F}_\kappa\|, \qquad (4)$$

where $\dot{F}_{\gamma\kappa}$ represents the element of matrix,

corresponding to γ th sequence of phases and κ th harmonic.

$$\dot{F}_{\gamma\kappa} = \alpha_m^{\gamma\kappa}\dot{F}_\kappa. \qquad (5)$$

In the case of the three phase system we have:

$$\begin{vmatrix} \dot{F}_{0\kappa} \\ \dot{F}_{1\kappa} \\ \dot{F}_{2\kappa} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha_3^{1\kappa} & \alpha_3^{2\kappa} \\ 1 & \alpha_3^{2\kappa} & \alpha_3^{4\kappa} \end{vmatrix} \begin{vmatrix} \dot{F}_{0\kappa} \\ \dot{F}_{1\kappa} \\ \dot{F}_{2\kappa} \end{vmatrix}. \qquad (6)$$

In order to represent the peculiarities of all cases of these matrixes, let us consider a three phase system of periodic non-sinusoidal functions.

For this case the equation (4) in expanded form is represented as:

$$f_I(x) = A^I_{\frac{\kappa}{\varkappa}}\sin\left(\frac{\omega t}{\varkappa} + \psi_{\frac{\kappa}{\varkappa}}\right) + \ldots + A^I_\kappa\sin\left(\kappa\omega t + \psi_\kappa\right) + \ldots$$

$$f_{II}(x) = A^{II}_{\frac{\kappa}{\varkappa}}\sin\left(\frac{\omega t - 120^\circ}{\varkappa} + \psi_{\frac{\kappa}{\varkappa}}\right) + \ldots + A^{II}_\kappa\sin\left(\kappa(\omega t - 120^\circ) + \psi_\kappa\right) + \ldots$$

$$f_{III}(x) = A^{III}_{\frac{\kappa}{\varkappa}}\sin\left(\frac{\omega t - 240^\circ}{\varkappa} + \psi_{\frac{\kappa}{\varkappa}}\right) + \ldots + A^{III}_\kappa\sin\left(\kappa(\omega t - 240^\circ) + \psi_\kappa\right) + \ldots \qquad (7)$$

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From these expressions of periodic functions it is apparent, that there are such components which may originate in different relations of phase displacement differing from 120°. If subharmonic components are displaced 360° or 720° from the first harmonic component, then neither the form, nor the amplitude of the resulting curve in that phase will be changed. Generally subharmonics of κ-th order in every phase may have the following displacements:

Phase I $0, \frac{360°}{\kappa}, \frac{2\cdot 360°}{\kappa}, \frac{3\cdot 360°}{\kappa}, \ldots, \frac{(\kappa-1)360°}{\kappa}$.

Phase II $\frac{120°}{\kappa}, \frac{360°+120°}{\kappa}, \frac{2\cdot 360°+120°}{\kappa}, \ldots, \frac{(\kappa-1)360°+120°}{\kappa}$.

Phase III $\frac{240°}{\kappa}, \frac{360°+240°}{\kappa}, \frac{2\cdot 360°+240°}{\kappa}, \ldots, \frac{(\kappa-1)360°+240°}{\kappa}$.

In the most general form we may show that phase displacements between adjacent phases of the three-phase system of subharmonic oscillations may be: $\frac{2\pi}{3\kappa}, \frac{4\pi}{3\kappa}, \frac{8\pi}{3\kappa}$.

By the way we shall note that the coincidence of phases of the three-phase system of high harmonic oscillations holds for $\kappa = \frac{2\pi}{3}$;

if: $\frac{\kappa}{3}$ = integer then harmonics coincide in time phase;

if: $\kappa = 3\gamma - 1$ - harmonics form a symmetric system of a ~~negative~~ inverted sequence of phases;

if: $\kappa = 3\gamma + 1$ - harmonics form a symmetric system of a ~~positive~~ direct sequence of phases

(γ is an integer).

Therefore, in symmetric systems of first and higher harmonics, subharmonics may form different combinations of asymmetric systems. Depending on the scheme of connection of three-phase systems some harmonics in phase, linear or zero conductors may ~~become apparent~~ appear.

As it is known, a symmetric three-phase system without a zero conductor refers to all multiple three high harmonics, as a disconnected system. If the zero point of the generator is grounded, and the neutral line of the star-connected load is isolated, then the potential of an isolated neutral will oscillate ~~according~~ with respect to ground with the frequency of multiple three high harmonics. Components of zero order sequence of phases of subharmonics, forming an asymmetric system may superpose upon these high harmonic oscillations. As a result the oscillation of neutral has a complicated form.

If zero points of the generator and receiver are connected, then the currents produced both by multiple three high harmonics and by components of subharmonics of zero sequence flow along a zero conductor. These currents may be very high.

In particular components of currents produced by three order subharmonics, on the zero conductor have a resulting value:

$$i_{\frac{I}{3}} = i_{\frac{I}{3}}^{I} + i_{\frac{I}{3}}^{II} + i_{\frac{I}{3}}^{III}.$$

The phase displacements of different values $i_{\frac{I}{3}}^{I}, i_{\frac{I}{3}}^{II}, i_{\frac{I}{3}}^{III}$ may be in different combinations, shown above.

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Under corresponding combinations of parameters, depending upon the value and ~~impedance~~ resistance characteristics of the zero conductor a high or low current may pass in it. If the system of applied e.m.f. is asymmetric, then one phase influences another; then submagnetization of ferromagnetic elements with currents of complicated form takes place and the process of excitation of different kinds of oscillations is clearly ~~pronounced~~ apparent. In this case the current curve in a zero conductor is of complicated form and may contain not only subharmonic components, but also components of polyharmonic oscillations.

Figure 1 shows oscillograms of subharmonic oscillations in three phase systems (a scheme goes with it).

2. Polyharmonic oscillations in three-phase systems.

Polyharmonic oscillations excited in nonlinear systems with ferromagnetic elements, are of complicated character. Even in those few works, which are dedicated to the investigation of these oscillations [2, 3, 4, 5, 6] there is no common point of view about the origin of physical nature and peculiarities of them.

General peculiarities of polyharmonic oscillations are:

a) The phenomenon has a character of periodically arising impulses with the "period" (the duration of intervals and impulses) changing from a part of a second to several scores of seconds (for the systems we have investigated).

b) The arising impulses are of oscillating character. The frequency of oscillations of impulses may be equal to the applied frequency or submultiple of the later.

c) The period and amplitude of impulses depends greatly on the value of the applied voltage and on the parameters of the system and load.

d) Polyharmonic oscillations may arise and disappear by the decreasing of intervals t_n and the increasing of duration of impulses. These processes are preceeded by non-stable oscillations with complicated form of curves.

e) In certain cases polyharmonic oscillations ~~pass~~ turn to subharmonic oscillations and vice versa.

Figure 2 shows the curves, characterizing polyharmonic oscillations in two core scheme circuits. (The scheme is also shown in that figure).

Polyharmonic oscillations are easily excited in three-phase electroferromagnetic oscillatory circuits with parallel-connected elements (figure 3). In such circuits, when the applied voltage amounts some critical value, the excited autoparametric oscillations jump in the circuit. In such circuits polyharmonic oscillations arise with swinging cycles equal 40 to 150 times of the cycle of the applied voltage. Changing the value of the applied voltage and capacity it is possible to change the "swinging period" of polyharmonic oscillations on a

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large scale. The currents in different phases adding in the zero conductor produce currents with a swinging period three times under that of the phases of a star-connected three-phase system.

To represent the polyharmonic oscillations in three-phase systems figures 3[a,b,c] show the oscillograms for different cases and with different speed of the tape in oscillograph.

Polyharmonic oscillations may cause supercurrents and overvoltages in the system. These oscillations are accompanied by a specific sound of vibrations of cores in ferromagnetic elements and the hunting of the pointers of measuring instruments. Asymmetric feeding in such circuits facilitates excitation of subharmonic and polyharmonic oscillations. For some relations between the parameters and the value of the applied voltage in a system such conditions may be created when owing to a little inrush of voltage the polyharmonic oscillations are either increasing or decreasing (fig.4). The impedance between zero points of star-connected capacities and star-connected ferromagnetic elements may greatly influence the autoparametric excitation regime of such circuits. In circuits with isolated zeros of capacity and inductive elements autoparametric excitation is suppressed completely.

Polyharmonic oscillations may also be excited in circuits of three-phase systems with successively connected elements.

3. Subharmonic and polyharmonic oscillations in power systems.

The presence of complex autoparametric oscillations in power installations was already observed in 1910 [7]. At that time scientists and engineers were interested in high harmonics /especially in the third harmonic/ in three phase circuits generally, and in transformers in particular [8, 9]. In 1915 L.Robinson was the first to have paid attention to the peculiarities of such phenomena in these systems [10,11].

A question of "abnormal" oscillations in three-phase systems was raised together with the question about high harmonics, generated by transformers.

The development of high voltage networks, the introduction to a system of different compensating devices (compensating reactors, longitudinal capacity compensation) offered new problems, connected with arising of internal overvoltages in such systems.

In connection with this several authors [12, 13, 14, 15, 16] investigated excitations of different oscillations in complex three-phase oscillating circuits (for instance in ~~simulated~~ system models).

Such oscillations were observed in several power installations [17, 18]. In particular, excitation of autoparametric oscillations were observed at substation of the Uzbekistan power system [19] and Leningrad power system [20]. These oscillations were accompanied by the aris-

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ing of "false ground" on substation tires, by the rising of voltages in different phases, and also by the unstable indication and hunting of pointers on the switchboard.

The detailed investigation of the arising and development of similar phenomena showed that:

a) The abnormal oscillations (huntings) in phase and line voltages, vary in different phases. (The range of hunting is 10-15% in phase voltages, 1 - 1,3% in line voltages).

b) The phase voltage increase is about 20-25% and line voltage drop is 2 - 2,5%.

c) Zero point voltage oscillates with the frequency equal to half of the applied voltage frequency.

d) The curve forms, phase and amplitude of hunting are nearly identical in different phases.

e) Phase voltage curves generally have components with frequencies: 1) equal to the frequency of applied voltage 2) nearly half of the applied voltage frequency 3) 100-120 submultiple of the applied voltage frequency.

f) Line voltages don't contain second order subharmonic components; fundamental harmonic component predominate in it.

In practice the conditions of supressing of such oscillations have a special significance, both in the laboratory tests and in field investigations; It was established, that such oscillations disrupt or don't excite with the change of the parameter relations and the applied voltage (in particular with the change of capacity or resistance of the circuit).

The excitation of subharmonic oscillations was also observed in Swedish long distance power transmission system.

In some cases of the investigation of peculiarities of the long distance power transmissions by the models complex autoparametrical oscillations were excited.

4. <u>The physical picture of polyharmonic oscillations in three-phase electroferromagnetic systems</u>.

We may distinguish two kinds of polyharmonic oscillations: those with three components and with periodic overturn (as in two core circuit [2]).

a) The following assumptions about physical picture of three component polyharmonic oscillations in three-phase systems may be admitted.

When the applied voltage varies, the system goes to a new steady state by means of levelling current of free oscillation frequency. If the circuit were linear, the levelling current would damp owing to the resistance of the system. However, because of the nonlinearity of the characteristics of ferromagnetic elements these currents lead to the excitation of autoparametric oscillations. Besides the magnetomotive forces with free oscillation frequency change the curve of magnetic flux in such a way, that the fundamental harmonic (with the applied voltage frequency) will be a little shifted in its phase.

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The phase-shift will be changed between the primary voltage and current in the circuit which characterise the energy inflow to the ferromagnetic element. This energy taking into account the loss in ferromagnetic material is expanded in circuit. Thus, the losses are compensated owing to a peculiar reconnection of circuit, and oscillations are held up in the system, determined mainly by its free frequency. Besides in energo-capacious elements of the system (in capacity and inductance) a sufficient large energy may be accumulated, which may increase considerably the voltage in it (to three multiple from normal).

The starting of hunting, i.e. oscillations of voltage and current in the system with a very low frequency may be explained in the following way. The connection of different phases between each other owing to the capacity define the conditions of superposing additional supermagnetization alternating field, formed by the currents of adjacent phases on fundamental core field, formed by the current of that phase. These additional fields change the characteristics of ferromagnetic elements, similarly to the change of their characteristics during the supermagnetization of direct current.

However, instead of direct current here we have a quasi-direct current, changing from period to period with the variation of phase displacement between the fundamental harmonic and harmonic, having flux free oscillation frequency. This quasi-constant component of flux varies

nearly sinusoidally and with a very low frequency, and at the same time induces e.m.f. in every phase, displaced to 120° between each other.

Experimental investigations show that the frequency of an additional e.m.f. decreases with the increase of the amplitude of the applied voltage. Indeed, the perimeter of hysteresis characteristic expands with the increase of the applied voltage. The total passing of full cycle is increasing in time, so that the hunting period increases or the frequency decreases.

In three-phase system conditions the second order subharmonic is able to keep up stable synchronized oscillations. The analysis of current curve oscillograms shows that at the second order subharmonics during one period of the applied voltage in every phase ferromagnetic element passes its positive (or negative) part of characteristic only once.

The moments of arising of current impulses in every phase are shifted to each other in such a way that on the whole they form one period of the applied voltage. In the following period the process goes to the negative (or positive) quadrant of characteristic. If the free oscillation frequency were not equal to the second order subharmonics, then passing to positive (or negative) parts of characteristics would overlap the next period with the change of sign. As a result the power balance in system would be broken and it would pass to an other regime of oscillations (disruption of oscillations).

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Thus the energy balance in all phases and in zero conductor may remain the same in the positive (or negative) part of characteristic, at first in all phases, afterwards the same process will be repeated in the second period; the full cycle of the process is repeated in two periods of the applied voltage, i.e. with the frequency of the second order subharmonic.

b) For some relations between system parameters and applied voltages in three-phase electroferromagnetic oscillation circuits polyharmonic oscillations are excited with periodic overturn. They are generally foregoing some stable synchronized states. For instance, in the case of capacity loading of the frequency tripler such oscillations are observed even with a very low applied voltages.

These oscillations as in two core circuits [2] are characterised by a periodical rebuilding of circuit. The system, being in one regime, after some interval of time (keeping the applied voltage constant) passes to another regime in a jump, and again to primary and so on.

The periodical rebuilding of the circuit is stable enough. Its cycle may be changed with the variation of applied voltage; increasing of the rebuilding frequency leads a system to a state of three component oscillations or to synchronized stable oscillations with any frequency (high, fundamental or subharmonics).

The nature of polyharmonic oscillations is defined by the continuous change of transient processes in oscillating circuits, tuned to some harmonics (fundamental, high or subharmonics). Since the transient process grows gradually the accumulation of energy in a system also occurs gradually and at a definite ratio of power values the rebuilding of the circuit takes place. Depending on the depth of parameter modulation, polyharmonic oscillations with periodic overturn may be excited by different values of the applied voltages.

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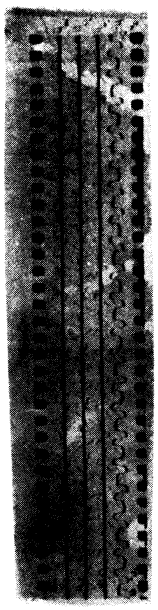

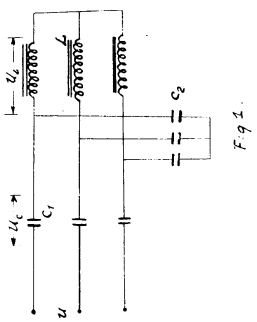

Фиг. 1

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Fig 2.

(a)

(b)

Fig 3

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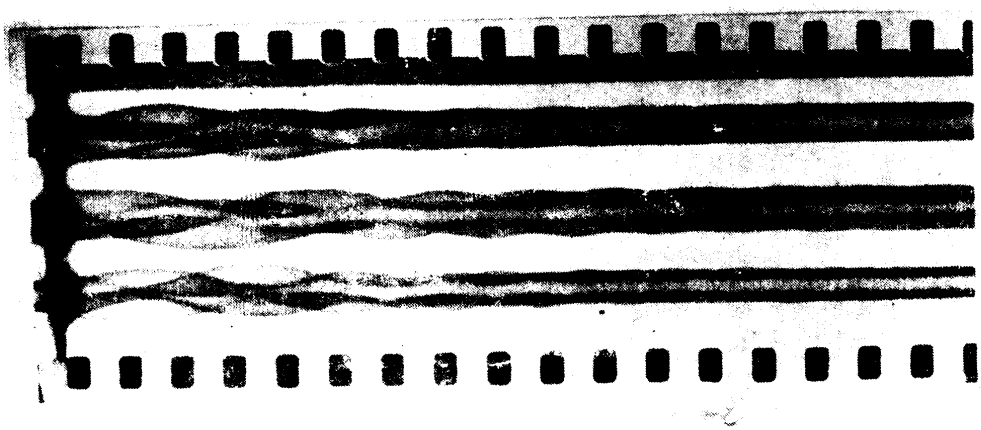

Fig. 4

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Ю. А. РЯБОВ

ОБ ОЦЕНКЕ ОБЛАСТИ ПРИМЕНИМОСТИ МЕТОДА МАЛОГО ПАРАМЕТРА В ЗАДАЧАХ ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ

In the theory of non-linear oscillations we have to deal often with the differential equations of the form

$$\frac{dz_s}{dt} = a_{s1}z_1 + \dots + a_{sn}z_n + \Psi_s(t) + \varepsilon F_s(t, z_1, \dots, z_n), \qquad (s = 1, 2, \dots, n) \qquad (1)$$

where a_{sk} are constants, ε is a positive small parameter, $\Psi_s(t)$ are continuous functions periodic in t of the period T and F_s are functions of $t, z_1, z_2, \dots, z_n$, continuous, periodic in t of the same period. For an autonomous system $\Psi_s(t) = 0$ and the functions F_s are independent of t. If for $\varepsilon = 0$ the system /1/ has a periodic solution $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$, then for $\varepsilon \neq 0$ the small parameter method of Liapunov-Poincare /see [1]/ or that developed by Krylov and Bogoliubov /see [2]/ are widely applied for the obtaining and investigation of the periodic solution of that system. In the applications one calculates only the first approximation /first terms of the series/ or, at the best, the second-third approximation/ the second - third terms of the series/. At the same time these methods are theoretically well -grounded, i.e. the sufficiently high precision of the approximate formula one obtains is assured only for sufficiently small values of ε, the concept of "sufficiently small value" not being connected with some numerical values of ε. Therefore the evaluations of the error of the obtained approximate formula /with respect to the strong solution/ for the fixed numerical values of ε would have been of importance for the applications.

The previous my papers [2 - 8] were dedicated mainly to the evaluation of the convergence area of the series or of the successive approximations obtained by using the Liapunov-Poincare method and the systems of sufficiently general form were considered there. The ways proposed are based

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$U^{(\kappa)} \geq |x^{(\kappa)}(t)|$

$x_\kappa(t)|, \; U_{\kappa+1}, \; U_\kappa \geq |x_{\kappa+1}(t) - x_\kappa$

her to calculate the upper

ius of the series /6*/ or o

secure the convergence of t

ximations $x_\kappa(t)$ for ε

a possibility to estimate

e error of the approximatio

the some initial terms of t

he strong periodic solution

of the successive approxima

equation for each approxima

functions $x_\kappa(t)$ admit th

urier polinomial. For $f(0,0,$

$M_\delta^{(\nu)} \cos\delta t + N_\delta^{(\nu)} \sin\delta t, \; x_1(t) = \varepsilon \sum_{\delta=0}^{m_1}$

$\kappa \geq 1$ we obtain

$\sum_{\nu=0}^{m_\kappa} M_\delta^{(\nu)} \cos\delta t + N_\delta^{(\kappa)} \sin\delta t, \; x_\kappa(t) = \varepsilon \sum_{\delta=0}^{m_\kappa}$

$N_\delta^{(\kappa)}$ are constant. Afte

he quantities

$= \varepsilon \sum_{\delta=0}^{m_\kappa} \frac{1}{|\omega^2 - \delta^2|} \sqrt{(x_\delta^{(\kappa)})^2 + (N_\delta^{(\kappa)})^2}$

functions $x_\kappa(t)$ are direct

the ideas of ...". Liapunov gave [illegible]. In Liapunov we are interested for the first example of the effective evaluation.

This question was also considered in the papers [illegible], but the concrete examples aren't given there and as the analytic and some calculations reveal (see [illegible]) the formula received there the following principle different from those of Liapunov/ are based on the more rough majoration.

In the present report there are pointed out some additional specifications to the way of obtaining the required majore relations and first of all the evaluations for the errors of the first-second approximations are considered. We touch also upon the subject of such evaluations in the case of Krylov - Bogoliubov's method. In the literature concerning the non-linear oscillation these questions weren't considered.

For the purpose of giving evidence for a possibility to obtain the effective evaluations I restrict myself to the equations of particular forms and give some examples.

§ 1. Consider the equation

$$\ddot{z} + \omega^2 z = \varepsilon \mathcal{F}(z, \dot{z}, t) + \Psi(t), \qquad /2/$$

describing the system with one freedom degree, where $\mathcal{F}(z,\dot{z},t)$, $\Psi(t)$ are continuous functions periodic in t of period 2π. Suppose $\mathcal{F}(z,\dot{z},t)$ to be a polinomial in $z, \dot{z}$ of the form

$$\mathcal{F}(z,\dot{z},t) = \sum_{j+\delta=1}^{m} P_{j\delta}(t) z^j \dot{z}^\delta, \qquad /2^*/$$

where $P_{j\delta}(t)$ are continuous, periodic in t and they could be represented by the Fourier polinomial. It is the case for sufficiently wide class of the systems.

The simplest case is that of the oscillations far of the resonance point when the value ω in $\varepsilon\varphi$ /2/ considerably differs from integer. We consider it first.

For $\varepsilon = 0$ the equation /2/ admits a unique periodic solution /"generating" after Poincare/ $x_0(t)$. After substitution in /2/ $z = z_0 + x$ we receive the equation

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, t), \qquad /3/$$

where $f(x,\dot{x},t)$ has the same form as $\mathcal{F}(z,\dot{z},t)$:

$$f(x,\dot{x},t) = \sum_{j+\delta=0}^{m} P_{j\delta}(t) x^j \dot{x}^\delta \qquad /3^*/$$

The periodic solution of /3/ that tends to zero as $\varepsilon \to 0$ one receives in the form of the series

$$x = \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \dots \qquad /4/$$

or by means of succesive approximations /see [1]/. The first approximation $x_1(t)$ is the periodic solution of the equation

$$\ddot{x}_1 + \omega^2 x_1 = \varepsilon f(0,0,t). \qquad /5/$$

The second approximation $x_2(t)$ is the periodic solution of the equation

$$\ddot{x}_2 + \omega^2 x_2 = \varepsilon f(x_1, \dot{x}_1, t) \qquad /5^*/$$

and so on. For obtaining the required evaluation one must after Liapunov construct the functional equation / accordingly to the right-hand side of differential equation /3/ /

$$u = \varepsilon \Phi(u), \qquad /6/$$

defining u as a function of the parameter ε. The positive solution of this equation in the form of the series

$$u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \dots \qquad /6^*/$$

or obtained by means of succesive approximations

$$u_1 = \varepsilon \Phi(0), \; u_2 = \varepsilon \Phi(u_1), \dots \qquad /6^{**}/$$

must majorise the given periodic solution of /4/. i.e. for

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taking into account that the Fourier series for $x_k(t)$ should not contain the first harmonic. Then consider the equation

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = \varepsilon f(x_k, \dot{x}_k, t) \qquad /18/$$

for the function $x_{k+1}(t) = \xi_{k+1}\cos t + \eta_{k+1}\sin t + \bar{x}_{k+1}$ and separate the right-hand terms linear in $x_k, \dot{x}_k$. We obtain $f(x_k, \dot{x}_k, t) = p_{00} + \left[\frac{1}{2}\beta_1 + \beta_0\cos t - \gamma_0 \sin t + \frac{1}{2}\beta_1\cos 2t - \frac{1}{2}\delta_1 \sin 2t\right]\xi_k + \left[\frac{1}{2}\gamma_1 + \gamma_0\cos t + \beta_0 \sin t + \frac{1}{2}\delta_1\cos 2t + \frac{1}{2}\beta_1 \sin 2t\right]\eta_k + p_{10}\bar{x}_k + p_{01}\dot{\bar{x}}_k + \sum_{j+\delta=2}^{m} p_{j\delta} x_k^j \dot{x}_k^\delta$ /19/

and then the inequalities

$$|\xi_{k+1}| \le \alpha_{1c} + \frac{\varepsilon}{|\omega^2 - 1|}\left[|\beta_0\xi_k| + |\gamma_0\eta_k| + |p_{10}\bar{x}_k|_{c1} + |p_{01}\dot{\bar{x}}_k|_{c1} + f_1(x_k, \dot{x}_k)\right],$$

$$|\eta_{k+1}| \le \alpha_{1s} + \frac{\varepsilon}{|\omega^2 - 1|}\left[|\gamma_0\xi_k| + |\beta_0\eta_k| + |p_{10}\bar{x}_k|_{s1} + |p_{01}\dot{\bar{x}}_k|_{s1} + f_2(x_k, \dot{x}_k)\right], \qquad (20)$$

$$|\bar{x}_{k+1}(t)|_* \le \bar{\alpha}_1 + \varepsilon\left[\left(\frac{1}{2\omega^2} + \frac{1}{2|\omega^2 - 1|}\sqrt{\beta_1^2 + \gamma_1^2}\right)|\xi_k| + \left(\frac{1}{2\omega^2}|\delta_1| + \frac{1}{2|\omega^2 - 1|}\sqrt{\delta_1^2 + \beta_1^2}\right)|\eta_k| + \varepsilon\bar{\rho}_k\left[|p_{10}\bar{x}_k|_* + |p_{01}\dot{\bar{x}}_k|_* + f_3(x_k, \dot{x}_k)\right]\right],$$

where $\bar{\rho}_k = \max\frac{1}{|\omega^2 - \sigma^2|}$, $\sigma \neq 1$ and the functions f_1, f_2, f_3 are such that

$$f_1 + f_2 + f_3 = \sum_{j+\delta=2}^{m} a_{j\delta}|x_k|_*^j |\dot{x}_k|_*^\delta, \quad a_{j\delta} \ge |p_{j\delta}(t)|_* \qquad /21/$$

/We don't need to define each of the functions f_1, f_2, f_3 /. If $x_1(t)$ being represented

$$x_1(t) = \xi_1\cos t + \eta_1 \sin t + \bar{x}_1(t),$$

then the numbers $\alpha_{1c}, \alpha_{1s}, \bar{\alpha}_1$ are such that $\alpha_{1c} \ge |\xi_1|, \alpha_{1s} \ge |\eta_1|, \bar{\alpha}_1 \ge |\bar{x}_1(t)|_*$. We obtain three majors equations instead of ore /14/:

$$\begin{aligned} U_1 &= \alpha_{1c} + \frac{\varepsilon}{|\omega^2 - 1|}\left[|\beta_0|U_1 + |\gamma_0|U_2 + a\bar{u} + f_1\right], \\ U_2 &= \alpha_{1s} + \frac{\varepsilon}{|\omega^2 - 1|}\left[|\gamma_0|U_1 + |\beta_0|U_2 + a\bar{u} + f_2\right], \\ \bar{u} &= \bar{\alpha}_1 + \varepsilon\left[\beta U_1 + C U_2 + \varepsilon\bar{\rho}(a\bar{u} + f_3)\right], \end{aligned} \qquad /22/$$

where $a = |\beta_0| + |\beta_1| + \delta_1|\delta_0| + \delta_1|\delta_1|$, $\beta = \frac{1}{2\omega^2}|\beta_1| + \frac{1}{2|\omega^2 - 4|}\sqrt{\beta_1^2 + \gamma_1^2}$, $C = \frac{1}{2\omega^2}|\delta_1| + \frac{1}{2|\omega^2 - 4|}\sqrt{\beta_1^2 + \gamma_1^2}$, $\bar{\rho} = \frac{1}{|\omega^2 - n^2|}$, $n \neq 1$, $\delta_1 = n$, if $n \ge 2$ and $\delta_1 = 2$ if $n = 0$. We have also that

$$f_1 + f_2 + f_3 = \sum_{j+\delta=2}^{m} a_{j\delta} u^j (\sigma, v)^\delta = V(u)$$

with $u = U_1 + U_2 + \bar{u}$. These three equations can easily reduced to one equation for u. For this purpose one solves /22/ regarding f_1, f_2, f_3 as known quantities. The relation obtained will have the form

$$u = \alpha_1\alpha_{1c} + \alpha_2\alpha_{1s} + \alpha_3\bar{\alpha}_1 + \varepsilon\left[S_1 f_1 + S_2 f_2 + S_3 f_3\right], \qquad /23/$$

where the numbers $\alpha_1, \alpha_2, \ldots, S_3$ depend only on the coefficients of the linear part of /22/. Denote $\alpha_1\alpha_{1c} + \alpha_2\alpha_{1s} + \alpha_3\bar{\alpha}_1 = \alpha$ and let S to be largest among the numbers S_1, S_2, S_3. Then the equation /23/ can be replaced by following /the majoration strengthening/:

$$u = \alpha + \varepsilon S(f_1 + f_2 + f_3)$$

or

$$u = \alpha + \varepsilon S \cdot V(u). \qquad /24/$$

It is possible to separate from $x_k(t)$ not one but two or three harmonics. Then we obtain the major equation also of the form /24/ and the calculations prove this equation to be more precise than that constructed without separating the harmonics.

This way represents some combination between the method of Liapunov and that of A.Wintner [16-18] envisaging the majoration of the Fourier coefficients for investigated periodic solution.

Consider as the illustrative example the Duffing equation of the form

$$\ddot{z} + 2z = -\varepsilon z^3 + \cos t, \qquad /25/$$

taking for generating solution $z_0 = \cos t$. Putting $z = z_0 + x$

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we obtain obtain the equation

$$\ddot{x} + \varepsilon x = -\varepsilon \Psi_1(t) - \varepsilon(\rho_1 x + \rho_2 x^2 + x^3), \qquad /26/$$

where $\Psi(t) = \frac{5}{4}\cos t + \frac{1}{4}\cos 3t$, $\rho_1 = 3\cos^2 t$, $\rho_2 = 3\cos t$

If periodic solution of the last equation being determined in the form of a power series in ε

$$x = \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \ldots, \qquad /27/$$

then we obtain

$$\ddot{x}^{(1)} + x^{(1)} = -\Psi_1(t), \quad x^{(1)} = -0{,}750\cos t + 0{,}036\cos 3t,$$
$$\ddot{x}^{(2)} + x^{(2)} = -\rho_1 x^{(1)}, \quad x^{(2)} = 3{,}348\cos t - 0{,}153\cos 3t + 0{,}001\cos 5t,$$
$$\ddot{x}^{(3)} + x^{(3)} = -\rho_1 x^{(2)} - \rho_2 x^{(1)2} - x^{(1)3}, \quad x^{(3)} = 8{,}646\cos t + 0{,}375\cos 3t + 0{,}006\cos 5t \qquad /28/$$

the precision of coefficients not exceeding 0,001/. It is easy to see that the Fourier polinomial for all of the functions $x^{(k)}$, $k \geq 1$, doesn't contain the even harmonics. Therefore the multiplier ρ for the right-hand side of major equation is equal 1 and this equation can be represented as follows

$$u = \varepsilon\alpha_1 + \varepsilon(\alpha_1 u + \alpha_2 u^2 + u^3), \qquad /29/$$

where $\alpha_1 = |x^{(1)}(t)|_* = 0{,}750$, $\alpha_1 = |\rho_1(t)|_* = 3$, $\alpha_2 = |\rho_2(t)|_* = 3$.

After simple calculations we obtain the upper limit $\bar{\varepsilon}$ of the values ε, when this equation has a positive solution. It be equal 0,16 /with precision to 0,01/. Therefore for $\varepsilon \sim 0{,}16$ we assure the conver ence of the series /27/ and it is possible to determine the required estimations. We have calculated, for example, the quantities $\alpha_1 - \delta^{(1)}\varepsilon u - \varepsilon\alpha_1$, $\delta^{(2)}\varepsilon^2 u - \varepsilon^2\alpha_1\alpha_1$, $\delta_2 = u - \varepsilon\alpha_1 - \varepsilon(\alpha_1\varepsilon\alpha_1 + \alpha_2\varepsilon^2\alpha_1^2 + \varepsilon^3\alpha_1^3)$ giving the estimations for differences between $z_0(t)$, $z_1(t) = z_0(t) + \varepsilon x^{(1)}(t) = z_0(t) + x_1(t)$, $z_0(t) + \varepsilon x^{(1)}(t) + \varepsilon^2 x^{(2)}(t)$, $z_2(t) = z_0(t) + z_2(t)$ and the strong periodic solution of /25/ respectively. We received for $\varepsilon = 0{,}10$; $0{,}12$; $0{,}14$ follows values

ε	0,10	0,12	0,14
α	0,114	0,137	0,161
$\delta_1 = \delta^{(1)}$	0,030	0,069	0,121
$\delta^{(2)}$	0,016	0,035	0,075
δ_2	0,015	0,032	0,070

/30/

a/ We shall construct the improved major equation using the substitition $x = \varepsilon x^{(1)} + y$. The equation for y has the form

$$\ddot{y} + 2y = \Psi_1(t) - \varepsilon(\bar{\rho}_1 y + \bar{\rho}_2 y^2 + y^3), \qquad /31/$$

where $\Psi_1(t) = -\varepsilon^2\rho_1 x^{(1)} - \varepsilon^3 p_2 x^{(1)2} - \varepsilon^4 x^{(1)3}$,

$$\bar{\rho}_1 = 1{,}500 - 2{,}250\varepsilon + 0{,}846\varepsilon^2 + (1{,}500 - 2{,}214\varepsilon + 0{,}762\varepsilon^2)\cos 2t + (0{,}108\varepsilon - 0{,}081\varepsilon^2)\cos 4t + 0{,}002\varepsilon^2\cos 6t, \qquad /31'/$$

$$\bar{\rho}_2 = (3 - 2{,}250\varepsilon)\cos t + 0{,}107\varepsilon\cos 3t.$$

Further we determine the function $y_1(t)$ satisfying the equation $\ddot{y}_1 + 2y_1 = \Psi_1(t)$ and representing the difference between second x_2 and first x_1 approximations for /26/:

$$y_1(t) = (3{,}348\varepsilon^2 - 1{,}228\varepsilon^3 - 0{,}303\varepsilon^4)\cos t + (-0{,}153\varepsilon^2 + 0{,}049\varepsilon^3 - 0{,}011\varepsilon^4)\cos 3t + (0{,}001\varepsilon^2 - 0{,}002\varepsilon^3)\cos 5t.$$

The major equation we obtain as followe

$$u = \alpha_2 + \varepsilon(\bar{\alpha}_1 u + \bar{\alpha}_2 u^2 + u^3), \qquad /32/$$

where

$$\alpha_2 = |y_1|_* = 3{,}502\varepsilon^2 - 1{,}279\varepsilon^3 - 0{,}292\varepsilon^4, \quad \bar{\alpha}_2 = |\bar{\rho}_2|_* = 3 - 2{,}118\varepsilon,$$
$$\bar{\alpha}_1 = |\bar{\rho}_1|_* = 3\ 4{,}356\varepsilon + 1{,}529\varepsilon^2. \qquad /32'/$$

We have calculated the quantities u, $u - \alpha_2$ /coincident with the quantities δ_1, δ_2 of the table /30/ for $\varepsilon = 0{,}14$; $0{,}18$; $0{,}20$:

ε	0,14	0,18	0,20
δ_1	0,105	0,220	0,394
δ_2	0,040	0,115	0,265

/33/

For $\varepsilon = 0{,}22$ the equation /32/ hasn't a positive solution.

b/ We shall now separate from the solution of /31/ the first harmonic putting $y = \xi\cos t + \bar{y}$. Then we write

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down $f_1(t)$ in the form $p_{10}+p_{12}\cos 2t+p_{14}\cos 4t+p_{16}\cos 6t$

where p_{10}, p_{16} are defined by /31*/, and for the product $\bar{p}_1\psi_k=p_1(\xi_k\cos t+\dot{\psi}_k)$ we obtain the expression

$$\bar{p}_1\psi_k=\left[\left(p_{10}+\frac{1}{2}p_{12}\right)\xi_k+\left[(\bar{p}_1-p_{10})\psi_k\right]_1\right]\cos t+\left[\frac{1}{2}(p_{12}+p_{14})\cos 3t+\frac{1}{2}(p_{14}+p_{16})\cos 5t+\frac{1}{2}p_{16}\cos 7t\right]\xi_k+(\bar{p}_1\psi_k)_6, \qquad /34/$$

where $\left[(\bar{p}_1-p_{10})\psi_k\right]_1$ is the coefficient of cost in Fourier polimial for the product $(\bar{p}_1-p_{10})\psi_k$ and $(\bar{p}_1\psi_k)_6$ is the set of terms of Fourier polinomial for $\bar{p}_1\psi_k$ not possesing the first harmonic.

According to the expression /34/ we construct the linear part of major equations for the quantities u_1, $\bar{u}$ majorising ξ and $\bar{y}(t)$ respectively. We obtain

$$u_1=\alpha_{21}+\varepsilon\left[\left(a_0+\frac{1}{2}a_2\right)u_1+\frac{1}{2}(a_2+a_4+a_6)\bar{u}+f_2\right],$$
$$\bar{u}=\bar{\alpha}_2+\varepsilon\left[\left(\frac{1}{4}(a_2-a_4)+\frac{1}{46}(a_4+a_6)+\frac{1}{94}a_6\right)u_1+\frac{1}{7}a_1u+\frac{1}{7}f_2\right], \qquad /35/$$

where $\alpha_{21}=3-3{,}348\varepsilon^2-1{,}228\varepsilon^3-0{,}303\varepsilon^4$, $\bar{\alpha}_2=0{,}154\varepsilon^2-0{,}051\varepsilon^3+0{,}011\varepsilon^4$, $a_0=|p_{01}|=1{,}500-2{,}250\varepsilon+0{,}846\varepsilon^2$, $a_2=|p_{12}|=1{,}500-2{,}214\varepsilon+0{,}762\varepsilon^2$, $a_4=|p_{14}|=0{,}108\varepsilon-0{,}081\varepsilon^2$, $a_6=|p_{16}|=0{,}002\varepsilon^2$, $f_1+f_2=\bar{\alpha}_2u^2+u^3$ and $\bar{\alpha}_1,\bar{\alpha}_2$ are defined by /32*/. These equations are more precise than /32/. For example putting $\varepsilon=0{,}20$ we represent /32/ in the form

$$u=0{,}229+0{,}917u^2+0{,}356u^3. \qquad /36/$$

After substituting the values $\varepsilon=0{,}20$ in /35/ we obtain

$$u_1=0{,}124+0{,}326u_1+0{,}111\bar{u}+0{,}200f_1,$$
$$\bar{u}=0{,}006+0{,}016u_1+0{,}063\bar{u}+0{,}029f_2,$$

and

$$u_1+\bar{u}=u=0{,}195+0{,}303f_1+0{,}0356f_2.$$

The last equation we replace by the following

$$u=0{,}195+0{,}303(f_1+f_2)$$

$$u=0{,}195+0{,}119u^2+0{,}305u^3. \qquad /37/$$

Comparing /36/ and /37/ we confirm said above. After solving /37/ we obtain $\delta_1=0{,}247$ /instead of $\delta_1=0{,}264$ after /36/ and $\delta_2=u\ \alpha_2=0{,}118$/ instead of $\delta_2=0{,}265$/. After calculations of δ_1,δ_2 also for $\varepsilon=0{,}14$; $0{,}18$; $0{,}22$ we obtain the following table

ε	0,14	0,18	0,20	0,22
δ_1	0,093	0,176	0,247	0,407
δ_2	0,028	0,070	0,118	0,252

/38/

For $\varepsilon=0{,}24$ the equation /35/ hasn't a positive solution.

From the numbers of table /38/ it follows that the error of the first approximation $x_1(t)$ doesn't exceed 10% of generating amplitude for $\varepsilon\leq 0{,}14$ and the error of the second approximation doesn't exceed 10-12% for $\varepsilon\leq 0{,}20$.

Notice that according to /2/ $\varepsilon^3\omega\rho/x^{(0)}(t)|=0{,}70$ for $\varepsilon=0{,}20$. Comparing this number with $\delta_2=0{,}118$ we can conclude that the numbers /36/ are not far from the real values of the errors considered.

§ 2. In the case of an autonomous system and also of the oscillations near the resonance point the principle of majoration is the same. But the major equations and others formulae are more complicated. This follows from necessity to define and then to majorise according to the periodicity conditions for every approximation the amplitude of generating solution and also the period of the periodic solution investigated /in the case of an autonomous system/. It doesn't seem worth while to write down the major equation and other relations in general form. Therefore I restrict myself to the examples from which it distinctly follows how to construct these relations.

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a/ Consider again the Duffing equation of the form

$$\ddot{z} + z = -\varepsilon z^3. \qquad /39/$$

Changing after Liapunov the independent variable by the formula $t=\theta\sqrt{1+h}$, where h is constant, we obtain

$$z'' + z = -h z - \varepsilon(1+h) z^3 \quad \left(z' = \frac{dz}{d\theta}\right) \qquad /40/$$

After substitution $z = z_0 + x$ with $z_0 = a\cos\theta$ we obtain the equation for x

$$x'' + x = \Psi(\theta, h, \varepsilon) - hx - \varepsilon(1+h)(\rho_1 x + \rho_2 x^2 + x^3), \qquad /40/$$

where $\Psi(\theta,h,\varepsilon) = -\left[h + \frac{3}{4}\varepsilon(1+h)\right]a\cos\theta - \frac{1}{4}\varepsilon(1+h)a\cos 3\theta$, $\rho_1 = 3a\cos\theta$, $\rho_2 = 3a\cos\theta$.

The first approximation satisfies the equation $x_1'' + x_1 = \Psi(\theta, h_1, \varepsilon)$, where the first approximation h_1 for the constant h is to be found from the condition that right-hand side of this equation doesn't contain the first harmonic. We obtain $1+h_1 = 1 + \frac{3}{4}\varepsilon a^2$, $x_1(t) = \frac{1}{32}\varepsilon a(1+h_1)\cos 3\theta$.

For the kth approximation we have the equation

$$x_k'' + x_k = \Psi(\theta, h, \varepsilon) - h_k x_{k-1} - \varepsilon(1+h_{k-1})(\rho_1 x_{k-1} + \rho_2 x_{k-1}^2 + x_{k-1}^3) \qquad /41/$$

and the first harmonic in the Fourier polinomial for the right-hand side of /41/ is absent only, if

$$1 + h_k = \frac{1}{1 + \frac{3}{4}\varepsilon + \varepsilon\left[(\rho_1 x_{k-1})_1 + (\rho_2 x_{k-1}^2)_1 + (x_{k-1}^3)_1\right]}, \qquad /41^*/$$

where by $(\varphi)_1$ we denote the coefficient of $\cos\theta$ in the Fourier polinomial for each function $\varphi(t)$.

It is easy to see that Fourier polinomial for each $x_k(t)$, $k \geq 1$, contains only the odd harmonic beginning with the third. Therefore for arbitrary k

$$|x_k(t)|_* \leq \frac{1}{8}|h_k x_{k-1}|_* + \frac{1}{8}\varepsilon\left|1 + h_{k-1}\left[\frac{3}{4}\rho_1 x_{k-1}|_* + \rho_2 x_{k-1}^2|_* + |x_{k-1}^3|_*\right]\right. \qquad /42/$$

In every case $|\rho_1 x_{k-1}|_* = 3|x_{k-1}|_*$, $|\rho_2 x_{k-1}^2|_* = 3|x_{k-1}^2|_* = 3|x_{k-1}|_*^2$, so /41*/ implies

$$|1+h_k| \leq \frac{1}{1 + \frac{3}{4}\varepsilon - \frac{9}{4}\varepsilon|x_{k-1}|_* - \frac{3}{4}\varepsilon|x_{k-1}|_*^2}, \quad |h_k| \leq \varepsilon\frac{\frac{3}{4}\varepsilon|x_{k-1}|_* + \frac{3}{4}\varepsilon|x_{k-1}|_*^2 + \frac{1}{4}\varepsilon|x_{k-1}|_*^3}{1 + \frac{3}{4}\varepsilon - \frac{9}{4}\varepsilon|x_{k-1}|_* - \frac{3}{4}\varepsilon|x_{k-1}|_*^2}$$

We can immediatly write down in accordance with /42,43/ the major equation for u

$$u = \frac{1}{8}Vu + \frac{1}{8}\varepsilon W\left(\frac{1}{4} + a_1 u + a_2 u^2 + u^3\right), \qquad /44/$$

where $a_1 = |\rho_1|_* = 3$, $a_2 = |\rho_2|_* = 3$,

$$W = \frac{1}{1 + \varepsilon\left(\frac{3}{4} - \frac{3}{4}u - u^2\right)}, \quad V = \varepsilon\,\frac{\frac{3}{4} + \frac{3}{4}u + \frac{3}{2}u^2 + u^3}{1 + \varepsilon\left(\frac{3}{4} - \frac{3}{4}u - u^2\right)}.$$

The expressions for W and V being substituted in /44/ we obtain after simple transformations

$$u = \mathcal{M}\left(\frac{1}{32} + \frac{39}{32}u^2 + \frac{5}{16}u^3 + \frac{9}{8}u^4\right), \qquad /45/$$

where $\mathcal{M} = \frac{\varepsilon}{1 + \frac{9}{32}\varepsilon}$.

The positive solution of this equation exists provided $\varepsilon \leq 2{,}49$ /with precision to 0.01/ or $\varepsilon \leq 0{,}29$.

By using the method of separating the individual harmonics we can improve the major equation. If we put

$$x = \xi_1\cos\theta + \xi_2\cos 3\theta + \xi_3\cos 5\theta + \ldots, \qquad /46/$$

then

$$\rho_1 x = \frac{3}{4}\xi_1\cos\theta + \left(\frac{3}{2}\xi_1 + \frac{3}{4}\xi_2\right)\cos 3\theta + \left(\frac{3}{4}\xi_1 + \frac{3}{2}\xi_2 + \frac{3}{4}\xi_3\right)\cos 5\theta + \frac{3}{4}\rho_2\cos 7\theta + \frac{3}{4}\rho(1+\cos 2\theta)\bar{x},$$

where the product $(1+\cos 2\theta)\bar{x}$ doesn't contain the first and third harmonics. In accordance with this expression and with the right-hand side of /41/ we construct the major equation for the quantities $u_1, u_2, \bar{u}$ majorising $|\xi_1|$, $|\xi_2|$ and $|x(\theta)|_*$ respectively:

$$\begin{aligned} u_1 &= \frac{1}{8}Vu + \frac{1}{8}\varepsilon W\left[\frac{1}{4} + \frac{3}{2}u_1 + \frac{3}{4}u_2 + f_1\right], \\ u_2 &= \frac{1}{25}Vu + \frac{1}{25}\varepsilon W\left[\frac{3}{4}u_1 + \frac{5}{2}u_2 + \frac{3}{4}u + f_2\right], \\ \bar{u} &= \frac{1}{48}Vu + \frac{1}{48}\varepsilon W\left[\frac{3}{4}u_2 + \frac{9}{4}u + f_3\right], \end{aligned} \qquad /47/$$

where

$$f_1 + f_2 + f_3 = f = 3u^2 + u^3, \quad u = u_1 + u_2 + \bar{u},$$

$$W = \frac{1}{1 + \varepsilon\left(\frac{3}{4} - \frac{3}{4}u_1 - u^2\right)}. \qquad /47^*/$$

We can reduce these three equations to a single one that coincides with /45/ except that ω is defined by the expres-

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We put $t=\theta(1+h)$, $z_0=M\cos\theta$, $z=z_0+x$ and obtain the equation $x''+x=\Psi(\theta,h,M)-(2h+h^2)x+\varepsilon(1+h)[M^2\rho_1 x+$

$+(1-M^2\rho_2)x'+M\rho_3 xx'+M\rho_4 x^2+x^3x']$, /52/

where $\Psi(\theta,h,M)=-(2h+h^2)M\cos\theta+\varepsilon(1+h)\left[\left(M+\frac{1}{4}M^3\right)\sin\theta+\frac{1}{4}M^3\sin 3\theta\right]$,

$\rho_1=\sin 2\theta$, $\rho_2=\cos^2\theta$, $\rho_3=-2\cos\theta$, $\rho_4=\sin\theta$. /52*/

In the first approximation $x_1''+x_1=\Psi(\theta,h_1,M_1)$, $h_1=0$, $-M_1+\frac{1}{4}M_1^3=0$, $M_1^2=4$, $M_1=2$ /the value $M_1=-2$ corresponds replacing θ by $\theta+\pi$/ and $x_1=-\frac{1}{4}\varepsilon\sin 3\theta$.

The kth approximation $x_k(t)$ satisfies the equation

$x_k''+x_k=\Psi(\theta,h_k,M_k)-(2h_k+h_k^2)x_{k-1}+\varepsilon(1+h_k)[M_k^2\rho_1 x_{k-1}+$

$+(1-M_k^2\rho_2)x_{k-1}'+M_k\rho_3 x_{k-1}'x_{k-1}+M_k\rho_4 x_{k-1}^2-x_{k-1}^2x_{k-1}']$ /53/

and the periodicity conditions imply the relations

$-(2h_k+h_k^2)M_k+\varepsilon(1+h_k)\left[M_k^2(\rho_1 x_{k-1}-\rho_2 x_{k-1}')_{c1}+M_k(\rho_3 x_{k-1}x_{k-1}'+\right.$

$\left.+\rho_4 x_{k-1}^2)_{c1}-(x_{k-1}^2x_{k-1}')_{c1}\right]=0$. /54/

$-M_k+\frac{1}{4}M_k^3+M_k^2(\rho_1 x_{k-1}-\rho_2 x_{k-1}')_{s1}+M_k(\rho_3 x_{k-1}x_{k-1}'+\rho_4 x_{k-1}^2)_{s1}-(x_{k-1}^2x_{k-1}')_{s1}=0$,

where by $(\varphi)_{c1}$, $(\varphi)_{s1}$ are denoted the Fourier coefficients of $\cos\theta$ and $\sin\theta$ for each function $\varphi(t)$ respectively. Putting $M_k=2+\nu_k$ we obtain from /54/:

$h_k=\frac{1}{2}\varepsilon\frac{Q_{kc}}{M_k}+\sqrt{1+\frac{1}{4}\varepsilon^2\left(\frac{Q_{ks}}{M_k}\right)^2}-1$, $\nu_k=\frac{1}{2}\left(-\frac{3}{8}\nu_k^2-\frac{1}{4}\nu_k^3-Q_{ks}\right)$, /55/

where

$Q_{kc}(M_k)=M_k^2(\rho_1 x_{k-1}-\rho_2 x_{k-1}')_{c1}+M_k(\rho_3 x_{k-1}x_{k-1}'+\rho_4 x_{k-1}^2)_{c1}-(x_{k-1}^2x_{k-1}')_{c1}$,

$Q_{ks}(M_k)=M_k^2(\rho_1 x_{k-1}-\rho_2 x_{k-1}')_{s1}+M_k(\rho_3 x_{k-1}x_{k-1}'+\rho_4 x_{k-1}^2)_{s1}-(x_{k-1}^2x_{k-1}')_{s1}$.

From /53/ we obtain the inequalities $|x_k(t)|_*\le\Phi$, $|x_k'(t)|_*\le 3\Phi$, where

$\Phi=\frac{1}{32}\varepsilon\left|(1+h_k)M_k^3\right|+\frac{1}{8}|2h_k+h_k^2|\cdot|x_{k-1}|_*+\frac{1}{8}\varepsilon|1+h_k|\left[|M_k^2|x_{k-1}|_*+\right.$

$\left.+|1-M_k^2|\left(|x_{k-1}'|_*+|M_k|\left(|x_{k-1}x_{k-1}'|_*+|x_{k-1}^2|_*\right)+|x_{k-1}^2x_{k-1}'|_*\right)\right]$.

These inequalities and formula /55/ imply the major equations of the form

$u=\frac{1}{32}\varepsilon(1+v)w^3+\frac{1}{8}\varepsilon(2v+v^2)u+\frac{1}{8}\varepsilon(1+v)\left[(4w^2-3)u+7wu^2+3u^3\right]$,

$v=\frac{1}{2}\varepsilon\frac{\mathcal{F}(u,w)}{2-\bar{w}}$, $1+\sqrt{1+\frac{1}{4}\varepsilon^2\frac{\mathcal{F}^2(u,w)}{(2-\bar{w})^2}}$, $\bar{w}=\frac{3}{4}\bar{w}^2+\frac{1}{8}\bar{w}^3+\frac{1}{2}\mathcal{F}(u,w)$, /56/

where $w=2+\bar{w}$, $\mathcal{F}(u,w)=\frac{5}{4}uw^2+\frac{7}{2}wu^2+3u^3$ and the quantities $|x(t)|_*$, $|h|$, $|M|$, $|M-M_1|=|M-2|$ are majorised by $u,v,w,\bar{w}$ respectively.

The major equations prove to be much more complicated than in the preceding case but it is still possible to receive a notion concerning the error of the first approximation. For example for $\varepsilon=0{,}1$ we obtain $\bar{w}<0{,}19$, $v<0{,}0184$, $u<0{,}046$. The analycity of solution is assured for $\varepsilon\le 0{,}1$. It is known that the periodic solution of /51/ exists for the arbitrary values $\varepsilon>0$. But we can not assure the convergence of the successive approximations for arbitrary $\varepsilon>0$ as it was in the preceding case of Duffing equation. Presumedly such convergence doesn't take place.

At the same time it is possible to affirm the analiticity of the discussed periodic solution being preserved only in a limited region. In fact the periodic solution being analitic in ε for $\varepsilon<\bar{\varepsilon}$ could have been obtained by means of the succesive approximations and each one of them will be analitic in ε. But if beginning from some k the approximations x_k, x_{k+1}, ... fail to be analitic in ε for $\varepsilon>\varepsilon_1$, then the analitic solution in ε for $\varepsilon>\varepsilon_1$ doesn't exist. The value of ε_1 we can estimate immediately after calculations of some first approximations. We obtain for example the quantities $M_2(\varepsilon)$, $h_2(\varepsilon)$ fail to be analitic for $\varepsilon>\sqrt{32}$ and $\varepsilon>4$ respectively, the quantity $M_3(\varepsilon)$ fails to be analitic at least for $\varepsilon\ge 2$. I didn't carry out the further calculations and the improving of the major functional equa-

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tions by the ways pointed out above.

b/ Consider the equation

$$\ddot{x}+x=-R(ax+bx^3)-[a\sqrt{\frac{L}{C}}+R\sqrt{\frac{C}{L}}+3b\sqrt{\frac{L}{C}}x^2]\dot{x} \qquad /57/$$

describing the oscillations of one radio systems and discussed in the paper [20], where the following values of the constants are given: $a=10^{\gamma}$, $b=1{,}74\cdot10^{-6}$, $L=2\cdot10^{-4}$

	I	2	3	4	5
R	3,74I	4,378	5,227	8,377	I5,325
$C\cdot10^{11}$	45	30	20	I0	5

The author puts $a=\mu\bar{a}$, $b=\mu\bar{b}$, $\frac{RC}{L}=\mu\left(\frac{R\bar{C}}{L}\right)$ and determines the periodic solution in the form of a power series in μ. Comparing the first approximation for "generating" amplitude

$$M_1=\sqrt{-\frac{4}{3b}\left(\frac{RC}{L}+a\right)}$$

with the experimental measurements of this amplitude M the author obtained the data:

	I	2	3	4	5
M_1	8,58	8,55	8,53	8,57	8,57
M	8,55	8,62	8,75	8,80	8,80

. /58/

The relative error $\frac{\delta M}{M}=\frac{M_1-M}{M}\,100$ of the first approximation is following

	I	2	3	4	5
$\frac{\delta M}{M}\%$	0,35	0,8I	2,5I	2,6I	2,6I

/58*/

Notice that the way of determination of periodic solution of /57/ in this paper is rather complicated. The calculations could be simplified. In fact, rewrite down the equation /57/ in the form

$$\ddot{x}+x=\alpha x-\gamma x^3+(\beta-\delta x^2)\dot{x},$$

where

$$\alpha=-Ra,\ \gamma=Rb,\ \beta=-a\sqrt{\frac{L}{C}}-R\sqrt{\frac{C}{L}},\ \delta=3b\sqrt{\frac{L}{C}}.$$

We change the independent variable t and the function x by formulae

$$t\sqrt{1-\alpha}=\tau,\quad x=y\sqrt{\frac{\beta}{\delta}}$$

Then the equation for y takes simple form

$$\frac{d^2y}{d\tau^2}+y=\varepsilon\left[-qy^3+(1-y^2)\frac{dy}{d\tau}\right], \qquad /59/$$

where $\varepsilon=\frac{\beta}{\sqrt{1-\alpha}}$, $q=\frac{\gamma}{\delta\sqrt{1-\alpha}}$ and values of ε, q are following

	I	2	3	4	5
ε	0,06II8	0,07644	0,09502	0,I36I	0,I938
q	0,001875	0,00I792	0,00I747	0,00I982	0,002574 .

This equation differs from Van-der Pol equation /5I/ only by the term qy^3, the number q being very small. The parameter ε changes from 0,06II8 up to 0,I9384. The major equation constructed for /59/ will differ from /56/ only by terms containing small coefficient q. The estimations obtained after /56/ remain without any essential change. For ε = 0,I we have obtained above that the error δM of the first approximation of "generating" amplitude M_1 doesn't exceed 0,I9, i.e. the relative error doesn't exceed 9,5%. The value ε =0,I corresponds to the third column of the tables /58,58*/. The measured relative error is equal about 2,5I%. The obtained estimation 9,5% could be recognised as satisfactory.

§ 3. The case of oscillations near the resonance point in regard to the construction of major relations and their structure is rather analogous to that of an autonomous system. Therefore I restrict myself to one example. Consider the Duffing equation

$$\ddot{z}+z=-\varepsilon z^3+\varepsilon\gamma\cos\omega t, \qquad /60/$$

where γ is a constant and ω is near to 1. We put

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$\omega=\frac{1}{\sqrt{1+h}}$, $\omega t=\theta$. Then the equation takes the form

$$z''+z=-hz-\varepsilon(1+h)(z^3-\gamma\cos\theta). \qquad /61/$$

After putting the "generating" solution to be equal $z_0=A\cos\theta$ we obtain after substitution $z=z_0+\beta x$ the equation

$$x''+x=\Psi(\theta,h,\varepsilon_1)-hx-\varepsilon_1(1+h)(\rho_1 x+\rho_2 x^2+x^3), \qquad /62/$$

where $\varepsilon_1=\varepsilon A^2$, $\rho_1=3\cos^2\theta$, $\rho_2=3\cos\theta$,
$\Psi(\theta,h,\varepsilon_1)=-[h+\varepsilon_1(1+h)](\cos\theta-\frac{1}{4}\varepsilon_1(1+h)\cos 3\theta$, $\bar{\gamma}=\frac{\gamma}{A^3}$. It is easy to see that the last equation doesn't differs from /40/ except that function Ψ has coefficient of $\cos\theta$ equal $-h-\varepsilon_1(1+h)(\frac{3}{4}-\bar{\gamma})$ instead of $-h-\frac{3}{4}\varepsilon(1+h)$. Therefore all the various relations obtained above will be true after replacing in /43, 44/ $\frac{3}{4}\varepsilon$ by $\varepsilon_1(\frac{3}{4}-\bar{\gamma})$ /in denominator/. We obtain, consequently, following major equation

$$u=\mu\left(\frac{1}{32}+\frac{39}{32}u^2+\frac{5}{16}u^3+\frac{9}{8}u^4\right), \qquad /63/$$

where approximately $\mu=\frac{\varepsilon_1}{1+(\frac{15}{32}-\frac{7}{8}\bar{\gamma})\varepsilon_1}$. For the frequency we obtain in accordance with /41*/ the relation

$$\omega^2=1+\varepsilon_1(\tfrac{3}{4}-\bar{\gamma})+\varepsilon_1\Delta, \qquad /64/$$

where $|\Delta|\le(\frac{3}{4}u+\frac{3}{2}u^2+u^3)$. Returning from $\varepsilon_1,\bar{\gamma}$ to ε, γ rewrite down /64/ in the form

$$\omega^2=1+\tfrac{3}{4}\varepsilon A^2-\frac{\varepsilon\gamma}{A}+\varepsilon A^2\Delta. \qquad /64^*/$$

This relation represents for $\Delta=0$ the known resonance curve in the first approximation. The quantity $\varepsilon A^2\Delta$ is the error of this curve on account of further approximations. The major equation allows to estimate errors of any approximation and the inequality for Δ allows to estimate the error of resonance curve of first approximation. It is possible also to obtain the estimations of the second or third approximations.

If $\frac{15}{32}-\frac{7}{8}\frac{\gamma}{A^3}>\frac{1}{2,49}$

then the successive approximation converge for arbitrary $\varepsilon>0$. This condition is fullfilled for the all branch of resonance curve corresponding to negative values of A and for the other part of branch where $\omega^2\gg 1$. For $\omega^2\ll 1$ the convergence becomes worse that is natural because of growing of multiplier $1+h$ of right-hand side of /61/.

§ 4. In conclusion I should like to notice that the methods described above could be useful for the estimations of the approximations obtained in accordance with Krylov-Bogoliubov's averaging method. In fact, consider the equation

$$\ddot{x}+x=\varepsilon f(x,\dot{x}) \qquad /65/$$

The kth approximation of periodic solution one obtains after this method /see[2]/ in the form

$$x=a\cos\psi+\varepsilon u_1(a,\psi)+\cdots+\varepsilon^k u_k(a,\psi), \qquad /66/$$

where $u_1(a,\psi),\dots,u_k(a,\psi)$ are periodic in ψ of period 2π, $\psi=\omega_k t$, $\omega_k=1+\varepsilon\beta_1(a)+\cdots+\varepsilon^k\beta_k(a)$, the quantity a satisfies the equation $\frac{da}{dt}=\varepsilon A_1(a)+\cdots+\varepsilon^k A_k(a)$, and $A_1(a),\dots,\beta_k(a)$ are the functions of a such that /66/ satisfies the equation /65/ with precision to quantities of order ε^{k+1}. As the obtaining of functions $u_1,\dots,u_k$ becomes more and more complicated when the order k grows, one calculates usually only the first or the second approximation. Therefore it is essential not the strong convergence of considered approximations for $k\to\infty$ but sufficient closeness of the first or second approximation to the strong solution.

Suppose that have obtained, for example, the periodic first approximation possesing the frequency $\omega_1=1+\varepsilon C$. Denote it by $x(\omega_1 t)$. Then we change independent variable t after formula $\theta=\bar{\omega}_1 t$, where $\bar{\omega}_1=\frac{1}{1-\frac{\varepsilon C}{1+\varepsilon C}+h}$,

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and $\bar{h}$ is an arbitrary constant. /For $\bar{h}=0$ we have $\bar{\omega}_1=\omega_1$ /. Denoting

$$\bar{h}=-\frac{\varepsilon c}{1+\varepsilon c}+h$$

we write down the equation for $x(\theta)$ in the form

$$x''(\theta)+x(\theta)=-(2\bar{h}+\bar{h}^2)x(\theta)+\varepsilon(1+\bar{h})^2 f\left(x(\theta),\frac{x'(\theta)}{1+\bar{h}}\right). \qquad /67/$$

We put further $x(\theta)=x_1(\theta)+y(\theta)$, then obtain following equation for $y(\theta)$: $y''(\theta)+y(\theta)=-(2\bar{h}+\bar{h}^2)y(\theta)+$

$$+\varepsilon(1+\bar{h})^2 f\left[x_1(\theta)+y(\theta),\frac{x_1'(\theta)+y'(\theta)}{1+\bar{h}}\right]-x_1''(\theta)-(2\bar{h}+\bar{h}^2)x_1(\theta). \qquad /68/$$

This equation doesn't differ essentially from those of the cases of autonomous sistems or of oscillations near the resonance point. In a way considered above one could construct for /68/ the major equations and then estimate $|y(\theta)|$ and the constant h , determining the error of frequency $\omega_1=$ /Certainly, if periodic solution of /65/ and, consequently, of /67/ exists.

In a case of systems with outer periodic forces it is possible to use the estimations considered above, if we have oscillations near the resonance point. Then also we could obtain the equation of the form /67/. But in the case of oscillations far of resonance point the averaging method leads in general to almost periodic solutions and then the method of estimations considered above fails. True, we have a good exception. Let the equations being reduced to standard form

$$\frac{dx_j}{dt}=\varepsilon f_j(x_1,\ldots,x_n,t), \quad j=1,2,\ldots,n, \qquad /69/$$

where f_j – is almost periodic in t , to possess such stationary solution $x_1^{(0)},x_2^{(0)},\ldots,x_n^{(0)}$ that matrix for corresponding linear functional equation has no one eigenvalue with real part equal to zero. Then we put $x_j=x_j^{(0)}+x_j^{(1)}+y_j$, where $x_j^{(1)}$ is, for example, the first approximation obtained by means of averaging method, and write down quasy-linear equations for y_j , $j=1,2,\ldots,n$ /it is possible that transformations of Krylov-Bogoliubov would applied for obtaining the terms of lower order in ε independent of t /. The matrix /constant/ of linear part of these equations has one eigenvalues with real part equal to zero. For these equations are applicable the above method of estimations.

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At present time there are worked out various methods of precise determination of periodic regimes in piece-wise automatic control systems, i.e. those described by different linear differential equations [1-4] during subsequent intervals of time. Calculations connected with the application of methods noted are immense though can be fulfilled successfully with the help of computors. In all practic calculations it is usually enough to limit the quantity of harmonics, by a few ones, of Fourier series expansion, for the investigated periodic regime. In particular the widely spread method of Harmonic balance is based on assumption that periodic regime is similar by its form to that of the sinusoidal one. The harmonic balance method due to its simplicity has found a wide use in calculations of non-linear automatic control systems [5]. However, owing to preconditions laid in the base, this method is far from being an exact one for all cases and sometimes gives wrong results. Really, suppose physical system with such properties that its periodical regimes have some near harmonics. In this case it would be wrong to expect that harmonic balance method in general will give correct results even from the quality point of view. All that proves the necessity in working out methods of more accuracy and at the same time simple enough to allow determination of periodic regimes. On the other hand a new problem of approximate methods verification is arising. It means that it is necessary to determine some additional conditions which would allow to assert that results of the approximate method concerned are similar to those of accurate method from the quality point of view. In this report some results concerning the problems noted above and having its source directly in the article [4] are given.

1. To simplify the question let us consider some plain non-linear system with lumped parameters which differs from a linear system only by presence of a non-linear link. These simplifications do not mean however that considerations given below could not transferred to cases of more

general character.

Consider the following system of equations

$$X = W(p)y = \frac{M(p)}{D(p)}\,y$$
$$y = f(x) + \varphi(t) \qquad (1)$$

where

$W(p)$ – linear part transfer function $\frac{M(p)}{D(p)}$ and $D(p)$ – polynomials; the power of $M(\lambda)$ is lower than that of $D(\lambda)$

$f(x)$ – some non-linear function (for instance, a piece-wise continuous function)

$\varphi(t)$ – a given piece-wise continuous function of time with a period T.

Let us suggest that equation

$$\Delta(\lambda) = 0 \qquad (2)$$

to have no multiple roots and complex roots with zero real parts.

It will be assumed that system (1) has a periodic solution of period T

$$y = y^T(t), \quad X = X^T(t) \qquad (3)$$

Let us introduce the periodic function

$$\Phi(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \frac{M(\gamma_k)}{D(\gamma_k)}\, e^{\gamma_k t}, \qquad (4)$$

where

$$\gamma_k = \frac{2\pi i}{T}k, \quad i = \sqrt{-1} \qquad (5)$$

In the interval $0 < \tau < T$ the function (4) may be written in a finite form

$$\Phi(\tau) = -\sum_{\rho=1}^{n} \frac{M(\lambda_\rho)}{D'(\lambda_\rho)} \cdot \frac{e^{\lambda_\rho \tau}}{e^{\lambda_\rho T} - 1}, \quad 0 < \tau < T \qquad (6)$$

where λ_ρ $(\rho = 1, 2, \ldots n)$ – roots of the equation (2).

If a periodic input of period T*)

$$F(t) = F(t+T)$$

acts on the linear part of the system the settled response of the linear part will be written in a form

$$X^T(t) = \int_0^T \Phi(t-\tau)F(\tau)\,d\tau \qquad (7)$$

*)In case of sustained vibrations $\varphi(t) = 0$ and T is unknown.

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Suppose that a periodic regime (3) exists in the system defined by equations (1). In this case it is possible to recognize that

$$F(t) = f(x(t)) + \varphi(t) \qquad (8)$$

Substituting (8) in equation (7) one comes to a non-linear integral equation of Hammerstein 6 type

$$x(t) = \int_0^\tau \Phi(t-\xi) f(x(\xi)) + \int_0^\tau \Phi(t-\xi)\varphi(\xi)\,d\xi \qquad (9)$$

In all practic calculations it is quite possible to limit a number of terms of Fourier series (4). Let $p_1, p_2 \dots p_m$ be numbers of harmonics playing the main role in (4).

Let

$$\tilde{\Phi}(t) = \frac{1}{T}\sum_{k=-m}^{m} \frac{M(\psi_{p_k})}{D(\psi_{p_k})} e^{\psi_{p_k} t}, \qquad (10)$$

and consequently using (10) instead of (4), we shall have an approximate integral equation

$$\tilde{x}(t) = \int_0^\tau \tilde{\Phi}(t-\xi) f(\tilde{x}(\xi))\,d\xi + \int_0^\tau \tilde{\Phi}(t-\xi)\varphi(\xi)\,d\xi \qquad (11)$$

It is not difficult to show that a solution of equation (11) (if one exists) has harmonics only of p_i $i = 1, 2 \dots m$ numbers.

Really, substituting (10) in (11) and integrating by , we shall have

$$\tilde{x}(t) = \sum_{k=-m}^{m} \frac{M(\psi_{p_k})}{D(\psi_{p_k})} y_k e^{\psi_{p_k} t} = \sum_{k=-m}^{m} x_{p_k} e^{\psi_{p_k} t}, \qquad (12)$$

where

$$y_k = \frac{1}{T}\int_0^T [f(x(\xi)) + \varphi(\xi)] e^{-\psi_{p_k}\xi}\,d\xi \qquad (13)$$

some constants to be determined.

Negleot of unknown harmonics of a periodic regime is obviously equivalent to be disregarded the same harmonics in Fourier series (4). Thus, fulfilment of condition

$$\left|\tilde{\Phi}(t)\right| \gg \left|\sum_{\substack{k=-m \\ k\neq p_m}}^{m} \frac{M(\psi_k)}{D(\psi_k)} e^{\psi_k t}\right| \qquad 0 < t < T \qquad (14)$$

may be considered as a practical criterion of the unknown periodic solution to be actually found in a form (12).

In particular for applicability of harmonic balance it should be

$$\left|\frac{M(0)}{D(0)} + \frac{M(\psi_1)}{D(\psi_1)} e^{\psi_1 t} + \frac{M(\psi_{-1})}{D(\psi_{-1})} e^{\psi_{-1} t}\right| \gg \left|\sum_{k\neq p_m} \frac{M(\psi_k)}{D(\psi_k)} e^{\psi_k t}\right| \qquad (15)$$

Conditions (14) and (15) differ from well-known filter-conditions which have for a general case a form

$$\left|\frac{M(\psi_{p_k})}{D(\psi_{p_k})}\right| \gg \left|\frac{M(\psi_z)}{D(\psi_z)}\right| \qquad z \neq p_k, \qquad (16)$$

and include them.

There is no difficulty in proving that the fulfilment of (16) in general case does not guarantee the small value of Fourier series sum in the right part of (14). That is why we shall call conditions (14) and (15) specified filter conditions. The check-up of filter conditions in the form of (14) is difficult and in practical calculations it should be substituted by a following condition

$$\left|\sum_{p=1}^{r} \frac{M(\lambda_p)}{D(\lambda_p)} \cdot \frac{e^{\lambda_p t}}{e^{\lambda_p T} - 1} + \tilde{\Phi}_n(t)\right| < \varepsilon \qquad 0 < t < T \qquad (17)$$

where $\varepsilon > 0$ - a beforehand given number, the value of which is conditioned by the precision of calculations concerned. In order to determine coefficients x_{p_k} from (12), equation (12) should be substituted in (11). Then equating the coefficients at the same harmonics we shall have the system of equations

$$x_{p_k} = \frac{1}{T}\frac{M(\psi_{p_k})}{D(\psi_{p_k})}\int_0^T [f(\sum_{k=-m}^{m} x_{p_k} e^{\psi_{p_k}\xi}) + \varphi(t)] e^{-\psi_{p_k}\xi}\,d\xi \qquad (18)$$

However it is difficult to recommend some methods of more general character to solve the system (18). In particular it may happen that a solution does not exist. Different numerical methods of solving the equations like (18) are given in works [7-8].

2. The system of equations (18) can be considerably simplified if the funotion $f(x)$ is of piece-wise linear type. In this case the system of equations (18) occures to be a linear one relatively x_{p_k} coefficients of Fourier series.

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Let us consider that the type of a periodic solution (i.e. sequence of a piece-wise characteristic parts passage) is given.

Let non-linear characteristic link equations which are passed subsequently during a period be

$$f(x) = a_i x + b_i$$

where a_i, b_i – given numbers, some of them may be equal.

It is possible to write

$$f(X^T(t)) = a_i X^T(t) + b_i \quad \text{при } t_{i-1} < t < t_i \quad (i = 1, 2 \ldots m) \tag{19}$$

where m – general number of passages from one part to another during a period;

$$t_0 < t_1 \ldots < t_m \qquad (t_m = T) \tag{20}$$

are unknown times to be determined.

Let us consider stepped periodic functions

$$A(t) = A(t+T), \quad A(t) = a_i \quad \text{при } t_{i-1} < t < t_i \tag{21}$$

$$B(t) = B(t+T), \quad B(t) = b_i \quad \text{при } t_{i-1} < t < t_i$$

From (20) it follows

$$A(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\psi_k t}, \quad \alpha_k = \frac{1}{2\pi i k} \sum_i a_i \left(e^{-\psi_k t_i} - e^{-\psi_k t_{i-1}}\right) \tag{22}$$

$$B(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{\psi_k t}, \quad \beta_k = \frac{1}{2\pi i k} \sum_i a_i \left(e^{-\psi_k t_i} - e^{-\psi_k t_{i-1}}\right)$$

The equation (19) with the help of (21) may be rewritten in the form of

$$f(X^T(t)) = A(t) X^T(t) + B(t) \tag{23}$$

and integral equation (9) transfers into Fredholm's equation

$$X^T(t) = \int_0^T \Phi(t-\tau) A(\tau) \sigma^T(t) d\tau + \int_0^T \Phi(t-\tau)[B(\tau) + \varphi(\tau)] d\tau \tag{24}$$

Considering (21) and periodicity of all integrand functions which are written in (24) it is possible to have

$$X^T(t) = \sum_i a_i \int_{t_{i-1}}^{t_i} \Phi(t-\tau) \sigma^T(\tau) d\tau + \sum_i b_i \int_{t_{i-1}}^{t_i} \Phi(t-\tau) d\tau + \int_0^T \Phi(t-\tau) \varphi(\tau) d\tau \tag{25}$$

Below we shall consider only those special cases when t_i numbers corresponding to a periodic regime do not turn

denominator of the resolvent into zero*).

$$K(t, \tau) = A(\tau) \Phi(t-\tau), \tag{26}$$

It is possible to show that the solution of equation (25) can be presented in the form of

$$X^T(t, t_0 \ldots T) = \int_0^T L(t, \tau)(B(\tau) + \varphi(\tau)) d\tau, \tag{27}$$

where $L(t, \tau)$ – function connected with the resolvent $R(t, \tau)$ of nucleus (26) by the relation

$$A(\tau) L(t, \tau) = R(t, \tau) \tag{28}$$

Assuming that a passage from one part of the characteristic to another (switching) takes place when $X^T(t)$ coordinate reaches its given value one will have equations of periods

$$X^T(t_i, t_i \ldots T) = X_{i\,зад} \tag{29}$$

where

$X_{i\,зад}$ – given numbers.

If function $\Phi(t-\tau)$ in (24) is substituted by $\tilde{\Phi}(t-\tau)$ an integral equation with degenerative nucleus will be obtained.

$$\tilde{X}^T(t) = \int_0^T \tilde{\Phi}(t-\tau)[A(\tau) X^T(\tau) + B(\tau)] d\tau \tag{30}$$

Then the system of equations (13) will take form of

$$\tilde{\chi}_{p_k} = \frac{M(\psi_{p_k})}{D(\psi_{p_k})} \sum_{e=-k}^{k} \alpha_{p_k - p_e} \tilde{\chi}_{p_e} + \frac{M(\psi_{p_k})}{D(\psi_{p_k})} (\beta_{p_k} + \varphi_{p_k}) \tag{31}$$

where φ_p – Fourier coefficients of function $\varphi(t)$.

Solving the system (30) we shall find

$$\tilde{\chi}_{p_k} = \frac{1}{\tilde{\Delta}} \sum_{e=-k}^{k} \frac{M(\psi_{pe})}{D(\psi_{pe})} (\beta_{pe} + \varphi_{pe}) \tilde{\Delta}_{ek}, \tag{32}$$

where

$$\tilde{\Delta} = \det \left| \delta_{ke} - \frac{M(\psi_{pk})}{D(\psi_{pk})} \alpha_{p_k - p_e} \right|, \tag{33}$$

$\tilde{\Delta}_{ek}$ – corresponding algebraic addition of the determinant (33).

Substituting (31) in (12) and using (29) we shall have approximate equations of periods

*) It is possible to show that the case $\varphi(t) = 0$ and $f(x)$ – a continuous function will always be a special one.

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$$\tilde{X}^{\tau}(t_i, t_{i+1}, \ldots \tau) = X_{i\,зад} \qquad (34)$$

3. It follows from above that a piece-wise system periodic regime presented in the form of a finite trigonometric sum is equivalent to the substitution of nucleus (26) by a degenerative one. Such a substitution is widely used in Fredholm equations theory. There are different ways allowing to estimate the value of the error permitted.

Let us consider space of m-dimensions of times $t_0, t_1, \ldots t_{m-1}$. Let $\tau(t_0, t_1, \ldots t_{m-1})$ - be a point of this space.

Let us consider a closed region $\mathcal{M}$ which is determined by the following correlations

$$P_i \le t_i \le Q_i \qquad i = 0, \ldots m-1 \qquad (35)$$

where P_i, Q_i - are given numbers.

Let us confront equations (29) and (30) with each point $\tau \in \mathcal{M}$. We shall suppose that resolvent denominators for $\tau \in \mathcal{M}$ (24) and (30) do not turn to zero. Denominator of degenerative nucleus

$$A(\tau)\tilde{\Phi}(t-\tau) \qquad (36)$$

resolvent is determinant (33).

Then it will be possible to prove the following theorems.

Theorem 1.

Let in the region $\mathcal{M}$ the function $\Phi(t-\tau)$ is chosen in such a way that

$$\left|(\Phi(t-\tau) - \tilde{\Phi}(t-\tau))A(\tau)\right| < \varepsilon \qquad (37)$$

We can call

$$\Delta = \max\left\{\left|\Phi(t-\tau)A(\tau)\right| + \varepsilon\right\} \qquad (38)$$

Then if point $\tau(t_0, \ldots t_{m-1})$ corresponds to a periodic regime of the system concerned, we shall have

$$X^{\tau}(t) - \tilde{X}^{\tau}(t) < \varepsilon K \qquad (39)$$

where

$$K = \frac{\Omega(\Delta T)\Omega'(\Delta T) + \Delta T\Omega'^{2}(\Delta T)\Omega''(\Delta T)}{|\tilde{\Delta}||\tilde{\Delta} - \varepsilon\Omega'(\Delta T)|} + \max|\tilde{R}(t,\tau)|T\max|B(\tau) + Y(\tau)| \qquad (40)$$

$\Omega(\Delta)$ - power series

$$\Omega(z) = \sum_{m=1}^{\infty} \frac{m^m}{m!} z^m, \qquad (41)$$

$\tilde{R}(t,\tau)$ - the resolvent of nucleus (36) which can easily be written in a finite form.

In this case in point $\tau(t_0, t_1, \ldots t_{m-1})$ approximate equations of periods (34) are satisfied up to ε_κ, i.e.

$$\left|\tilde{X}^{\tau}(t_i, t_{i+1}, \ldots t_{m-1}) - X_{i\,зад}\right| < \varepsilon_\kappa \qquad (42)$$

Theorem 2.

If point $\tilde{\tau}(\tilde{t}_0, \tilde{t} \ldots \tilde{t}_{m-1}) \in \mathcal{M}$ according to given above assumptions corresponds to a solution of approximate equations of periods (34), exact equations of periods will be satisfied up to ε_κ at the same point, i.e.

$$\left|X(\tilde{t}_i, \tilde{t}_0 \ldots \tilde{t}_{m-1}) - X_{i\,зад}\right| < \varepsilon_\kappa \qquad (43)$$

Theorem 3.

For the points $\tau(t_0, t_{m-1}) \in \mathcal{M}$ where at least for one takes place inequality

$$\left|\tilde{X}(t_i, t_0 \ldots t_{m-1}) - X_{i\,зад}\right| > \varepsilon_\kappa \qquad (44)$$

exact equations of periods cannot be satisfied and consequently periodic solution of the type investigated does not exist.

The demonstration of the theorems follows from estimations given in [6]. The given theorems allow to localize points $\tau(t_0 \ldots t_m)$ supposed to be "pretenders" to periodic equations solution with any precision and to estimate the form of unknown periodic regime.

If the value ε_κ of $\tau \in \mathcal{M}$ is small enough, we shall be forced to consider in limits of calculations (made) precision that solutions of exact and approximate equations of periods presented in the region coincide with one another. The question of found number systems real correspondance to periodic regimes of the system concerned should be examined specially on the basis of direct construction of the periodic regime or on the basis of physical considerations.

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It is not difficult particularly to use the theorems proved in the case of harmonic balance method. In this it is quite enough to restrict the function $\varphi(t-\tau)$ with a free member and the first harmonic only.

In particular it is possible to get sufficient conditions on the absence of periodic equations solutions corresponding to harmonic balance method to provoke that for exact periods equations solutions. In the total all these conditions come to the statement that the exact filter conditions (15) should be fulfilled to enough degree of precision.

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The gyroscopic compass with mercury ballistic tanks is a nonlinear mechanic system. The presence of nonlinear terms in the differential equations of the motion leads to the azimuthal deviations of the gyrocompass, appearing on the rolling ship. If the parameters of the gyrocompass are choosen unsatisfactorily, these deviations may reach very large values, as far as 10-15 degrees.

The securing of the accuracy of the gyrocompass on the rolling ship, may be reached, as this was pointed by A.L. Rawlings /1/, by the properly choise of the free oscillations period of the mercury in the ballistic tanks. The theory of Rawlings was based on the assumption that the rolling of the ship is sinusoidal.

In the present work the motion of the gyrocompass in studying under assumption that the rolling of the ship is a stationary stochastic process /2/ with a rational spectral density.

The equations of the motion of the gyro-

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scopic compass are /1/,

$$H\beta' + Hu\cos\varphi\ \alpha - c_1\vartheta + c\gamma\vartheta = H\frac{v_x}{R},$$
$$H\alpha' + c\vartheta = Hu\sin\varphi,$$
$$C\vartheta'' + m\vartheta' + \vartheta + \beta = -\frac{W_2}{g},$$
$$A\gamma'' + J\chi' + \ell\mathcal{P}\gamma = \ell\mathcal{P}\frac{W_1}{g} \qquad (I.1)$$

Here, α is the angle of turn of the gyrocompass in azimuth, β is the angle on which the gyroaxle is tilted above gorisontal plane, γ is the angle between the gyroaxle and the surface of the mercury in the ballistic tanks, ϑ is the angle on which the spider of the gyrocompass is turning about an axis, which is parallel to the gyroaxle, H is the angular momentum of the gyroscope, u is the angular velocity of the earth, φ is the latitude of the place of observation, v_y is the north component of the ship's velocity, R is the earth's radius, W_1 and W_2 are the east and north components of the fulcrum's of the gyrocompass translate acceleration.

By the straight line motion of the ship with a constant velocity

$$W_1 \approx -z\vartheta''\cos\psi \quad W_2 \approx z\vartheta''\sin\psi, \qquad (I.2)$$

where ϑ is the rolling angle of the ship, ψ is the course of the ship, z is the distance from the fulcrum of the gyrocompass to a line, which is passing through the centre of rolling parallel to the longitudinal axis of the ship.

Let us now introduce new variables

$$X_1 = \alpha - \alpha^*,\ X_2 = \beta - \beta^*,\ X_3 = \vartheta - \vartheta^*,\ X_4 = \gamma - \gamma^*, \qquad (I.3)$$

where

$$\alpha^* = \frac{v_x}{Ru\cos\varphi} = \lambda\operatorname{tg}\varphi,\ \beta^* = -\vartheta^* = \frac{Hu\sin\varphi}{c}$$
$$\gamma^* = 0,\quad \lambda = \frac{c_1}{c} \qquad (I.4)$$

Then the equations (I.I) are transformed to

$$X_1' + \frac{\kappa^2}{u\cos\varphi}X_3 = 0,$$
$$X_2' + u\cos\varphi X_1 - \lambda\frac{\kappa^2}{u\cos\varphi}X_3 - u\sin\varphi X_4 + \frac{\kappa^2}{u\cos\varphi}X_3X_4 = 0$$
$$X_3'' + \zeta X_3' + n^2X_3 + n^2X_2 = -\frac{z\sin\psi}{g}n^2\Theta'', \qquad (I.5)$$
$$X_4'' + \varkappa X_4' + \rho^2X_4 = -\frac{z\cos\psi}{g}\rho\Theta'',$$

where

$$\kappa^2 = \frac{cu\cos\varphi}{H},\ n^2 = \frac{1}{C},\ \rho^2 = \frac{C\mathcal{P}}{A},\ \zeta = \frac{m}{C},\ \varkappa = \frac{J}{A} \qquad (I.6)$$

The first approximation equations, which we obtain from (I.5) by neglecting the second order small terms, may be presented by the matrix equation

$$f(x)\dot{X}^{(0)} = \Theta(x)h(\Theta(x)), \quad \left(x = \frac{d}{dt}\right) \qquad (I.7)$$

where

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$$f(D)=\begin{vmatrix} D & 0 & \frac{\kappa^2}{U\cos\varphi} & 0 \\ 0 & D & -\lambda\frac{\kappa^2}{U\cos\varphi} & -U\sin\varphi \\ 0 & n^2 & D^2+\xi D+n^2 & 0 \\ 0 & 0 & 0 & D^2+\varkappa D+\rho^2 \end{vmatrix} \tag{I.8}$$

$$X^{(0)}=\begin{vmatrix} X_1^{(0)} \\ X_2^{(0)} \\ X_3^{(0)} \\ X_4^{(0)} \end{vmatrix}, \qquad e(D)=\begin{vmatrix} 0 \\ 0 \\ -\frac{z\sin\psi}{g}n^2D^2 \\ -\frac{z\cos\psi}{g}\rho^2D^2 \end{vmatrix}$$

From the equation (I.7) it follows, that

$$X^{(0)}=Y(D)\theta(t) \quad \left(Y(D)=\frac{F(D)e(D)}{\Delta(D)}\right). \tag{I.9}$$

Here $F(D)$ is the adjoint matrix for the matrix $f(D)$ and $\Delta(D)$ is the determinant of the matrix $f(D)$

$$\Delta(D)=(D^2+\varkappa D+\rho^2)\Delta_1(D), \tag{I.10}$$

where

$$\Delta_1(D)=D^4+\zeta D^3+n^2D^2+\xi D+\kappa^2n^2,$$
$$\xi=\lambda n^2\frac{\kappa^2}{U\cos\varphi}. \tag{I.11}$$

We shall need below the values of $X_3^{(0)}$ and $X_4^{(0)}$. According to (I.9) they are defined by following operational expressions:

$$X_3^{(0)}=\left[-\frac{z\sin\psi}{g}\frac{n^2D^4}{\Delta_1(D)}+\frac{z\cos\psi}{g}\frac{\rho^2n^2U\sin\varphi D^3}{\Delta(D)}\right]\theta(t), \tag{I.12}$$

$$X_4^{(0)}=-\frac{z\cos\psi}{g}\frac{\rho^2D^2}{D^2+\varkappa D+\rho^2}\theta(t)$$

From (I.12) it follows that

$$X_3^{(0)}=\Phi(D)X_4^{(0)}, \tag{I.13}$$

where

$$\Phi(D)=\frac{n^2tg\psi}{\rho^2}\frac{D^2(D^2+\varkappa D+\rho^2)}{\Delta_1(D)}-\frac{n^2}{\Delta_1(D)}\frac{U\sin\varphi D}{\Delta_1(D)}. \tag{I.14}$$

Assuming /2/ the rolling of the ship to be a stationary stochastic process with the spectral density

$$S(\omega)=L\frac{4\mu\omega^2}{(\omega^2-\rho^2)^2+4\mu^2\omega^2} \tag{I.15}$$

Let us determine the expectations of the stochastic processes $X_1,\ldots,X_4$ which we shall denote by

$$x_i=M[X_i] \quad (i=1,\ldots,4). \tag{I.16}$$

For determining the functions x_i $(i=1,\ldots,4)$ we deduce from (I.5) a system of differential equations

$$x_3'+\frac{\kappa^2}{U\cos\varphi}x_3=0,$$
$$x_2'+U\cos\varphi x_1-\lambda\frac{\kappa^2}{U\cos\varphi}x_3-U\sin\varphi x_4=-\frac{\kappa^2}{U\cos\varphi}E, \tag{I.17}$$

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$$x_3'' + \zeta x_3' + n^2 x_3 + n^2 x_2 = 0,$$
$$x_4'' + \varkappa x_4' + \rho^2 x_4 = 0,$$

where

$$E = M[X_3 X_4]. \tag{I.18}$$

Restricting ourselves to the first approximation, we shall replace E in the equations (I.17) by the expression

$$E_0 = M[X_3^{(0)} X_4^{(0)}]. \tag{I.19}$$

The integrals of the system of homogeneous equations which may be obtained from (I.17) when $E=0$ will tend asimptotically to zero as $t \to \infty$, because all the zeros of the determinant (I.10), according to (I.10), have negative real parts. Therefore, as the rolling is sufficiently prolonged, the expectations $x_1, \ldots, x_4$ will take their stable values $x_1^*, \ldots, x_4^*$, which, according to (I.17), are

$$x_1^* = -\left(\frac{\kappa}{U\cos\varphi}\right)^2 E_0, \quad x_2^* = 0, \quad x_3^* = 0, \quad x_4^* = 0. \tag{I.20}$$

To determine E_0 we remarke, that the spectral density of the stochastic process $X_4^{(0)}$

$$S_{44}(\omega) = |Y_4(i\omega)|^2 S(\omega)$$

in accordance with (I.12) and (I.15) will have the form

$$S_{44}(\omega) = 4\mu\nu^2 \frac{\rho^4 r^2 \cos^2\psi}{g^2} \frac{\omega^4}{[(\omega^2-\rho^2)^2 + \varkappa^2\omega^2][(\omega^2-\nu^2)^2 + 4\mu^2\omega^2]} \tag{I.21}$$

The cross spectral density $S_{43}(\omega)$ of the random processes $X_4^{(0)}$ and $X_3^{(0)}$, according to (I.13) has the form

$$S_{43}(\omega) = \Phi(i\omega) S_{44}(\omega)$$

According to (I.14) and (I.21) one can transforme the expression (I.22) into the form

$$S_{43}(\omega) = 4\mu\nu^2 \left[z_1 \frac{G_1(\omega) + i K_1(\omega)}{Z(\omega)} + z_2 \frac{G_2(\omega) + i K_2(\omega)}{Z(\omega)} \right], \tag{I.23}$$

where

$$z_1 = \frac{\rho^2 n^2 r^2 \sin 2\psi}{2g^2}, \qquad z_2 = \frac{\rho^4 n^2 U \sin\varphi\, r^2 \cos^2\psi}{g^2},$$

$$G_1(\omega) = \omega^4 [\omega^6 + (\varkappa\zeta - \rho^2 - n^2)\omega^4 + (\rho^2 n^2 + \kappa^2 n^2 - \varkappa\zeta)\omega^2 - \rho^2\kappa^2 n^2],$$

$$G_2(\omega) = \omega^6 (\zeta\omega^2 - \xi),$$

$$K_1(\omega) = \omega^7 [(\xi - \varkappa)\omega^4 + \varkappa n^2 - \zeta\rho^2\xi)\omega^2 - \xi\rho^2 - \varkappa\kappa^2 n^2], \tag{I.24}$$

$$K_2(\omega) = \omega^5 (\omega^4 + n^2\omega^2 - \kappa^2 n^2),$$

$$Z(\omega) = \Delta_1(i\omega)\Delta_1(-i\omega)[(\omega^2-\rho^2)^2 + \varkappa^2\omega^2][(\omega^2-\nu^2)^2 + 4\mu^2\omega^2].$$

The cross correlation function $R_{43}(\tau)$ of the random processes $X_4^{(0)}$ and $X_3^{(0)}$ has the form

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$$R_{43}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{43}(\omega)e^{i\omega\tau}d\omega \qquad (I.25)$$

and, as according to (I.2I) we have $E_0=R_{43}(0)$ then

$$E_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{43}(\omega)\,d\omega \qquad (I.26)$$

Substituting the expression $S_{43}(\omega)$ from (I.23) and taking into account that if the function under the sign of the integral is odd, then the integral of the form (I.26) becomes zero, one can reduce the expression (I.26) to the following form

$$E_0=4\mu\nu^2L(z,I+z_2J), \qquad (I.27)$$

where

$$I=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g_1(i\omega)}{h(i\omega)h(-i\omega)}d\omega,\quad J=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{g_2(i\omega)}{h(i\omega)h(-i\omega)}d\omega \qquad (I.28)$$

Here

$$g_1(i\omega)=\sum_{s=0}^{7}B_s(i\omega)^{14-2s},\quad g_2(i\omega)=\sum_{s=0}^{7}B_s(i\omega)^{14-2s},$$
$$h(i\omega)=\sum_{s=0}^{8}a_s(i\omega)^{8-s} \qquad (I.29)$$

and the polynominal coefficients have tne form

$$B_0=0,\ B_1=1,\ B_2=\rho^2+n^2-\varkappa\xi,\ B_3=(\rho^2+\kappa^2)n^2-\varkappa\xi,$$
$$B_4=\rho^2\kappa^2n^2,\ B_5=B_6=B_7=0,$$
$$B_0=B_1=B_2=0,\ B_3=\xi,\ B_4=\xi,\ B_5=B_6=B_7=0,$$
$$a_0=1,\ a_1=2\mu+\varkappa+\xi,\ a_2=\nu^2+\rho^2+n^2+2\mu(\varkappa+\xi)+\varkappa\xi,$$
$$a_3=2\mu(\rho^2+n^2+\varkappa\xi)+\xi(\rho^2+\nu^2)+\varkappa(\nu^2+n^2)+\xi,$$
$$a_4=2\mu(\xi+\varkappa n^2+\xi\rho^2)+\varkappa(\xi+\xi\nu^2)+n^2(\nu^2+\rho^2+\kappa^2)+\nu^2\rho^2,$$
$$a_5=2\mu(\kappa^2n^2+\rho^2n^2+\varkappa\xi)+\xi(\rho^2+\nu^2)+\varkappa n^2(\nu^2+\kappa^2)+\xi\nu^2\rho^2,$$
$$a_6=2\mu(\xi\rho^2+\varkappa\kappa^2n^2)+n^2(\nu^2\rho^2+\kappa^2\nu^2+\kappa^2\rho^2)+\varkappa\xi\nu^2,$$
$$a_7=(2\mu\rho^2+\varkappa\nu^2)\kappa^2n^2+\xi\nu^2\rho^2. \qquad (I.30)$$
$$a_8=\kappa^2n^2\rho^2\nu^2.$$

All the zeros of the polynome $h(p)$ having the negative real parts, the integrals (I.28), according to Phillips[3], have the form

$$I=-\frac{M_1}{2a_0N},\quad J=-\frac{M_2}{2a_0N}, \qquad (I.3I)$$

where

$$N=\begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0\\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0\\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0\\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0\\ 0 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2\\ 0 & 0 & 0 & a_8 & a_7 & a_6 & a_5 & a_4\\ 0 & 0 & 0 & 0 & 0 & a_8 & a_7 & a_6\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 \end{vmatrix} \qquad (I.32)$$

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Here M and M_1 are the determinants obtained by substituting of the first column in the determinant (I.32) by the column with the elements $b_0, b_1, \ldots, b_7$ or $B_0, B_1, \ldots, B_7$ accordingly.

As it follows from (I.20) the gyrocompass has the azimuthal deviation in the presence of rolling. From (I.20) and (I.27) this deviation is defined by the expression

$$x_1^* = -a I^* \sin 2\psi - b J^* \cos^2\psi, \qquad (I.33)$$

where

$$a = 2\mu\nu^2 L \rho^2 \frac{z^2}{g^2}\left(\frac{\kappa}{U\cos\varphi}\right)^2, \qquad (I.34)$$

$$b = 4\mu\nu^2 L \rho^4 U \sin\varphi \frac{z^2}{g^2}\left(\frac{\kappa}{U\cos\varphi}\right)^2,$$

$$I^* = n^2 I, \quad J^* = n^2 J. \qquad (I.35)$$

As an example let us define the deviation of the gyroscopic compass which has $\kappa = 1{,}24 \cdot 10^{-3}\ sec^{-1}$; that is equal to the free oscillation period $T_\kappa = 84{,}4\ min$. Let the latitude of the place of observation [illegible] degrees, then $U\cos\varphi = 3{,}646 \cdot 10^{-5}\ sec^{-1}$. [illegible] tance between the fulcrum and the [illegible] line passing through the centre of [illegible] the ship parallel to the longi- [illegible] the ship $z = 2m$. [illegible] parameters let us fix the va-

$\rho = 6\ sec^{-1}$, $\varkappa = 1{,}2\ sec^{-1}$, $\zeta = 0{,}2n\ sec^{-1}$, $\xi = 10^{-3} n^2 sec^{-3}$.

The parameters which define the spectral density of the rolling are

$$\mu = 0{,}1\ sec^{-1}, \quad \nu = 0{,}8\ sec^{-1}.$$

The rolling angle dispersion $L = 0{,}03$, then the root mean square value of rolling angle $\sqrt{L} = 0{,}173$ i.e., about 10 degrees.

Under these conditions, the compass deviation according to (I.33) is defined by the expression

$$x_1^* = -6{,}72 I^* \sin 2\psi - 3{,}03 \cdot 10^{-2} J^* \cos^2\psi, \qquad (I.36)$$

where I^* and J^* according to (I.35) and (I.31) depend from the frequency n of the free oscillations of the mercury surface in the ballistic tanks. The graphs of the functions $I^*(T)$ and $J^*(T)$, where

$$T = 2\pi/n$$

are plotted on the figure.

The maximal value of J^* in this example is $0{,}0037$, so the second term in the expression (I.37) does not exceed $0{,}11 \cdot 10^{-3}$ or $0{,}4'$. Neglecting this term as a small, we shall obtain from (I.36) the following expression for the deviation

$$x_1^* = -6{,}72 I^* \sin 2\psi \qquad (I.38)$$

The deviation x_1^* is an intercardinal one. It vanishes on the cardinal courses of

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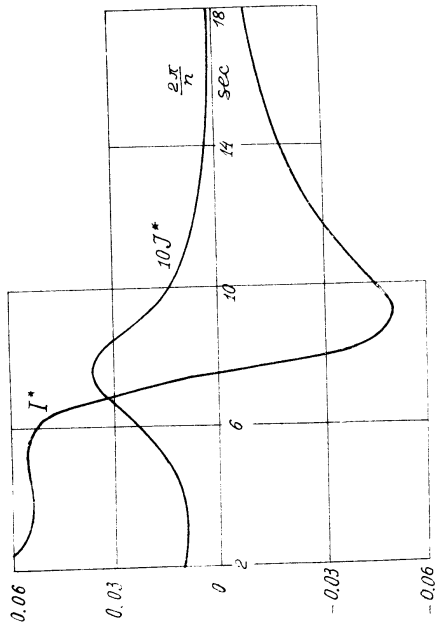
the shi, $\psi = 0°, 90°, 180°, 270°$ and attains its maximal value on the intercardinal courses $\psi = 45°, 135°, 225°, 315°$.

To prevent the intercardinal deviation one must choose the mercury tanks parameters accordingly to the conditions that the frequency n of free oscillations of the mercury surface be suffitiently near to the value n^* , for which $I^*(n)$ becomes zero. As it is seen from the graph, the zero of the function $I^*(n)$ is located near the point $n = \nu$. In our example $n^* = 0,843$.



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НЕКОТОРЫЕ ВОПРОСЫ ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ НЕАВТОНОМНЫХ ОДНОМЕРНЫХ СИСТЕМ

Introduction

In 1833 C.Sturm [1] raised the problem of oscillations for linear non-autonomous differential equations of the second order

$$x'' + \alpha(t)x' + \beta(t)x = 0. \qquad (*)$$

This problem consists in that it is required to establish the conditions to be superposed on the functions $\alpha(t)$ and $\beta(t)$, for which the solutions of equation (*) have a given number of zeros over the time interval in question.

The comparison method as proposed by Sturm, its various modifications, and other methods, in particular, the phase method, the method of successive approximations, made it possible to obtain a set of sufficient, necessary as well as necessary and sufficient conditions for which the solutions of equation (*) are oscillatory (non-oscillatory).

The above conditions are not connected with the initial values due to the fact that the character of the solutions of equation (*) is indepen-

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dent of the initial conditions in the sense suggested by the corollaries to Sturm's theorem [1].

As it is known, the solution of a great number of applied problems would lead us beyond the scope of the theory of oscillations of linear non-autonomous systems, since the linear approximation does not allow to establish some phenomena accompanying nonlinear oscillations and may lead to erroneous conclusions.

Some applied problems of modern physics and engineering raise new mathematical problems of the theory of nonlinear non-autonomous systems. As an illustrative example, we might mention the problem of the determination of the region of capture[1] of charged particles into the synchronous state of acceleration in some types of modern accelerators.

Over the past 5-7 years, a number of authors have obtained some interesting results in the domain of the theory of oscillation of the solutions of nonlinear non-autonomous systems. Thus, C.T.Taam [2], by applying the M.Picone identity, established the theorem of oscillation, and also found the conditions when the oscillating solutions of the equation

$$(m(t)x')' + \sum_{i}^{n} f_i(t)x^{2i-1} = 0$$

are non-increasing (non-damping).

[1] The totality of initial conditions determining the solutions of some nonlinear non-autonomous system, which oscillate/in a definite sense/about a given position of equilibrium.

R.V. Petropavlovskaya [3] stated the comparison theorem for the equation

$$u'' = f(u, u', t) \qquad (**)$$

and obtained a series of sufficient conditions of oscillation for all the solutions of some of the more particular cases of equation (**). In particular, some of the well known results due to A.Wintner and A.Kneser were generalized by her for the equation

$$u'' + L(u)p(t) = 0.$$

R.Bass [4] has shown the conditions under which the solutions of the vector equation

$$x'' = f(x, x', t)$$

either tending to infinity or bounded, will pass through a given initial position x_0 under various initial impulses x_0'.

Generalizing the works of physicists on the determination of the region of capture of charged particles into the synchronous state of acceleration, V.K.Melnikov [5] has suggested a method for determining the separatrix analogy for the equation

$$(m(\varepsilon, t)x')' + \kappa(\varepsilon, t)p'(x) = \varepsilon f(\varepsilon, t, x, x')x', \qquad (***)$$

assuming that there exist no node-type equilibrium positions. Yu.S.Sayasov and V.K.Melnikov [6] have made an attempt at investigating the effect of the non-autonomy of the equation of phase motions of particles in some types of accelerators (as a particular case of equation (***)) on the size of the capture region.

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Let it be empnasized tnat by giving a short review of works for the purpose of comparing results, the autnors do not claim to nave included a l l the works dealing with the theory of oscillation of the solutions of nonlinear non-autonomous equations. It should be noted here that in [2,3], as well as in some other works of the same kind, no mention was made of the fact that nonlinear equations allow, generally speaking, a great number of equilibrium positions (which was not really the aim of the authors of those works). However, to this fact, as follows from studies in the qualitative theory of nonlinear non-autonomous systems, started by A.Poincaré [7] and A.M. Liapunov [8], is due the effect of initial conditions on the character of motion, in general, and the character of oscillation, in particular: some initial conditions will determine oscillating motions, others - non-oscillating motions. In the case of linear non-autonomous systems, as pointed out above, initial conditions have no effect upon the character of oscillation. It is easy to construct examples of nonlinear non-autonomous systems for which the character of solutions is also independent of the initial conditions.

Taking into account the above considerations, the authors of this paper approach the problem of oscillations of nonlinear non-autonomous equations from the point of view of there being available many equilibrium positions: an attempt is made to establish the conditions which allow to classify

nonlinear non-autonomous second order equations according to whether the character of oscillation of the solutions is dependent or independent of the initial conditions; some questions are investigated concerned with the determination of the regions of oscillation (non-oscillation) and with the behaviour of the analogue of the amplitude of oscillating motions; examples (of the motion of a non-autonomous pendulum) are considered, illustrating some of the results obtained.

Fundamental Definitions. Statement of Problem

Consider the equation

$$(\kappa(t)x')' + f(x, x', t) = 0 \qquad (1)$$

for $t \geq t_0 \geq 0$, assuming the continuity of the functions $\kappa(t) \geq \kappa_0 > 0$ ($\kappa = \text{const}$), $\kappa'(t)$, $f(x, x', t)$ with respect to their arguments, and the existence of a unique solution.

We shall study the behaviour of the solutions of equation (1) in the vicinity of the equilibrium positions $x_r = \text{const}$ ($r = 1, 2, \ldots$) which are the roots of the equation $f(x_r, 0, t) = 0$ for any t proceeding from the following fundamental definitions.

The solution of equation (1) $x(t)$ will be referred to as n o n - o s c i l l a t i n g if $y_r(t) = x(t) - x_r$ for any r has no more than one zero.

The solution $x(t)$ for which $y_r(t)$ ($r = 1, 2, \ldots$) has a countable number of zeros, will be referred to as o s c i l l a t i n g.

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If $y_r(t)$ has a countable number of zeros only for a unique $r=r_0$, then the solution $x(t)$ will be said to be **oscillating about a given equilibrium position** x_{r_0}.

The solution $x(t)$ oscillating about a given equilibrium position x_{r_0} will be said to be **uniformly oscillating** if $y_r(t)$ $(r \neq r_0)$ has no zeros at all.

The solution $x(t)$ for which $y_r(t)$ has n $(2 \leq n < \infty)$ zeros for a given $r=r_0$ will be said to be **mixed** with reference to the equilibrium position x_{r_0}.

It would be inexpedient to regard as mixed solutions having the following two zeros: $y_r(t_1)=0$, $t_1 < \infty$, $y_r(t) \to 0$ for $t \to \infty$. Therefore, the zero attained at infinity is ruled out of consideration both here and in what follows.

Note that if system (1) admits of a unique equilibrium position, then the concepts "oscillating", "oscillating about a given equilibrium position", and "uniformly oscillating motions" will coincide.

Equation (1) will be referred to as oscillating, non-oscillating, and mixed, if its solutions are, respectively, oscillating, non-oscillating, or mixed.

The totality of initial conditions for which oscillating, non-oscillating or mixed solutions are obtained, will be referred to as the region of oscillating, non-oscillating or mixed solutions (motions), respectively.

It will be remembered that the motions being considered can in no way be regarded as a detailed classification of motions characteristic of non-autonomous systems. In particular, such interesting types of oscillating motions as periodic, almost periodic, or motions close to these, are not brought into classification for the time being.

Proceeding from the above definitions, and by making use of a number of applied problems and publications on the theory of nonlinear oscillations, we now state some cases of the problem of nonlinear non-autonomous systems (1):

a) To classify the positions of equilibrium as allowed by system (1), from the point of view whether that or other type of motion about these equilibrium positions is possible or impossible of existence.

b) To find the conditions imposed on the functions $\kappa(t)$ and $f(x,x',t)$ for which equation (1) will be of the oscillating, non-oscillating, or mixed type.

c) For fixed $\kappa(t)$ and $f(x,x',t)$ to determine the region of oscillating, non-oscillating, or mixed motions about a given position of equilibrium.

d) Given the region of initial values, to find the conditions imposed on the coefficients of equation (1), which determine the given type of oscillation.

The solution of the second problem is to be found in a number of publications, in particular

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[2,3]. The third problem has been solved for the case of a nonlinear, non-autonomous equation of the second order close to the conservative equation, in publications [5,6]. The solution of the fourth problem will be of interest from the point of view of controlling the process described by equation (1); this problem can be extended by introducing the conditions for the optimum occurrence of the process. The solution of the first problem will be of interest, in particular, from the point of view of the correctness of statement of the third problem.

On the Solution of Some of the Problems Stated Above

We shall now attempt to establish the conditions which allow to isolate a class of nonlinear equations

$$x'' + F(x, x', t) = 0, \quad t \in [t_0, \infty) \qquad (1')$$

for which the initial conditions have no effect on the character of oscillation of the solutions. In so doing, we shall proceed from assumptions made for equation (1) and, without restricting generality, let us assume that system (1') admits of a position of equilibrium $x_r = 0$.

Theorem 1. a) If $\frac{F(x, x', t)}{x} \geq \gamma_1 > 0, \gamma_1 = const$ for all x, x', t

then whole plane (x_0, x_0') is the region of oscillating solutions of equation (1').

b) If $\frac{F(x, x', t)}{x} \leq -\gamma_2 < 0, \gamma_2 = const$ for all x, x', t then the whole plane (x_0, x_0') is the region of non-oscillating solutions of equation (1').

Theorem 1 is an analogue to Sturm's theorem [1].

Remark 1. Statement a) is valid for the case of equation (1), when

$$\frac{f(x, x', t)}{x} \geq \gamma > 0, \quad \gamma = const, \quad 0 < \alpha \leq \kappa(t) \leq \beta < \infty, \quad \alpha, \beta - const.$$

Remark 2. The conditions of theorem 1 exclude the possibility of system (1') having many positions of equilibrium.

Remark 3. Statement b) is valid for system (1) if $\frac{f(x, x', t)}{x} \leq 0$, $\kappa(t) \geq \alpha > 0$ which conditions admit, however, many positions of equilibrium for system (I).

Thus, the existence of one position of equilibrium implies a uniform character of the motions of system (1'). The reverse is not true: uniformity of motions may occur also when there exist many positions of equilibrium.

For the proof of theorem 1 let us give an additional statement on the relation between the zeros of the two following equations:

$$(\kappa_1(t)u')' + f_1(u, u', t) = 0 \qquad (2)$$

$$(\kappa_2(t)v')' + f_2(v, v', t) = 0 \qquad (3)$$

under assumptions as those made for the case of equation (1).

If $\kappa_1 \geq \kappa_2$, $\frac{f_1(v, v', t)}{v} \geq \frac{f_2(v, v', t)}{v}$ for all

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u, u', v, v' and $0 \le t_0 \le t < \infty$ then between any two consecutive zeros of each solution $u(t)$ of equation (2) there lies at least one zero of the solution $v(t)$ of equation (3).

Writing down M.Picone's identity [9] for equations (2) and (3)

$$\left[\frac{u}{v}\left(\kappa_1 u'v - \kappa_2 uv'\right)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} u^2\left(\frac{f_2}{v} - \frac{f_1}{u}\right)dt + \int_{t_1}^{t_2}\left[(\kappa_1 - \kappa_2)u'^2 + \kappa_2\left(u' - \frac{uv'}{v}\right)^2\right]dt \qquad (4)$$

which is valid on any interval (t_1, t_2) on which $v(t) \neq 0$ it is easy to prove the above statement.

Indeed, let us assume that between any two consecutive zeros $u(t_1) = u_1 = u(t_2) = v_2 = 0$ of any solution $v(t)$ of equation (2) there lies no zero of any solution $v(t)$ of equation (3). Then it is evident that the left-hand side of (4) becomes zero, while the right-hand side is positive. This contradiction proves the statement. In the case of $v_1 = 0$ or $v_2 = 0$ with $v(t) \neq 0$ $(t_1 < t < t_2)$ the statement above is also valid by virtue of the same contradiction, which is readily seen by taking the limit of the left-hand side of (4).

By making use of the well-studied oscillator equation

$$y'' + \gamma_1 y = 0 \qquad (2')$$

as standard equation (2) and then comparing with equation (1'), the validity of case a) of theorem 1 becomes obvious. The proof of case b) of theorem 1 follows if equation

$$y'' - \gamma_2 y = 0 \qquad (2'')$$

is taken as standard equation.

It is seen from the proof of case a), theorem 1, that if the function $\frac{F(x, x', t)}{x}$ for all x, x', t is bounded from below by some constant $\gamma > 0$ on an unbounded time interval $t \ge t_0 \ge 0$, then the solutions of equation (1') have more frequent zeros than some sinusoid.

Consider equation (1') for the case when the function $\frac{F(x, x', t)}{x}$, remaining positive for all $x, x', t \in [t_0, \infty)$ is bounded from below by the value $\gamma = 0$.

Theorem 2.

a) If $\frac{F(x, x', t)}{x} \ge \frac{1+\alpha}{4t^2}$ $(\alpha > 0 - const)$ for all x, x', t then equation (1') is of the oscillating type.

b) If $\frac{F(x, x', t)}{x} \le \frac{1}{4t^2}$ for all x, x', t then equation (1') is of the non-oscillating type.

Theorem 2 is an analogue to Kneser's theorem [10].

The proof of case a) is accomplished by taking equation

$$x'' + \frac{1+\alpha}{4t^2}x = 0$$

as standard equation. If equation

$$x'' + \frac{1}{4t^2}x = 0$$

is used as standard equation, then the proof of case b) follows.

Of interest is also the case when the coefficients of equation (1') change sign or vary in a definite manner.

For equation

$$v'' + p(t) f(v) = 0 \qquad (1'')$$

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the following theorem holds.

Theorem 3. Assume that in equation (1'')
$r(t)=O\left(\frac{1}{t^{k+1}}\right), k>0$-const, $|f(v_1)-f(v_2)|\le N|v_1-v_2|$, N- const
for all v_1, v_2. Then, for any solution $v(t)$ the following relation
$$v(t)-(\alpha t+\beta)=O\left(\frac{1}{t^k}\right),\ \alpha,\beta\text{ - const}$$
is true. The proof follows.

Under the above assumptions it is not difficult to show that the series $\sum_{n=0}(v_{n+1}-v_n)$ where
$$v_{n+1}=v_0-\int_t^\infty\int_t^\infty r(t)f(v_n)\,dt\,dt,\quad v_0=\alpha t+\beta$$
is uniformly convergent.

Indeed, since $|r(t)|\le\frac{A}{(1+t)^{k+1}}$, $|v_0|\le B(1+t)$, A- const, $B=\max(|\alpha|,|\beta|)$ the following estimate is obtained by induction
$$|v_{n+1}-v_n|\le\frac{(Ai)^{n+1}B}{k(k+1)\dots[(n+1)k+n][(n+1)k+(n+1)]}\cdot\frac{1}{(1+t)^{(n+1)k+n}}$$
whence results the relation:
$$\sum_{n=0}(v_{n+1}-v_n)=O\left(\frac{1}{t^k}\right).$$
In a similar manner we prove the uniform convergence of the series $v_0'+\sum_{n=0}(v_{n+1}'-v_n')$, $\sum_{n=0}(v_{n+1}''-v_n'')$ whence it follows that the series $v_0+\sum_{n=0}(v_{n+1}-v_n)$ is the solution of equation (1'').

Remark 1. Theorem 3 is valid for the solutions of equation (1'') for $f=f(v,v',t)$, if the Lipschitz conditions are satisfied for all v uniformly in t and v'.

Remark 2. From theorem 3 it follows that under the above assumptions equation (1'') possesses an innumerable quantity of solutions which are not oscillatory.

Note that the above-made statement concerning the relation of the zeros of the solutions of equations (2) and (3) represents the comparison theorem which is true when (2) and (3) have no more than one position of equilibrium. Let us now attempt to extend the comparison theorems to systems with many positions of equilibrium.

Theorem 4. If $k_1>k_2>0$, $\frac{f_2}{v}-\frac{f_1}{u}\ge 0$ for all u, v, $0\le t_0\le t<\infty$, $u\in U(-\bar u,\bar u)$, $v\in U(-\bar v,\bar v)$, $0<\bar u\le\infty$, $0<\bar v\le\infty$ then between any two consecutive zeros of each solution $u\in U(-\bar u,\bar u)$ of equation (2) there lies at least one zero of any solution $v\in U(-\bar v,\bar v)$ of equation (3).

Proceeding from the standard equation (2') and assuming without any restriction on generality the existence of $v_n=0$ among the positions of equilibrium of system (1''), we establish the following statements for the solutions of equation (1'') as corollaries to theorems 2 and 4:

If $\frac{f(v)}{v}\ge f_0>0$ (f_0 = const), for $v\in U(-\bar v,\bar v)$, $0<\bar v$ and $r(t)\ge r_0>0$ (r_0 = const) or if $\frac{f(v)}{v}\ge f_0>\frac{1}{4}$ (f_0 = const) for $v\in U(-\bar v,\bar v)$, $0<\bar v$ and $r(t)>\frac{1+\alpha}{t^2}$ ($\alpha>0$-const) then the solutions $v(t)\in U(-\bar v,\bar v)$ are uniformly oscillating about $v=0$.

If $\frac{f(v)}{v}\le 0$ for $v\in U(-\bar v,\bar v)$, $0<\bar v$ and $r(t)>0$ or if $\frac{f(v)}{v}\le\frac{1}{4}$ for $v\in U(-\bar v,\bar v)$, $0<\bar v$ and $0\le r(t)\le\frac{1}{4}$ then the solutions $v(t)\in U(-\bar v,\bar v)$ will not be oscillating about $v_n=0$.

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Making use of the previous results, theorem of the number of zeros of the nonlinear non-autonomous equation (3) is established, given the number of zeros of the solutions of the standard equation (2).

Theorem 5.

a) Let $K_1 > K_2 > 0$, $\frac{f_2(v,v',t)}{v} - \frac{f_1(u,u',t)}{u} \geq 0$ for any u, u', v, v', t. Then, if the solution $u(t)$, $u_0 \neq 0$, $u_0' \neq 0$ has m zeros on the interval $t_0 < t \leq t_0 + T$ there exist solutions $v(t)$ having on the same interval no less than m zeros, the K-th zero of $v(t)$ being less than the K-th zero of $u(t)$.

b) Let $K_1 > K_2$, $\frac{f_2}{v} - \frac{f_1}{u} \geq 0$ for $u \in U(-\bar{u}, \bar{u})$, $0 < \bar{u}$, $v \in U(-\bar{v}, \bar{v})$, $0 < \bar{v}$ and for any u, v, t. Then, if the solution $u(t) \in U$, $u_0 \neq 0$ $u_0' \neq 0$ has m zeros on the interval $t_0 < t \leq t_0 + T$ the solutions $v(t) \in U$ will have no less than m zeros on the same interval, the K-th zero of $v(t)$ being less than the K-th zero of $u(t)$.

2) It seems natural to attempt to isolate those classes of non-autonomous systems for which some corresponding autonomous systems can be given. Bearing in mind further generalizations, let us illustrate this approach by giving an example of a special case of system (I)

$$x'' + \alpha(t)x' + \beta(t)f(x) = 0. \qquad (5)$$

It readily seen that if $\operatorname{sign}\beta(t) = const$, $\varphi = \sqrt{\frac{\beta(t_0)}{\beta(t)}}$, $\int_0^\infty \frac{1}{\varphi} dt = \infty$, $\alpha\varphi \cdot \varphi' = \gamma$, $\gamma = const$ in equation (5), then the latter equation is equivalent to the autonomous equation

$$x_\tau'' + \gamma x_\tau' + \beta(t_0)f(x_\tau) = 0 \qquad (5')$$

where $\varphi(t)d\tau = dt$.

Equation (5'), hence also equation (5), can be investigated by making use of the methods of the qualitative theory [11]. In particular, for $\gamma = 0$ system (5) behaves as a conservative one (non-autonomy of the system in this case has no effect on the size of the regions of oscillation (non-oscillation).

In case system (5) does not reduce to an autonomous one, some information concerning the regions of oscillation/non-oscillation can be obtained [12] by means of a certain integro-differential equation which may be considered as the analogue of the first integral of system (5).

3. Theorems 3 and 4 can be applied in some neighbourhood of each of the positions of equilibrium and thus the question will be solved whether or not the system under consideration admits oscillating /non-oscillating/motions with respect to some position of equilibrium. However, the question of the behaviour of the amplitude of oscillations, as well as the question of the determination of the regions of oscillation, remains open. We shall now attempt to establish the damping conditions of the amplitude of oscillating motions by investigating the stability of the positions of equilibrium using A.M.Liapunov's second method [18]. Here, we confine ourselves to equation (5), assuming the existence of an expansion of function $f(x)$ into Taylor's series in the neighbourhood with $f'(x_r) \neq 0$.

Theorem 6.

If $0 < \alpha_1 \leq \alpha(t) \leq \alpha_2 < \infty$, $0 < \delta_1 \leq \delta(t) < \infty$ $(\delta(t) = \beta(t)f'(x_r))$ and $4\alpha_1\delta_1(\alpha_1 + \alpha_2) > (\delta_2 - \delta_1)(\alpha_2^2 + 4\delta_1)$ then the position of

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equilibrium $x = x_r$ is asyptotically stable.

As corollary it will be concluded that by fulfilment of the conditions of this theorem is guaranteed the damping of the amplitude of oscillating motions of the system described by equation (5), at any rate beginning with some $t, t \geq \tau$.

The conditions of damping (rise) of the amplitude of oscillations can be obtained by other methods as well. We give here a theorem established for (5) by using a method analogous to that in [2], assuming that function $F(x^2)$ does not decrease with respect to its argument ($f(0) = 0$, $F(x^2) = \int f(x)dx$).

Theorem 7.

If $F(x^2) > 0$ for $x \neq 0$, $|x| \leq c$, $c = const$ then for $\beta' + 2\alpha\beta \leq 0$ the amplitude of the solutions $|x(t)| \leq c$ of equation (5), oscillating about $x = 0$ does not decrease, while for $\beta' + 2\alpha\beta \geq 0$ the amplitude does not rise.

Example. - Motion of a Non-autonomous Pendulum.

A number of problems in the dynamics of a non-autonomous pendulum has been studied in [13] and [14], using asymptotical methods. Some results connected with the determination of the regions of oscillation (rotation) of a non-autonomous pendulum, have been obtained in [15]. At this point, we shall consider the motion of the pendulum as described by equation

$$(ml^2\theta')' + mgl\sin\theta = 0 \qquad (6)$$

assuming its mass and length to be bounded ($0 < m_1 \leq m(t) \leq m_2 < \infty$, $0 < l_1 \leq l(t) \leq l_2 < \infty$, m_1, m_2, l_1, l_2 const).

for the purpose of illustrating the conclusions which follow from the theorems stated above.

In the linear approximation ($\sin\theta \approx \theta$) all the motions of the pendulum are uniformly oscillating. In the second approximation ($\sin\theta \approx \theta - \frac{\theta^3}{3!}$) the system has three positions of equilibrium: $\theta_0 = 0$, $\theta_{1,2} = \pm\sqrt{6}$. From the conclusions following from theorem 2 it follows that in this case the solutions of (6) in the interval $(\sqrt{6}, \sqrt{6})$ are uniformly oscillating. With reference to the position of equilibrium $\theta_{1,2} = \pm\sqrt{6}$ there are no oscillatory solutions. It turns out that the next approximation ($\sin\theta \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$) admits of a unique position of equilibrium $\theta = 0$ and, consequently, by virtue of theorem 1 all the motions of the pendulum are uniformly oscillating. Thus, qualitatively, the third approximation is not so good a representation of the dynamics of the pendulum as the second approximation. In the general case, the positions of equilibrium of system (6) are $\theta_r = r\pi$ ($r = 0, \pm1, \pm2, \ldots$). From the conclusions of theorem 2 it follows that if there exist $\theta(t)$ in the intervals $((2r-1)\pi, (2r+1)\pi)$, they will be uniformly oscillating with reference to $\theta_{2r} = 2r\pi$. With reference to the positions of equilibrium $(2r+1)\pi$ there can be no oscillating motions.

It follows from theorems 1,4, and 7 that in case of all the approximations considered, the uniformly oscillating solutions of equation (6) do not decrease when $S(t) = ml^2\theta^2$ does not increase, and do not increase when $S(t)$ does not decrease.

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§ I. Definition

Denote by $q_1, \ldots, q_n$ the Lagrange co-ordinates of a mechanical system, and by T the kinetic energy of the system

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q_1, \ldots, q_n)\dot{q}_i\dot{q}_j \qquad (a_{ij} = a_{ji},\ i, j = 1, \ldots, n)$$

To the generalized forces add the resistance forces and we obtain

$$Q_j(q_1, \ldots, q_n) - R_j(\dot{q}_1, \ldots, \dot{q}_n) \qquad (j = 1, \ldots, n),$$

where Q_j and R_j are assumed to be continuos and differential functions of its arguments in the domain of their definition.

For the apportioned resistance forces we will suppose, that their work is negative on an arbitrary virtual removal:

$$\sum_{j=1}^{n} R_j(\dot{q}_1, \ldots, \dot{q}_n)\,\delta q_j > 0 \qquad (1)$$

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The motion equations of Lagrange form of II race can then be written in the form of

$$\sum_{i=1}^{n_1} a_{\nu i}\ddot{q}_i + \sum_{i,j=1}^{n_1}\left(\frac{\partial a_{\nu i}}{\partial q_j} - \frac{1}{2}\frac{\partial a_{ij}}{\partial q_\nu}\right)\dot{q}_i\dot{q}_j = Q_\nu - R_\nu \qquad (\nu = 1, \ldots, n_1). \qquad \text{(I.1)}$$

Let us assume the equations (I.1) have the solution

$$q_\nu = q_{\nu 0}(t), \quad \dot{q}_\nu = \dot{q}_{\nu 0}(t) \quad (\nu = 1, \ldots, n). \qquad \text{(I.2)}$$

Our aim is to study the stability of the unperturbed motion (I.2) relative to the variable $q_1, \ldots, q_{n_1}; \dot{q}_1, \ldots, \dot{q}_{n_1}$ $(n_1 \leq n)$ in the sense of Liapunov (I.2). The co-ordinates and velocities for a perturbed motion denote by

$$q_\nu = q_{\nu 0}(t) + x_\nu, \quad \dot{q}_\nu = \dot{q}_{\nu 0}(t) + \dot{x}_\nu \quad (\nu = 1, \ldots, n).$$

The differential equation of a perturbed motion in the first approximation (the equation in variations) will be present in the form of:

$$A(t)\frac{d^2x}{dt^2} + B(t)\frac{dx}{dt} + C(t)x = 0 \qquad \left(x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) \qquad \text{(I.3}$$

Write out the matrices:

$$A(t) = \|a_{ik}\|_1^n, \quad B(t) = \|b_{ik}(t)\|_1^n, \quad C(t) = \|c_{ik}(t)\|_1^n,$$

[illegible] give the [illegible] their values $q_\nu = q_{\nu 0}(t)$ [illegible] mechanical sense – "the [illegible]" relative to the [illegible] it [illegible] co-ordinates, [illegible] matrices $A(t)$, $B(t)$ and $C(t)$ [illegible] the direct sum of the [illegible] matrix [illegible] as is shown in the following [illegible]:

[illegible] (I.4)

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$$\sum_{i=1}^{m}(a_{\nu i})_0 \frac{d^2x_i}{dt^2} + \sum_{i=1}^{m}\left(\frac{\partial R_\nu}{\partial \dot q_i}\right)_0 \frac{dx_i}{dt} + \sum_{i=1}^{m} c_{\nu i}(t)\,x_i = 0$$

$$(\nu = 1, \ldots, m), \qquad \text{(I.5)}$$

$$\sum_{i=m+1}^{n} (a_{\nu i})_0 \frac{d^2x_i}{dt^2} + \sum_{i=m+1}^{n}\left(\frac{\partial R_\nu}{\partial \dot q_i}\right)_0 \frac{dx_i}{dt} + \sum_{i=m+1}^{n} c_{\nu i}(t)\,x_i = 0$$

$$(\nu = m+1, \ldots, n) \qquad \text{(I.6)}$$

Admit the system of equations (I.3), and consequently the systems of equations (I.5) and (I.6) ~~there are~~ to be correct in the sense of Liapunov /I/ . If among the characteristic numbers of the system (I.5) (or system(I.6)) there are one or any negatives, then the unperturbed motion (I.2) is unstable relative to the variable $q_1, \ldots, q_n$; $\dot q_1, \ldots, \dot q_n$.

§ 2. The whole [illegible]astic free oscillating [illegible].

A simple example of an "[illegible]scillating chain" there is the whole elastic free oscillating chain /[illegible]/ [illegible] to vertical vibrations (It [illegible] when for [illegible] unperturbed motion [illegible] vertical [illegible] given [illegible]. On the fig.1 the [illegible] of [illegible] material points is [illegible] ,con- [illegible] with [illegible]

The springs have a hardness $C_1, \ldots, C_\nu$ and a length in the free state [illegible] .The beginning of the first spring is fastened in point [illegible], the beginning of each of the following are fastened to massless [illegible] their axis are square to vertical [illegible] , which makes the motion plane [illegible] system there are [illegible] points [illegible] Lagrange co-ordinate [illegible] material points [illegible] energy [illegible]

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$$\frac{\partial V}{\partial x_k} = C_k x_{k-1} + (C_k + C_{k+1}) x_k - C_{k+1} x_{k+1} - C_{k+1} \left[(x_k - x_{k-1})^2 + (y_k' - y_{k-1}')^2 \right]^{-1/2} (x_k - x_{k-1}) + C_{k+1} \ell_{k+1} \left[(x_{k+1} - x_k)^2 + (y_{k+1}' - y_k')^2 \right]^{-1/2} (x_{k+1} - x_k) = 0,$$

$$\frac{\partial V}{\partial y_k} = m_k g - C_k y_{k-1}' + (C_k + C_{k+1}) y_k - C_{k+1} y_{k+1}' - C_k \ell_k \left[(x_k - x_{k-1})^2 + (y_k' - y_{k-1}')^2 \right]^{-1/2} (y_k' - y_{k-1}') + C_{k+1} \ell_{k+1} \left[(x_{k+1} - x_k)^2 + (y_{k+1}' - y_k')^2 \right]^{-1/2} (y_{k+1}' - y_k') = 0 \quad (k = 1, \ldots, N). \qquad (2.1)$$

[illegible] $C_{N+1} \ell_{N+1} = 0$. This system admits the solution (the inferior state of equilibrium)

$$x_k = 0, \quad y_k' = (\ell_1 + \lambda_1) + \ldots + (\ell_k + \lambda_k) \quad (k = 1, \ldots, N) \qquad (2.2)$$

where λ_j denotes the statical lengthening of the j spring

$$\lambda_j = \frac{1}{C_j} (m_j + \ldots + m_N) g \quad (j = 1, \ldots, N)$$

Further let us lead in the variables y_k, which are vertical declinations of the k material point from the inferior state of equilibrium

$$y_k = y_k' - (\ell_1 + \lambda_1) - \ldots - (\ell_k + \lambda_k) \quad (k = 1, 2, \ldots, N)$$

The inferior state of equilibrium will

[illegible] to the values of the Lagrange co-ordinate $x_1, \ldots, x_N$; $y_1, \ldots, y_N$ equal zero. The Jacobi determinant of the system of equations (2.1) at values of variables (2.2) does not equal zero. Consequently, the inferior state of equilibrium is isolated (though it is not the single).

§ 3. The asymtotic stability "in large" inferior state of equilibrium of the whole elastic free oscillating chain in the presence of effective resistance forces.

The equations (1.1) in the examined case has form

$$m_k \ddot{x}_k = -\frac{\partial V}{\partial x_k} - R_k(\dot{x}_1, \ldots, \dot{x}_N; \dot{y}_1, \ldots, \dot{y}_N),$$

$$\ldots = -\frac{\partial V}{\partial y_k} - R_{N+k}(\dot{x}_1, \ldots, \dot{x}_N; \dot{y}_1, \ldots, \dot{y}_N). \qquad (3.1)$$

[illegible] us take the complete energy of system [illegible] v of the theorem 14.1 of N.N. [illegible]asovsky [4]

$$v = T + V - V(0, \ldots, 0).$$

Calculate $V(0, \ldots, 0)$ – the value of potential energy in the inferior state of equilibrium:

$$V(0, \ldots, 0) = g \sum_{k=1}^{N} m_k (\ell_1 + \lambda_1) + \ldots + (\ell_k + \lambda_k) + \frac{1}{2} \sum_{k=1}^{N} C_k (\ell_k^2 + \lambda_k^2)$$

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Show, that $V-V(0,\dots,0)$ is the positive- definite function of Lagrange co-ordinates in the sense of Liapunov [1] . Reform $V-V(0,\dots,0)$ to the form

$$V-V(0,\dots,0)=\frac{1}{2}\sum_{\kappa=1}^{\mathcal{N}}c_\kappa\left[(x_\kappa-x_{\kappa-1})^2+(y_\kappa-y_{\kappa-1})^2+\right.$$

$$\left.2\ell_\kappa(\ell_\kappa+\lambda_\kappa+y_\kappa-y_{\kappa-1})-2\ell_\kappa\sqrt{(x_\kappa-x_{\kappa-1})^2+(\ell_\kappa+\lambda_\kappa+y_\kappa-y_{\kappa-1})^2}\right].$$

Evidently that at any physical realizable values $x_1,\dots,x_{\mathcal{N}}, y_1,\dots,y_{\mathcal{N}}$ we will have

$$V(x_1,\dots,x_{\mathcal{N}},y_1,\dots,y_{\mathcal{N}})-V(0,\dots,0)\geq 0$$

and $V-V(0,\dots,0)$ equal zero is possible only at $x_1=\dots=x_{\mathcal{N}}=y_1=\dots=y_{\mathcal{N}}=0$. Consequently, the complete energy of system is the positive- definite function of Lagrange co-ordinates and velocites in the sense of Liapunov. Its derivative, taken in virtue of equations (3.I) equal

$$\frac{d}{dt}\left[T+V\,V(0,\dots,0)\right]=-\sum_{\kappa=1}^{\mathcal{N}}(R_\kappa\dot{x}_\kappa+R_{\mathcal{N}+\kappa}\dot{y}_\kappa)\leq 0.$$

In virtue of (I) in the domain of modification of variables $x_1,\dots,x_{\mathcal{N}}, y_1,\dots,y_{\mathcal{N}}$, $\dot{x}_1,\dots,\dot{x}_{\mathcal{N}}$, $\dot{y}_1,\dots,\dot{y}_{\mathcal{N}}$, for which the inferior state of equilibrium is single, the sign of equality in the last inequality is possible only at $x_1=\dots=x_{\mathcal{N}}=y_1=\dots=y_{\mathcal{N}}=0$.

Such, the conditions of N.N. Krasovskii theorem [2] are satisfied.

Thus the following theorem is demonstrated: At the presence of effective resistance forces, which satisfy the condition (I), the inferior state of equilibrium of the whole elastic free oscillating chain is asymptotic stable in that domain G of the initial declination, for which the inferior state of equilibrium is single.

Remark. Let V_{inf} be the least value of the potential energy among the states of equilibrium, which distinct from the inferior. The closed domain G in space of the initial declination $x_{1o},\dots,x_{\mathcal{N}o}; y_{1o},\dots,y_{\mathcal{N}o}$, $\dot{x}_{1o},\dots,\dot{x}_{\mathcal{N}o}; \dot{y}_{1o},\dots,\dot{y}_{\mathcal{N}o}$ is defined by the inequality

$$T_o+V_o\leq V_{inf}$$

where in the expressions T_o and V_o the values $x_1=x_{1o},\dots,\dot{y}_{\mathcal{N}}=\dot{y}_{\mathcal{N}o}$ are substituted.

§ 4. The equations in variations for vertical vibrations of the whole elastic free oscillating chain

Let us write out the detailed equations (3.I) :

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$$m_\kappa \ddot{x}_\kappa = -c_\kappa (x_\kappa - x_{\kappa-1}) + c_{\kappa+1}(x_{\kappa+1} - x_\kappa) + c_\kappa l_\kappa \Big[(x_\kappa - x_{\kappa-1})^2 + (l_\kappa + \lambda_\kappa + y_\kappa - y_{\kappa-1})^2 \Big]^{-\frac{1}{2}} (x_\kappa - x_{\kappa-1}) - c_{\kappa+1} l_{\kappa+1} \Big[(x_{\kappa+1} - x_\kappa)^2 + (l_{\kappa+1} + \lambda_{\kappa+1} + y_{\kappa+1} - y_\kappa)^2 \Big]^{-\frac{1}{2}} (x_{\kappa+1} - x_\kappa) - R_\kappa (\dot{x}_1, \dots, \dot{x}_N, \dot{y}_1, \dots, \dot{y}_N),$$

(4.I)

$$m_\kappa \ddot{y}_\kappa = -c_\kappa l_\kappa + c_{\kappa+1} l_{\kappa+1} - c_\kappa (y_\kappa - y_{\kappa-1}) + c_{\kappa+1}(y_{\kappa+1} - y_\kappa) + c_\kappa l_\kappa \Big[(x_\kappa - x_{\kappa-1})^2 + (l_\kappa + \lambda_\kappa + y_\kappa - y_{\kappa-1})^2 \Big]^{-\frac{1}{2}} (l_\kappa + \lambda_\kappa + y_\kappa - y_{\kappa-1}) - c_{\kappa+1} l_{\kappa+1} \Big[(x_{\kappa+1} - x_\kappa)^2 + (l_{\kappa+1} + \lambda_{\kappa+1} + y_{\kappa+1} - y_\kappa)^2 \Big]^{-\frac{1}{2}} (l_{\kappa+1} + \lambda_{\kappa+1} + y_{\kappa+1} - y_\kappa) - R_{N+\kappa} (\dot{x}_1, \dots, \dot{x}_N, \dot{y}_1, \dots, \dot{y}_N) \quad (\kappa = 1, \dots, N).$$

Let us assume, that the projections on the axis Ox of the resistance forces satisfy the conditions

$$R_\kappa (0, \dots, 0, \dot{y}_1, \dots, \dot{y}_N) = 0 \quad (\kappa = 1, \dots, N)$$

The system (4.I) allows the solution (unperturbed motion(I.2)) :

$$x_\kappa = x_{\kappa 0}(t) = 0, \quad y_\kappa = y_{\kappa 0}(t) \quad (\kappa = 1, \dots, N) \tag{4.2}$$

and these $y_{\kappa 0}(t)$ satisfy the system of equations

$$\ddot{y}_{\kappa 0}(t) + \frac{1}{m_\kappa} R_{y\kappa}(0, \dots, 0, \dot{y}_{n0}(t), \dots, \dot{y}_{N0}(t)) = p_\kappa + (p_\kappa + q_{\kappa+1} p_{\kappa+1}) y_{\kappa 0}(t) - q_{\kappa+1} l_{\kappa+1} \dots$$

$(\kappa = 1, \dots, N)$

, where

$$p_\kappa = \frac{c_\kappa}{m_\kappa} (\kappa = 1, \dots, N), \quad q_\kappa = \frac{m_\kappa}{m_\kappa} \quad \kappa = 2, \dots, N \quad q_{\kappa+1} \dots$$

It is easy to examine, that the conditions (I.4) are fulfilled, if

$$\left(\frac{\partial R_\kappa}{\partial \dot{y}_j} \right)_0 = 0, \quad \left(\frac{\partial R_{N+\kappa}}{\partial x_j} \right)_0 = 0 \quad (j, \kappa = 1, \dots, N), \tag{4.3}$$

where the index zero denotes, that after differentiation, the values of arguments from (4.2) are substituted. We will assume the conditions (4.3) are executed. Consequently, the equations in variations (I.5) and (I.6) are just for perturbed motion ($x_\kappa = 0 + \xi_\kappa$, $y_\kappa = y_{\kappa 0}(t) + \eta_\kappa$, $\kappa = 1, \dots, N$) :

$$\frac{d^2 \xi_\kappa}{dt^2} + \frac{1}{m_\kappa} \sum_{j=1}^{N} \left(\frac{\partial R_\kappa}{\partial \dot{x}_j} \right)_0 \frac{d\xi_j}{dt} + \frac{1}{m_\kappa} \sum_{j=1}^{N} \left(\frac{\partial^2 V}{\partial x_\kappa \partial x_j} \right)_0 \xi_j = 0,$$

$$\frac{d^2 \eta_\kappa}{dt^2} + \frac{1}{m_\kappa} \sum_{j=1}^{N} \left(\frac{\partial R_{N+\kappa}}{\partial \dot{y}_j} \right)_0 \frac{d\eta_j}{dt} + \frac{1}{m_\kappa} \sum_{j=1}^{N} \left(\frac{\partial^2 V}{\partial y_\kappa \partial y_j} \right)_0 \eta_j = 0,$$

or developped

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$$\frac{d^2\xi_\kappa}{dt^2}+\frac{1}{m_\kappa}\sum_{i=1}^{N}\left(\frac{\partial R_\kappa}{\partial \dot{x}_i}\right)_0\frac{d\xi_i}{dt}-\rho_\kappa\left\{1-\left[1+\gamma_\kappa+(y_{\kappa 0}(t)-\right.\right.$$

$$\left.\left.-y_{(\kappa-1,0}(t))/\ell_\kappa\right]^{-1}\right\}(\xi_{\kappa-1}-\xi_\kappa)+\mu_{\kappa+1}\rho_{\kappa+1}\left\{1-[1+\right.$$

$$+\gamma_{\kappa+1}+(y_{\kappa+1,0}(t)-y_{\kappa 0}(t))/\ell_{\kappa+1}]^{-1}\Big\}(\xi_\kappa-\xi_{\kappa+1})=0, \qquad (4.5)$$

$$\frac{d^2\eta_\kappa}{dt^2}+\frac{1}{m_\kappa}\sum_{i=1}^{N}\left(\frac{\partial R_{N+\kappa}}{\partial y_i}\right)_0\frac{d\eta_i}{dt}-\rho_\kappa\eta_{\kappa-1}+$$

$$+(\rho_\kappa+\mu_{\kappa+1}\rho_{\kappa+1})\eta_\kappa-\mu_{\kappa+1}\rho_{\kappa+1}\eta_{\kappa+1}=0 \qquad (4.6)$$

$(\kappa=1,\dots,N)$, besides $\gamma_\kappa=\lambda_\kappa/\ell_\kappa\ (\kappa=1,\dots,N,\ \gamma_{N+1}=0)$.

Let us proceed to study of a conservative case, when the theorem § 3 is not just.

§ 5. The conservative case

At the absence of the resistance forces the whole elastic free oscillating chain is a conservative system. The declinations $y_{\kappa 0}(t)$ of their mass for the inferior state of equilibrium at the vertical vibrations (the unperturbed motion) will satisfy the system (4.3) at $R_{N+\kappa}=0\ (\kappa=1,\dots,N)$, which describes small vibrations of the Sturm system /5/. The frequencies equation of this system is a characteristic equation of some Jacobi matrix

$$\begin{vmatrix} \omega^2(\rho_1+\mu_2\rho_2) & -\mu_2\rho_2 & 0 & \dots & 0 \\ -\mu_2\rho_2 & \omega^2(\mu_2\rho_2+\mu_3\rho_3) & -\mu_3\rho_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \omega^2\rho_N \end{vmatrix}=0.$$

The equations (4.5) from the system of equations in variations at $R_{\kappa+N}=0\ (\kappa=1,\dots,N)$ will have periodic coefficients, if the frequencis $\omega_1,\dots,\omega_N$ are commensurable, or almost - periodic, - otherwise. Let us consider the simpliest example.

Example. The one link whole elastic free oscillating chain is a mathematical pendulum with a mass m on the spring at a relaxed length ℓ and a harmness C (fig.2). The system of equations (4.3) results in

$$m\ddot{y}_0+cy_0=0$$

and will be for unperturbed motion

$$x=0,\ y=y_0(t)=\gamma\cos\omega t\quad(\omega=\sqrt{\tfrac{c}{m}}).$$

The equations in variations (4.5) and (4.6) will be written in the form

$$\frac{d^2\xi}{dt^2}+\omega^2\left[1-\frac{1}{1+\vartheta+\gamma\cos\omega t}\right]\xi=0, \qquad (5.1)$$

$$\frac{d^2\eta}{dt^2}+\omega^2\eta=0.$$

The transformation of independent variable [illegible] the differential equa-

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tions of perturbed motion into

$$\frac{d^2\xi}{d\tau^2} + \frac{\gamma + a\cos\tau}{1+\gamma+a\cos\tau}\xi + E(\tau,\xi,\eta) = 0,$$

$$\frac{d^2\eta}{d\tau^2} + \eta + H(\tau,\xi,\eta) = 0 \quad \left(\gamma = \frac{\lambda}{\ell},\ a = \frac{Y}{\ell}\right).$$

The stability or instability of the trivial solution in variations (5.I) is defined by the first equation (5.I). In general in the examined conservative case the stability of the trivial solution of systems (5.I) does not define , the stability of unperturbed motion relative to the variables $x, y, \dot{x}, \dot{y}$, as we have here one of the critical case. But the instability of trivial solutions of system (5.I) attracts (perhaps excluding boundary cases) the instability of unperturbed motion relative to variables $x, y, \dot{x}, \dot{y}$ ([2] § 70). This is specified by the first equation in variations (5.I) has as coefficient a periodic function and if this trivial solution is instable, the characteristic number in the sense of Liapunov [I] is negative.

For definition of the domains of instability by the method of the small parameter in the form given by V.A.Yakubovic [6] let us admit that a is the small parameter and write down the first equation (5.I) in the form

$$\frac{d^2\xi}{d\tau^2} + \left[p_0(\gamma) + a p_1(\tau,\gamma) + a^2 p_2(\tau,\gamma) + \dots\right]\xi = 0,$$

where

$$p_0(\gamma) = \frac{\gamma}{1+\gamma},\quad p_1(\tau,\gamma) = 2p_1^{(1)}(\gamma)\cos\tau,\quad p_1^{(1)} = \frac{1}{2(1+\gamma)^2}.$$

In the scalar case the domains of instability in the parameters plan $a\gamma$ may adjoin on axis a to these γ_k points, which are the roots of equation

$$2\sqrt{p_0(\gamma_k)} = k, \quad \text{hence} \quad \gamma_k = -\frac{k^2}{4-k^2} \quad (k = 1, 2, \dots).$$

At $\gamma > 0$ the broad domain of instability (it is at an angle differing from zero formed by the tangents) adjoins but only to point $\gamma_1 = \frac{1}{3}$. The tangent of declivity angle of the tangent in our example is defined by the formula, obtained from formula (6) [6].

$$\alpha^{\pm} = \pm \left.\frac{p_1^{(1)}}{\frac{dp_0}{d\gamma}}\right|_{\gamma=\gamma_1} \pm \frac{1}{2}$$

Hence we define the domain of instability of the vertical vibrations pendulum on the spring in the first approximation:

$$\frac{1}{3} - \frac{1}{2}a + \dots \gamma \quad \frac{1}{3} + \frac{1}{2}a,$$

The straights continius this domain, are dotted on fig. 2.

In the second approximation the domain

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of instability is defined by inequalities (calculated by B.G. Pittel)

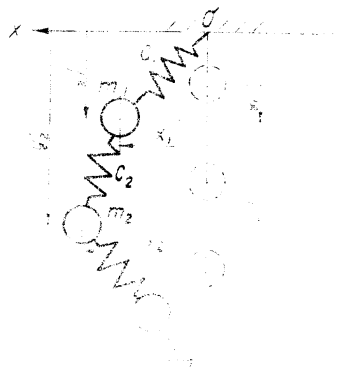
$$\frac{1}{2} - \frac{1}{2}a + \frac{5}{128}a^2 + \dots < \gamma < \frac{1}{3} + \frac{1}{2}a + \frac{5}{128}a^2 + \dots$$

The curves, limiting this domain, are dash- dotted on fig.3.

The investigation of the transition of the longitudinal vibrations into transversal ones at the values of parameters of the system, pertaining to the domains of instability, are started by E. Mettler [3].

The method of low alteration of amplitude and phase, proposed by N.N. Bogoliubov and N.N.Krylov [4] and developped by Q.A. Mitropolsky [6] is used.

Conclusion. The domains of "conservative instability" will create in the dissipative case the domains of "large perturbance" on the final time - interval. Nevertheless the asymptotic damping of the vibration, which will be slowly at a little dissipation, practically great alteration due to the autoresonance in the oscillating [illegible] of great importance in the evaluation of the [illegible].



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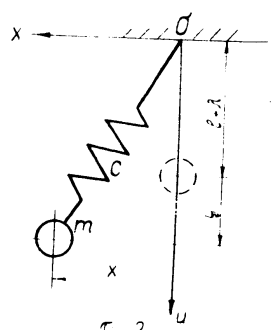

Fig 2.

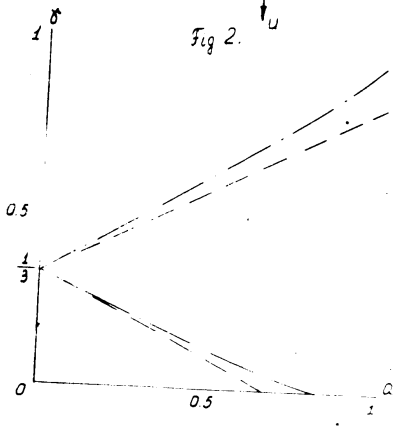

Fig 3

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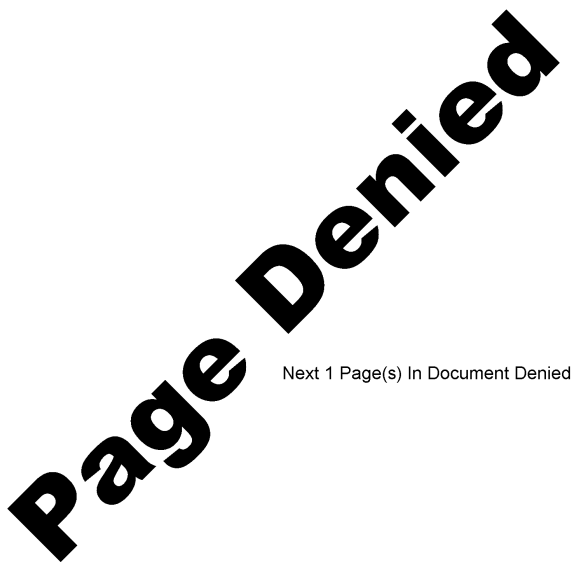

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ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ НЕЛИНЕЙНЫХ КОНЕЧНО-РАЗНОСТНЫХ УРАВНЕНИЙ И ИХ УСТОЙЧИВОСТЬ

The development of nonlinear impulse systems and in particular of digital automatic systems, described by nonlinear finite-difference equations put forward the problem to study the solution of sucn equations [1].

Tnis report deals with precise methods of determining of symmetrical periodic solutions and studying of their stability.

To find low-frequency periodic solutions laborious computations are needed. To this end it proves to be convenient to use approximate function technique.

Let us consider nonlinear finite-difference equation of this type

$$\sum_{\nu=0}^{\ell} \alpha_\nu x[n+\nu] = \sum_{\nu=0}^{\ell_1} \beta_\nu y[n+\nu]$$
$$y[n] = \Phi(x[n]) \qquad (1)$$

Here $\Phi(x)$ is nonlinear symmetrical function and $\ell > \ell_1$. Let us find symmetrical periodic so-

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lutions $\tilde{x}[n]$ of frequence $\bar{\omega} = \frac{\pi}{N}$ or half-period N where N is an integer.

Periodic solutions of finite-difference equations can be represented by Fourier polinomial *)

$$\tilde{x}[n] = \frac{1}{2} \sum_{\kappa=-N_1}^{N_1}{}' a_\kappa e^{j \frac{\kappa \bar{\omega}}{N} n} \tag{2}$$

where

$$a_\kappa = a_\kappa e^{j \varphi_\kappa} \tag{3}$$

are the required complex amplitudes. The prime means that summation involves only odd indexes and

$$N_1 = \begin{cases} N-1 & \text{if } N \text{ is even} \\ N & \text{if } N \text{ is odd} \end{cases} \tag{4}$$

Under periodic changing $\tilde{x}[n]$ the value $\tilde{y}[n]$ will be periodic also. We write it in the form of fourier polinomial and we have:

$$\tilde{y}[n] = \Phi(\tilde{x}[n]) = \Phi\left(\frac{1}{2} \sum_{\kappa=-N_1}^{N_1}{}' a_\kappa e^{j \frac{\kappa \bar{\omega}}{N} n}\right) = \frac{1}{2} \sum_{\kappa=-N_1}^{N_1}{}' b_\kappa e^{j \frac{\kappa \bar{\omega}}{N} n} \tag{5}$$

The coefficients of this polinomial, being complex amplitudes, are defined by well-known formula of harmonic analysis, Bessel's formula :

$$b_\kappa = b_\kappa e^{j \varphi_\kappa} = \frac{2}{N} \sum_{\nu=0}^{N-1} \Phi(\tilde{x}[\nu]) e^{-j \frac{\kappa \bar{\omega}}{N} \nu} \tag{6}$$

and, if N is odd, for $\kappa = \pm N$

$$b_{-N} = b_{+N} = \frac{1}{N} \sum_{\nu=0}^{N-1} \Phi(\tilde{x}[\nu]) e^{-j \kappa \bar{\omega} \nu} \tag{7}$$

It is evident that b_κ are functions of a_ν,

*) See [2] about physical reasons for this form of solution.

φ_ν ($\nu = 1, 3, \ldots, N_1$), i.e

$$b_\kappa = b_\kappa (a_{\nu_1}, \varphi_{\nu_1}, N) \tag{8}$$

We substitute $\tilde{x}[n]$ from (2) and $\tilde{y}[n]$ from (5) into the equation (1) and equating coefficients at equal powers of $e^{j \frac{\bar{\omega}}{N} n}$ we obtain the system of nonlinear algebraic equations defining desired complex amplitudes

$$a_\kappa Q^*\left(j \frac{\kappa \bar{\omega}}{N}\right) = b_\kappa P^*\left(j \frac{\kappa \bar{\omega}}{N}\right) \quad \kappa = 1, 3, \ldots N_1 \tag{9}$$

where

$$Q^*\left(j \frac{\kappa \bar{\omega}}{N}\right) = \sum_{\nu=0}^{\ell} \alpha_\nu e^{j \frac{\kappa \bar{\omega}}{N} \nu}, \quad P^*\left(j \frac{\kappa \bar{\omega}}{N}\right) = \sum_{\nu=0}^{\ell_1} \beta_\nu e^{j \frac{\kappa \bar{\omega}}{N} \nu} \tag{10}$$

We can find also the equation for periodic solution $\tilde{x}[n]$ in the interval $0 \le n \le N-1$ To this end we substitute into (2) the values a_κ from (9); then we have

$$\tilde{x}[n] = \frac{1}{2} \sum_{\kappa=-N_1}^{N_1}{}' b_\kappa \frac{P^*\left(j \frac{\kappa \bar{\omega}}{N}\right)}{Q^*\left(j \frac{\kappa \bar{\omega}}{N}\right)} e^{j \frac{\kappa \bar{\omega}}{N} n} \tag{11}$$

Then, substituting for b_κ the values from (6), (7) and reversing the order of summation, the equation (11) will be of the form:

$$\tilde{x}[n] = \frac{1}{N} \sum_{\nu=0}^{N-1} A_{n-\nu}(N) \Phi(\tilde{x}[\nu]) \tag{12}$$

where

$$A_{n-\nu}(N) = \sum_{\kappa=-N+1}^{N-1}{}' \frac{P^*\left(j \frac{\kappa \bar{\omega}}{N}\right)}{Q^*\left(j \frac{\kappa \bar{\omega}}{N}\right)} e^{j \frac{\kappa \bar{\omega}}{N} (n-\nu)} \tag{13}$$

if N is odd and

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$$A_{n-\nu}(N)=\sum_{\kappa=-N+2}^{N-2}\frac{P^*(j\frac{\kappa\bar{x}}{N})}{Q^*(j\frac{\kappa\bar{x}}{N})}e^{j\frac{\kappa\bar{x}}{N}(n-\nu)}+\frac{P^*(j\bar{x})}{Q^*(j\bar{x})}e^{j\bar{x}(n-\nu)} \qquad (14)$$

if it is even.

The system of nonlinear algebraic equations (12) defines the values of periodic solution.

Thus, the problem of defining of desired periodic solution is reduced to the solution of finite system of nonlinear algebraic equations (9) and (12).

In special cases $N=1,2$ these equations are easily solved by graphic methods [2].

Thus, for $N=1$ (the most high frequency periodic solutions) we get from (12) and (13)

$$\tilde{x}[0]=A_0(1)\,\Phi(\tilde{x}[0]) \qquad (15)$$

where

$$A_0(1)=\frac{P^*(j\bar{x})}{Q^*(j\bar{x})} \qquad (16)$$

and periodic solution is determined as abscissa of the point of intersection of the line with slop $1/A_0(1)$ on the plot of nonlinear function $\Phi(x)$ (fig.1). This result can be obtained from the equation (9) too, under $k=1$.

Under $N=2$ it is more convenient to use the equations (4). Putting $\kappa=1$ and $N=2$ we have

$$\frac{Q^*(j\frac{\bar{x}}{2})}{P^*(j\frac{\bar{x}}{2})}=\frac{\beta_1}{a_1}\,S^*(a_1,\varphi_1,2) \qquad (17)$$

where in virtue of the equations (6), (5) and (2)

$$S^*(a_1,\varphi_1,2)=\frac{1}{a_1}\sum_{\nu=0}^{1}\Phi\left(a_1\cos\left(\frac{\bar{x}}{2}\nu+\varphi_1\right)\right)e^{-j\left(\frac{\bar{x}}{2}\nu+\varphi_1\right)} \qquad (18)$$

Let us plot S^* in complex plane in the form of a family of curves of the parameter φ_1 (every point on the curves is identified by digits a_1) and put on this family the point corresponding to $Q^*(j\frac{\bar{x}}{2})/P^*(j\frac{\bar{x}}{2})$. Then the digits and the parameter of the curve on which this point occurs define the amplitude and the phase of the desired periodic solution (fig.2).

For $N\geq 3$ the equations (9) and (12) can be solved generally speaking by well-known numerical methods. However, these methods are often very laborious [3]. Therefore in a number of cases it is convenient to limit oneself with receiving approximate periodic solution. Suppose, periodic solution is close to harmonic, that is complex amplitudes α_κ $(\kappa=3,5,\dots,N-2)$ are so small that they can be neglected; then

$$\tilde{x}[n]\approx a_1\cos\left(\frac{\bar{x}}{N}n+\varphi_1\right). \qquad (19)$$

This supposition will be appropriate (9) if the following inequalities hold:

$$\left|\frac{P^*(j\frac{\kappa\bar{x}}{N})}{Q^*(j\frac{\kappa\bar{x}}{N})}\right|\ll\left|\frac{P^*(j\frac{\bar{x}}{N})}{Q^*(j\frac{\bar{x}}{N})}\right| \qquad \kappa=3,5,\dots,N-2, \qquad (20)$$

as it is readily seen from (9).

In this case the system of equations (9) is reduced to one, corresponding to $\kappa=1$ and this

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one can be represented as follows:

$$\frac{Q^*(j\frac{\pi}{N})}{P^*(j\frac{\pi}{N})} = \frac{\beta_1}{a_1} = S^*(a_1,\varphi_1,N) \qquad (21)$$

where taking into account the equations (5) and (3) we have:

$$S^*(a_1,\varphi_1,N) = \frac{1}{Na_1}\sum_{\nu=0}^{N-1}\Phi\left(a_1\cos(\frac{2\pi}{N}\nu+\varphi_1)\right)e^{-j(\frac{2\pi}{N}\nu+\varphi_1)} \qquad (22)$$

The equation (21) is of the form of the equation (17) and the parameters of the periodic solution a_1 and φ_1 can be found in the way described above. The equations (21) are the same thing for nonlinear finite-difference equations the well-known equations of harmonic balance are for nonlinear differential equations. It is curious to note that for $N=1,2$ these equations give a precise result, not approximate. For physical sampled-data nonlinear systems $P^*(j\frac{\pi}{N})/Q^*(j\frac{\pi}{N})$ is frequency characteristic, and $S^*(a_1,\varphi_1,N)$ is equivalent gain coefficient of nonlinear element (describing function) [2].

Let us consider now stability of periodic solution. For this we shall write the equations of variations. Denote $\xi[n]$ the small variation of the periodic solution $\tilde{x}(n)$.

Then disturbed solution will be equal to

$$x[n] = \tilde{x}[n] + \xi[n] \qquad (23)$$

Substituting (23) into the equation (1) and substracting (23) the equation of periodic regime under $|\xi[n]| << 1$ we obtain the finite difference equation with periodically changing coefficients:

$$\sum_{\nu=0}^{\ell}\alpha_\nu\xi[n+\nu] = \sum_{\nu=0}^{\ell}\beta_\nu\Phi'\left(\tilde{x}[n+\nu]\right)\xi[n+\nu] \qquad (24)$$

where $\Phi'(x) = \frac{d\Phi(x)}{dx}$.

The stability of the solutions of this equation determines the stability of periodic solution [4]. As the function $\Phi(x)$ is odd $\Phi'(x)$ will be even and this function can be represented in the form of Fourier polinomial:

$$\Phi'(\tilde{x}[n]) = \Phi'\left(\frac{1}{2}\sum_{k=-M}^{M}a_k e^{j\frac{k2\pi}{N}n}\right) = \frac{1}{2}\sum_{k=-M}^{M}d_{2k}e^{j\frac{2k2\pi}{N}n} \qquad (25)$$

where $M=[\frac{N}{2}]$ is the greatest integer contained in $N/2$ and $d_{2k}=d_{2k}e^{j\vartheta_k}$ are complex amplitudes of harmonics, defined by formula similar to (6), (7).

We seek for the solution of the equation (24) in the form:

$$\xi[n] = \frac{e^{\beta n}}{2}\sum_{z=-M}^{M}c_{2z}e^{j\frac{2z2\pi}{N}n} \qquad (26)$$

Substituting (25) and (26) into (24) and equating the coefficients under the equal $e^{j\frac{k2\pi}{N}n}$ we obtain after simple transformations the finite system of equations:

for odd $N=2M+1$

$$\frac{Q^*(q+j\frac{2z\pi}{N})}{P^*(q+j\frac{2z\pi}{N})}c_{2z} + \frac{1}{2}\sum_{k=-M}^{M}d_{2k}c_{2(z-k)} = 0 \qquad (27)$$

for even $N=2M$

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$$\frac{Q^*(q+j\frac{2\pi k}{N})}{P^*(q+j\frac{2\pi k}{N})}\,\dot{\Xi}_{2l} + \frac{1}{2}\sum_{k=M+1}^{M-1} d_{2k}\,\dot{\Xi}_{2(l-k)} + \frac{d_{2m}+d_{2m}}{2}\,\dot{\Xi}_{2(l-M)} = 0 .$$

Moreover

$$\dot{\Xi}_{\pm 2|k|} = \dot{\Xi}_{\pm 2|k|}$$

under

$$|k| + |2| = N .$$

The systems of linear algebraic equations (2.7), (2,8) have non trivial solution, if the determinants of the system are equal to zero*), that is

$$\Delta_N^*(q) = 0 .$$

Denote the roots of this equation by q_k. Then as one can see from the form of the solution $\xi[n]$ (26), the latter will be stable, and consequently the periodic solution will be stable, if the roots of the characteristic equation (29) will have negative real parts. For studying stability one can use the stability criterions well-known from the theory of sampled-data systems.

Let us note that the similar approach to the problem of studying stability of periodic solutions for nonlinear differential equation gives us a characteristic determinant of infinite order [7].

*) Somewhat different approach of receiving the determinant, based on discrete Laplace transformation, is described in [5].

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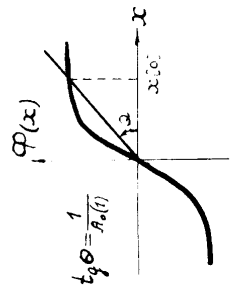

Fig. 1

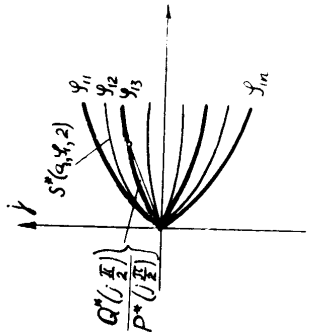

Fig.

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Introduction

In this report the authors consider some of their results concerning the theory of differential equations, containing a small parameter.

The question of dependence upon parameters of solutions of some differential equations arises in the investigation of some problems of automatic control, nonlinear oscillations, etc.

The right parts of the equations depending continuously on parameters, the solutions of the equations are also continuous functions of the parameters.

But some problems lead us to the system where the dependence upon the parameters is not continuous. The general class including such problems may be reduced to the investigation of the systems of differential equations, containing small parameters in the derivatives.

These systems may be written in the form

$$\mu \frac{dz}{dt} = F(z,y,t), \quad \frac{dy}{dt} = f(z,y,t) \qquad /1/$$

where $y=\{y_1, \dots, y_n\}$, $z=\{z_1, \dots, z_n\}$, $\mu = \begin{Vmatrix} \mu_1 & & 0 \\ & \mu_2 & \\ 0 & & \mu_n \end{Vmatrix}$ and μ_i - small parameters. When $\mu = 0$ the equation /1/ becomes a degenerate system

$$F(z,y,t)=0, \quad \frac{dy}{dt} = f(z,y,t). \qquad /2/$$

One of the problems dealing with the systems /1/ is to investigate the dependence of solutions upon μ when $\mu \to 0$. It is necessary to find out whether the solution of the original system /1/ which satisfies some initial conditions

$$y(t_0,\mu)=y_0, \quad z(t_0,\mu)=z_0, \qquad /3/$$

tends to the solution of the degenerate system /2/ when $\mu \to 0$, and if it does then we should find out what additional conditions define this solution and in particular how

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to choose the root of the equation $F(z, y, t) = 0$ if this equation is solvable with respect to z and has several roots. The problems of this kind were considered by A.N.Tychonov [5], I.S.Gradstein [7] and other authors.

In the papers of A.N.Tychonov [5] there is a treatment of some conditions under which the solutions of /1/ tend to the solutions of /2/ when $\mu \to 0$. We give the results of A.N.Tychonov below § 1.

In those cases when the solutions of /1/ tend to the solutions of /2/ we come to the problem of finding the asymptotic approximations with respect to small parameter μ for the solution of the original system /1/.

These problems were discussed in the papers of A.B.Vasilyeva /4/. In the papers of A.B. Vasilyeva /4/ for the solutions of /1/ the following expansion is given:

$$x(t, \mu) = x(t, 0) + \mu x_{\mu}(t, 0) + \ldots + \frac{\mu^n}{n!} x_{\mu^n}(t, 0) + \ldots, \quad x = \{y, z\}. \qquad /4/$$

/for simplicity we now regard the case when /1/ contains only one small parameter $\mu = \mu_1$ /. The formula /4/ is similar to Maclaurin's formula, but in some respect it is considerably different from it. In Maclaurin's formula we get the derivatives x_{μ^k} from variation equations, however it turns out that if x_{μ^k} calculated under usual initial values $x_{\mu^k}(t_0, 0) = 0$, the results will be wrong - the expansion /4/ will not suit the solution of the system /1/ when $\mu \to 0$. It is shown that the derivatives x_{μ^k} are to be calculated with some initial values unequal to zero. A.B.Vasilyeva gets the formulas for the above mentioned "displacements" of initial values. However Maclaurin's formula even modified in this way does not give us an asymtotic on the whole interval where the solution is considered. In her papers A.B.Vasilyeva also deduced asymptotic formulas which are uniform on the whole segment under consideration. These formulas make it possible to investigate some boundary value problems for the systems /1/ and to construct asymtotic for them.

A.B.Vasilyeva's results will be given in § 2. In the case when the equation $F(z, y, t) = 0$ is insolvable with respect to z one of the trends of investigation of the systems of equations containing a small parameter is connected with the problem of averaging the right parts of equations. For the systems

$$\dot{x} = \varepsilon X(x, t) \qquad /1/$$

/the so-called systems in a standard form / and for the systems of the type

$$\dot{x} = \varepsilon X(x, \psi), \quad \dot{\psi} = \omega(x) + \varepsilon Y(x, \psi) \qquad /1/$$

/the systems with a rotating phase/ where ε - a small parameter, the averaging problem was discussed in the works of N.N.Bogoliubov and his disciples [3]. For the system of more general type

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon), \quad \dot{y} = Y(x, y, t, \varepsilon) \qquad /7/$$

$(x = \{x_1, \ldots, x_n\}, \; y = \{y_1, \ldots, y_m\})$, the particular cases of which /5/ and /6/, the averaging problem may also be discussed. This problem was considered by V.M.Volosov [6].

It can be easily seen that the system /7/ may be obtained from the form /1/ by means of changing the independent variable.

The notion of averaging in the papers of V.M.Volosov has been generalized in the following way. Let $F(x, y, t)$ be a certain function. Then we assume that its average value is the limit

$$\bar{F} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0 + T} F(x, y(x, t), t) \, dt, \qquad /8/$$

where $y(x, t)$ is the solution of the unperturbed system

$$\dot{y} = Y_0(x, y, t) \equiv Y(x, y, t, 0), \quad x = const, \qquad /9/$$

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into which /7/ is transformed when $\varepsilon = 0$.

By means of such process of averaging along the degenerate trajectory we get averaged systems which give us asymptotic approximations with respect to the parameter ε for the solution of /7/ on a long interval of time $t \sim \frac{1}{\varepsilon}$.

The results of V.M.Volosov are given in §§ 4, 6-10. The above mentioned method of averaging was also applied in the paper of V.M.Volosov to the investigation of some problems of the theory of nonlinear oscillation systems, containing slowly varying parameters.

§ 1. The Limit in the System I under $\mu \to 0$

Now we shall consider the results obtained by A.N.Tychonov /16/. In the paper /Nat. sb. 31 /73/, Nu.3, 1952/ the general case is investigated when the system /I/ contains several small parameters of different orders of smallness. For simplicity we shall consider here the particular case when the system /I/ contains only one small parameter ($\mu_1 = \dots = \mu_m = \mu$). Let $z = \varphi(y,t)$ be a solution of the equations $F(z,y,t) = 0$ defined in the bounded closed domain $D(y,t)$. The system of equations

$$\frac{dz}{d\tau} = F(z, y^*, t^*) \qquad /10/$$

in which y^*, t^* are regarded as parameters we shall call the adjoining system corresponding to the system /I/. The root $z = \varphi(y,t)$ is considered to be stable if the points $\varphi(y^*,t^*)$ for all values $(y^*, t^*) \in D$ are Lyapunov's asymptotic stability points of the system /10/. Then, we [illegible] the neighbourhood of the surface $z = \varphi(y,t)$ which satisfies the following conditions: the point (z^*, y^*, t^*) belonging to this neighbourhood, the solution $z(\tau)$ of the system /10/ which satisfies the initial conditions $z_{\tau=0} = z^*$ tends to $\varphi(y^*,t^*)$ when $\tau \to \infty$. A.N.Tychonov calls this neighbourhood the domain of influence of the stable root $z = \varphi(y,t)$.

The idea of investigating the solutions $z(t,\mu), y(t,\mu)$ of the system /I/, satisfying the initial conditions z_0, y_0, t_0 is the following: if the initial point z_0, y_0, t_0 is outside the surface $F(z,y,t) = 0$ then the function z changes rapidly in the neighbourhood of $t = t_0$ /in comparison with the speed of y the speed of changing z has the order $\frac{1}{\mu}$ /. That is why the properties of the system /I/ can be sufficiently characterized by means of a simpler system

$$\mu \frac{dz}{dt} = F(z, y_0, t_0), \quad z_{t=t_0} = z_0 \quad \text{or}$$

$$\frac{dz}{d\tau} = F(z, y_0, t_0), \quad z_{\tau=0} = z_0 \qquad /11/$$

i.e. with the help of the adjoint system /10/, with the initial point $z^* = z_0, y^* = y_0, t^* = t_0$. If the point (z_0, y_0, t_0) belongs to the domain of influence of the stable root $z = \varphi(y,t)$ then the solution $z(\tau)$ of the system /11/ together with $z(t,\mu)$ will quickly approach the value $\varphi(y_0, t_0)$ /with a speed $\sim \frac{1}{\mu}$ /. After this speed of z and that of y will have equal orders. The root $z = \varphi(y,t)$ being stable, the integral curve of the system /I/ which has reached a small neighbourhood of the root will never leave it, and the behaviour of the integral curve of the system /I/ will be described further by another auxiliary system, namely, the degenerate system /2/.

The investigation of A.N.Tychonov shows that whenever the root $z = \varphi(y,t)$ is stable and the initial point (z_0, y_0, t_0) belongs to its domain of influence, then

$$\lim_{\mu \to 0} y(t,\mu) = y(t) \qquad /12/$$

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$$\lim_{\mu \to 0} z(t,\mu) = \bar{z}(t) = \varphi(\bar{y}(t),t) \,, \qquad /13/$$

where $\bar{y}(t)$ is the solution of the degenerate system /2/ corresponding to the root $z = \varphi(y,t)$ and satisfying the initial condition $\bar{y}|_{t=t_0} = y_0$. The convergence /13/ is uniform on the segment $t_0 \le t \le T$ ($T < T_0$, where T_0 - is the value of t, when the curve $\bar{y}(t)$ leaves the domain $\mathcal{D}$). The convergence /13/ takes place on the interval $t_0 < t \le T$ but it is not uniform. At the initial point we have

$$\lim_{} z(t,\mu) = z_0 \,,$$

and therefore the limit function corresponding to z, is discontinuous on the segment $t_0 \le t \le T$. The behaviour of the integral curve near the initial point differs from its behaviour far from it. As a result we consider two domains: a small neighbourhood around the initial point in which $z(t,\mu)$ varies quickly and rapidly moves from the initial point z_0, y_0, t_0 to the root $z = \varphi(y,t)$, the other part of the segment $t_0 \le t \le T$ - the domain of slow movement /2/ with the order of the speed equal to y / where the curve "clings" along the root $z = \varphi(y,t)$. In the domain of slow movement, i.e. on the segment $t_1 \le t \le T$, where $t_1 > t_0$ is fixed and is arbitrarily close to t_0, the convergence /13/ is also uniform.

This result concerns the bounded interval of the argument t. Some problems of the limit when $\mu \to 0$ on the unbounded interval of time were considered by I.S.Gradstein /7/.

§ 2. On Discontinuous Periodic Solutions

When the system /2/ has several stable roots defined in bounded domains, discontinuous periodic solutions of the system /2/ may appear. This phenomenon is connected with the fact that as $\mu \to 0$ the solution of /1/ placed in the neighbourhood of a certain root, on reaching the boundary of its domain of definition may rapidly go over to the region of the neighbouring stable root. There is a treatment of these problems in the works of A.A.Dorodnitzin /8/, A.N.Tychonov /15/, N.Levinson /17/, L.S.Pontryagin /14/, E.F. Mischenko /15/.

§ 3. Construction of Asymptotic Approximations of Solutions of the Systems /1/

Now after we have stated the conditions when the solution of the system /1/ converges to the solution of the degenerate system /2/, it is natural to raise a question of constructing of asymtotic approximations for a solution with respect to the small parameter μ. These problems have been investigated by A.B.Vasilyeva.

If the right parts of the system regularly depend on the parameter μ:

$$\frac{dx}{dt} = f(x,t,\mu) \qquad /14/$$

then Maclaurin's usual formula may serve as an asymptotic expansion for the solution, satisfying the initial condition $x|_{t=t_0} = x_0$:

$$x(t,\mu) = x(t,0) + \mu x_\mu(t,0) + \dots + \frac{\mu^n}{n!} x_{\mu^n}(t,0) + R_{n+1} \,. \qquad /15/$$

The coefficients of this expansion can be found by means of variation equations which we get by inserting /15/ in /14/ and then equalling the coefficients in the similar powers of μ. The initial conditions of variation equations are the following:

$$x_{\mu^k}(t,0)|_{t=t_0} = 0 \qquad /16/$$

i.e. when $t = t_0$ the functions $x(t,0), x_\mu(t,0), \dots, x_{\mu^n}(t,0)$

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have the same initial values as the functions $x_{\mu\kappa}(t,\mu)$ when $\kappa \neq 0$. When $\kappa = 0$ the residual R_{n+1} has the uniform order μ^{n+1} with respect to t on the segment $t_0 \leq t \leq T$ if the right parts of /I4/ are sufficiently differentiable.

The question arises whether the formula /I5/ is applicable to the system /I/ when there is no regular dependence on κ in the right parts. Indeed, as it was shown by A.B.Vasilyeva, one can use the formula /I5/ as an asymptotic formula for the solution of the system /I/ satisfying the condition /3/ since the derivatives of the solution /I/ with respect to the parameter μ have finite limits when $\mu \to 0$. Variation equations for these limit functions are constructed in the same way as in the regular case /it schould be noted that these equations will have a lower order than /I/ and therefore the initial values should be assigned only for y /. However if we try take the usual initial values /I6/, i.e. if we assume $y_{\mu\kappa}(t,0)|_{t=t_0} = 0$, then the obtained solution of the variation equation will not be the limit function for $y_{\mu\kappa}(t,\mu)$, and therefore the expansion /I5/ will not describe the solution of /I/. A.B.Vasilyeva shows that the function $y_{\mu\kappa}(t,\mu)$ has in limit the point of discontinuity as $t = t_0$ as well as $z(t,\mu)$, therefore to get the correct expressions for $\lim_{\mu\to 0} y_{\mu\kappa}$ with the help of variation equations one must supply them with initial values unequal to zero. It turns out that $y_{\mu\kappa}(t,0)|_{t=t_0}$ are some constants unequal to zero. Some formulas have been deduced to calculate them /see 24/.

So, taking into consideration this displacement of initial values for $y_{\mu\kappa}$, we can alsopreserve the formula /I5/ for the systems /I/. However the uniform estimation of the residual will take place only on the segment $t_0 < t_1 \leq t \leq T$. If it is necessary to get an asymptotic formula with a uniformly estimated residual having the order μ^{n+1} then one must use some additional terms of a special type constructed by A.B.Vasilyeva /4/. Now we shall give a more detailed description of this result. To simplify the formulas let us take $t_0 = 0$. We assume that the real parts of the roots of a characteristic equation

$$Det \left\| \frac{\partial F}{\partial z}(\varphi(y,t), y, t) - \lambda E \right\| = 0$$

are negative in $\bar{D}$. This assumption guarantees the stability of the root $z = \varphi(y,t)$ /the expressions of the type $\frac{\partial F}{\partial z}$ mean here further those particular matrixes composed of the derivatives of vector - function with respect to their arguments/. Let us consider the auxiliary system of equations

$$\frac{d\bar{z}}{d\tau} = F(\bar{z}, \bar{y}, \tau\mu)$$
$$\frac{d\bar{y}}{d\tau} = \mu f(\bar{z}, \bar{y}, \tau\mu) \qquad /17/$$

which one can obtain from /I/ by means of formal substitution $\frac{t}{\mu} = \tau$. We shall take the formal expansion of the solution of /I7/ in the form

$$\bar{x} = \bar{x}_0(\tau) + \mu \bar{x}_1(\tau) + \mu^2 \bar{x}_2(\tau) + \cdots \quad (x = \{y, z\}). \qquad /18/$$

It is evident that

$$\frac{d\bar{z}_0}{d\tau} = F(\bar{z}_0, \bar{y}_0, 0)$$
$$\frac{d\bar{y}_0}{d\tau} = 0 \qquad /19/$$

and $\bar{x}_k$ $(k = 1, 2, \ldots)$ satisfy the systems of linear equations with coefficients depending on $\bar{x}_i$ $(i < k)$.
To find the solutions of all these systems in consecutive order, beginning with /I9/, we assume the following initial conditions

$$\bar{x}_0|_{\tau=0} = x_0, \quad \bar{x}_k|_{\tau=0} = 0 \quad (k > 0). \qquad /20/$$

Thus, the expansion /I8/ may be formally constructed.

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Let us write now the system /I/ in the form

$$\mu \frac{d\overset{(2)}{z}}{dt} = F(\overset{(2)}{z}, \overset{(2)}{y}, t)$$
$$\frac{d\overset{(2)}{y}}{dt} = f(\overset{(2)}{z}, \overset{(2)}{y}, t). \qquad /2I/$$

/the index /2/ is used here to make it convenient to write the formulas which will occur further/. We shall find now the formal solution of /2I/ in the form of the following expansion:

$$\overset{(2)}{x} = \overset{(2)}{x}_0(t) + \mu \overset{(2)}{x}_1(t) + \mu^2 \overset{(2)}{x}_2(t) + \dots, \qquad /22/$$

where

$$F(\overset{(2)}{z}_0, \overset{(2)}{y}_0, t) = 0 \quad (\text{or } \overset{(2)}{z}_0 = \varphi(\overset{(2)}{y}_0, t))$$
$$\frac{d\overset{(2)}{y}_0}{dt} = f(\overset{(2)}{z}_0, \overset{(2)}{y}_0, t).$$

The equations which define $\overset{(2)}{x}_k$ $(k = 1, 2, \dots)$ will be linear and their coefficients depend on $\overset{(2)}{x}_i$ $(i < k)$.

We shall take the initial values in the form

$$\overset{(2)}{y}_0|_{t=0} = y_0$$
$$\overset{(2)}{y}_k|_{t=0} = \frac{(-1)^k}{k!} \int_0^\infty \tau^k \frac{d^k}{d\tau^k} \overset{(1)}{f}_{k-1}(\tau)\, d\tau, \qquad /24/$$

where $\overset{(1)}{f}_{k-1}$ denotes the corresponding coefficient of the formal expansion

$$f(\overset{(1)}{z}_0 + \mu \overset{(1)}{z}_1 + \dots, \overset{(1)}{y}_0 + \mu \overset{(1)}{y}_1 + \dots, \tau\mu) = \overset{(1)}{f}_0 + \mu \overset{(1)}{f}_1 + \dots$$

We needn't define initial values for $\overset{(2)}{z}$ as $\overset{(2)}{z}_k$ are expressed algebraically by means of $\overset{(2)}{y}_k$. The initial values for $\overset{(2)}{x}_k$ thus chosen these functions will express /with the error of k/ the limits for the derivatives of k-th order of the solutions of the system /I/ with respect to the parameter μ. Thus the formula /24/ gives us a displacement of the initial values of the derivatives with respect to the parameter which has been mentioned before.

We shall need one more formal expansion of the solutions of the system /2I/ in the form of double series:

$$\overset{(2)}{x} = \overset{(2)}{x}_{00} + \mu \overset{(2)}{x}_{01} + t \overset{(2)}{x}_{10} + \dots \qquad /25/$$

Here we have

$$\overset{(2)}{y}_{00} = \overset{(2)}{y}_0|_{t=0} = y_0$$
$$\overset{(2)}{y}_{0k} = \overset{(2)}{y}_k|_{t=0} = \frac{(-1)^k}{k!} \int_0^\infty \tau^k \frac{d^k}{d\tau^k} f_{k-1}(\tau)\, d\tau \qquad /26/$$

and $\overset{(2)}{y}_{ik}$ $(i > 0)$, $\overset{(2)}{z}_{ik}$ are expressed by means of /26/ with the help of the system /2I/ itself.

We denote the partial sums of the formal series /I8/, /22/ and /25/ as $(\overset{(1)}{x})_n, (\overset{(2)}{x})_n, (\overset{(2)}{x})_n$ respectively. Let us construct the expression

$$x_n = (\overset{(1)}{x})_n + (\overset{(2)}{x})_n - (\overset{(2)}{x})_n. \qquad /27/$$

This expressions /27/ are the asymptotic approximations for the solutions of the system /I/, satisfying the initial conditions /3/. When the right parts of /I/ are sufficiently differentiable we have the following estimation for $R_{n+1} = x(t, \mu) - x_n$

$$|R_{n+1}| < C\, \mu^{n+1} \qquad /28/$$

where C – is a positive constant which does not depend on t, μ. The estimation /28/ is applicable to the whole interval $0 \le t \le T$ including the initial point if μ is sufficiently small.

If one takes an interest in asymptotic approximation on the segment $t_1 \le t \le T$, where $t_1 > t_0$ the asymptotic formulas will be considerably simpler: one can use the values $(\overset{(2)}{x})_n$ instead of /27/ and the estimation /28/ will be correct for the values $R_{n+1} = x - (\overset{(1)}{x})_n$. So, the formula /27/ turns to Maclaurin's formula on the interval $t_1 \le t \le T$.

Similar formulas are also deduced for the general case of several small parameters of different orders of smallness

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... Acad. Nauk USSR, 128, N 4, 1959 /.

The asymptotic formulas /27/ contain the initial values x_0 as parameters. This makes it possible to solve various problems of the systems /1/ with some additional conditions, the boundary value problems, in particular, by choosing suitable values of parameters x_0.

The boundary value problems for the systems of type /1/ were considered from different points of view by many authors M.S.Brish /3/, M.I. Visnik, L.A.Lusternik /6/, W.Wasow /10/, N. Levinson /12/ and others/.

In our papers for the equations /1/ A. B.Vasilyeva gives asymptotic formulas of arbitrary order for some types of the solutions of the boundary value problem

$$R\,(\,y(0),y(1),z(0),z(1))=0 \qquad /29/$$

by using the formulas /27/. The ~~questions~~ problems of the existence and the uniqueness of solutions are regarded by A.B.Vasilyeva. Some examples show that limit properties of solutions of boundary value problems are rather various and it is not always possible to describe them with formulas of type /27/. A.B.Vasilyeva considers such cases when there exist solutions of boundary value problems /29/ for sufficiently small value of ε. Then are constructed according to definite schemes. These schemes are characterized by rules of choice of initial values x_0. In her papers A.B.Vasilyeva gives sufficient conditions for the existence of a certain formal expansion, further she proves that in a neighbourhood of the zero-th term of this expansion there exists the unique solution of the boundary value problem /29/ when ε is sufficiently small and she also proves that the constructed formal expansion is the asymptotic expansion for this solution.

Two schemes of this kind are given for the general case /1/, /29/ which have different qualitative behaviour of the solution in limit respectively, namely, in the first scheme the limit functions corresponding to z have a point of discontinuity at one of the end - points of the segment, in the second scheme the limit functions corresponding to z have a similar point of discontinuity and the limit functions corresponding to y have an angle point inside the segment.

We shall explain that the solutions of the first type may exist in the particular case when x and y are one-dimensional.

Let us take the boundary conditions

$$z(0)=z^0, \quad y(1)=y^0 \qquad /30/$$

Let us consider the ~~solutions of~~ Cauchy's problem $z|_{t=0}=z^0$, $y|_{t=0}=y_0=\bar{y}(0)$, where $\bar{y}(t)$ - is the solution of the equation $\frac{dy}{dt}=f(\bar{y},z,t)$ defined by the condition $\bar{y}_{t=1}=y^0$. According to what is shown above this solution will be infinitely close to $\bar{y}$ as $\varepsilon \to 0$ when $0 \le t \le 1$, therefore it will have at the point $t=1$ the value infinitely close to $y(1)=y^0$, and it means that this solution will approximately satisfy the second boundary condition. It is natural to expect that in a neighbourhood of $y_0=\bar{y}(0)$ there will exist such value y^*, that the solution /1/ defined by the condition $z|_{t=0}=z^0$, $y_{t=0}=y^*$ will fully satisfy the boundary condition at the end-point $t=1$.

The particular case /30/ can also illustrate the possibility of the existence of the second type solutions. It should be noted that if the point of motionlessness $\bar{y}(y^*,t^*)$ of adjoining system is an asymptotic stability point of the system /10/ as $\tau \to +\infty$, then the solution of the equation /1/ will converge to the solution of ... when $\varepsilon \to 0$,

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but on the interval lying to the left of the initial point $t = t_0$. Let us assume that the equation $F(z, y, t) = 0$ has two roots $z = \varphi_1(y, t)$ and $z = \varphi_2(y, t)$ /for definiteness, let $\varphi_1 > \varphi_2$ /let the root φ_1 correspond to an asymptotic stability point as $\tau \to +\infty$. It means that the solution of Cauchy's problem with initial conditions taken at the right end-point of the segment tends to the degenerate solution of the system /2/ defined by this root. Let us suppose that root φ_2 corresponds to an asymptotic stability point when $\tau \to +\infty$. Let $\bar{y}_1$ and $\bar{y}_2$ be the solutions of degenerate equations

$$\frac{dy}{dt} = f(\varphi_i(y, t), y, t), \quad i = 1, 2,$$

corresponding to the roots given above respectively. We shall define $\bar{y}_1$ by means of the initial condition $\bar{y}_1|_{t=0} = y_1^0$ where y_1^0 is to be found from the requirement $\varphi_1(y_1^0, 0) = z^0$ and $\bar{y}_2$ will be defined by means of the condition $\bar{y}_2|_{t=1} = y_2^0 = y^0$. Now we assume that the curves $\varphi_1(t)$ and $\bar{y}_2(t)$ cross at a certain point t_0 $(0 < t_0 < 1)$. Let us consider the solution of Cauchy's problem with the initial conditions at the point $t = t_0$ in the form of

$$y|_{t=t_0} = y_0 = \bar{y}_1(t_0) = \bar{y}_2(t_0), \quad z|_{t=t_0} = z_0,$$

where z_0 is a value lying between $\varphi_1(y_0, t_0)$ and $\varphi_2(y_0, t_0)$. According to limit properties of the solutions of Cauchy's problem y is close to $\bar{y}_2(t) = y_0$ at the point $t = t_0$ and z is close to $\varphi_1(\bar{y}_1(0), 0) = z^0$ at the point $t = 0$. Therefore this solution composed of two arcs $\bar{y}_1$ and $\bar{y}_2$ will approximately satisfy the given boundary conditions at the end - points of the segment $[0, 1]$. It is natural to expect that in a neighbourhood of y_0 there may occur such value y^* and between $\varphi_1(y_0, t_0)$ and $\varphi_2(y_0, t_0)$ there may be found such value z^* that the solution defined by the condition $\varphi_1(z^*, y^*, t_0) = z^*$ will fully satisfy the boundary conditions /3/.

As the investigation shows the ideas given above in connection with the particular problem /3/ are also applicable to a general case and it is possible to construct the asymptotic for the solutions of the boundary problems of both types by means of the asymptotic formulas /2/ /Dokl. Akad. Nauk SSSR, 135, N 6, 1960/.

It should be remarked that if the characteristic equation $\det\left\|\frac{\partial F}{\partial z}\right\| = 0$ has roots both with negative and positive real parts the point of motionlessness $z = \varphi(y^*, t^*)$ will not be stable and the solution of Cauchy's problem, generally speaking, will have no finite limit when $\varepsilon \to 0$. However the solutions of some boundary problems may have a limit when $\varepsilon \to 0$. In the papers of A.B.Vasil'eva, V.A.Tupchiev the asymptotic formulas similar to /2/ are deduced for some cases of this kind.

§ 4. Rapidly Oscillating Solutions

We have just considered the problems in which the limit properties of solutions of the system /1/ as $\varepsilon \to 0$ are defined by the properties of roots of the degenerate equations /2/. But if the equations /2/ are insolvable with respect to z then the given construction cannot be applied and the continuous dependence of solutions on ε, generally speaking, does not take place. The rapidly oscillating solutions which have a frequency of oscillation with the order of $\frac{1}{\sqrt{\varepsilon}}$ may occur in such cases. V.M.Volosov /6/ has considered a number of problems ~~a number of problems~~ dealing with the investigation of such solutions, the problem of amplitude curves which describe extremes of oscillating solutions in particular.

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§ 5. Asymptotic Methods dealing with Averaging

By changing the variable the system /1/ may be reduced to the form

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon), \quad \dot{y} = Y(x, y, t, \varepsilon),$$
$$x = \{x_1, \ldots, x_n\}, \quad y = \{y_1, \ldots, y_m\}, \qquad /31/$$

where ε is a small parameter /we again denote the new argument as t and we denote slow and rapid variables as x and y /. The systems of type /31/ are usually considered on a large interval $t \sim \frac{1}{\varepsilon}$. In investigation of these systems there arises question about averaging of the right parts of /3/. The method of averaging has been used in solving some problems for a long time, for instance in astronomy. This method was based in the works of N.N.Bogoliubov and his followers /3/. N.N.Bogoliubov considered the so-called systems in standard from

$$\dot{x} = \varepsilon X(x, t, \varepsilon) \qquad /32/$$

which present a particular case of /31/. If the right parts of /32/ allow averaging with respect to t, i.e. there exists the limit

$$\overline{X} = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} X(x, t, \varepsilon)\, dt,$$

then we take the averaged systems

$$\dot{x} = \varepsilon \overline{X}_1(\bar{x}) = \varepsilon \overline{X}_{\varepsilon=0} \qquad /33/$$

corresponding to the system /31/. N.N.Bogoliubov has proved that under some restrictions the solutions of /33/ are close to the solutions of /32/ on a large interval of time $\tau \sim \frac{1}{\varepsilon}$ when $\varepsilon \to 0$ and has also worked out the method of constructing the approximations of higher order. In their paper N.N.Bogoliubov and D.N.Zubarev have investigated the so-called systems with a rapidly rotating phase which may be written in the form

$$\dot{x} = \varepsilon X(x, \psi, \varepsilon), \quad \dot{\psi} = \omega(x) + \varepsilon Y(x, \psi, \varepsilon) \qquad /34/$$

where X, Y are periodical with respect to a scalar phase ψ of the period 2π. There has been worked out a method of eliminating ψ, equivalent to averaging with respect to ψ. In the first approximation the solutions of /34/ are described by the system

$$\dot{\bar{x}} = \varepsilon \overline{X}_1(\bar{x}) = \varepsilon \frac{1}{2\pi} \int_0^{2\pi} X(\bar{x}, \psi, 0)\, d\psi.$$

A detailed description of these questions can be found in books by N.N.Bogoliubov and Yu.A. Mitropolsky /3/, /4/.

§ 6. Generalization of Averaging Method

It can be easily seen that the systems /34/ are the generalization of the systems /32/ and on the other hand they present a particular case of the more general system /31/. In the papers of V.M. Volosov /5/ generalization of the averaging method was developed which extends this method to the systems of the general type /31/. The problem of averaging of systems /31/ is considered as a generalization of averaging of system /32/ as /31/ built with respect to /32/.

Together with the system /1/ we discuss the so-called unperturbed system

$$\dot{x} = X_0(x, y, t), \quad \dot{y} = Y_0(x, y, t) \qquad /35/$$

into which the system /1/ is transformed at $\varepsilon = 0$. It is assumed that the solutions of the system /35/ under consideration is

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the right parts of /31/ and some other functions along integral curves of the degenerate system /35/. As the average value of a certain function $F(x, y, t)$ in the above mentioned sense we assume the following value:

$$\bar{F} = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} F(x, y, a, t), t)\,dt. \qquad /36/$$

We suppose that for the given function F the limit $\bar{F}$ exists and depends only on the parameters $a_1, \ldots, a_n$ and does not depend on the initial values x_0, t_0 of the system /35/. As it is shown in what follows /see §7/, the assumption that limit /36/ does not depend on initial values, i.e. on the choice of a trajectory of /35/, along which the averaging goes, does not restrict the universality of the problem. The averaged system of the first approximation corresponding to the systems /31/ is the following:

$$\dot{\bar{x}} = \varepsilon \bar{X}_1(\bar{x}) + \varepsilon^2 \bar{X}_2(\bar{x}) + \ldots, \qquad /37/$$

Under some general restrictions it is proved that the solutions of the system /37/ approximate the solutions x of the system /31/ with the same initial conditions on a large interval of time $t \sim \frac{1}{\varepsilon}$.

Besides the establishment of averaging in the first approximation /the system /37/ V.M.Volosov also constructs the averaged systems for higher approximations and proves that these systems approximate the solutions of /31/ more precisely than /37/ does. The problem of higher approximations is considered in the following way: for the system /31/ which may be written in the form

$$\dot{x} = \varepsilon X_1(x, y, t) + \varepsilon^2 X_2(x, y, t) + \varepsilon^3 \ldots,$$
$$\dot{y} = Y_0(x, y, t) + \varepsilon Y_1(x, y, t) + \varepsilon^2 Y_2(x, y, t) + \varepsilon^3 \ldots \qquad /38/$$

we find the transformation of variables of type

$$x = \bar{x} + \varepsilon u_1(\bar{x}, \bar{y}, t) + \varepsilon^2 u_2(\bar{x}, \bar{y}, t) + \varepsilon^3 \ldots$$
$$y = \bar{y} + \varepsilon v_1(\bar{x}, \bar{y}, t) + \varepsilon^2 v_2(\bar{x}, \bar{y}, t) + \varepsilon^3 \ldots \qquad /39/$$

which transforms /36/ into the averaged system of type

$$\dot{\bar{x}} = \varepsilon \bar{X}_1(\bar{x}) + \varepsilon^2 A_2(\bar{x}) + \varepsilon^3 \ldots,$$
$$\dot{\bar{y}} = Y_0(\bar{x}, \bar{y}, t) + \varepsilon B_1(\bar{x}) + \varepsilon^2 B_2(\bar{x}) + \varepsilon^3 \ldots \qquad /40/$$

It is necessary to find both the right parts of /40/, i.e. the functions $A_2, B_2, \ldots$ and the coefficients of the transformation /39/, i.e. the functions $u_1, u_2, \ldots, v_1, v_2, \ldots$

We have found the algorithm by means of which we can get any number of coefficients of /39/ and /40/ in consecutive order, i.e. transformation /39/ and the system /40/ may be formed with arbitrary precision. The property of the averaged system is that if we leave in it the terms of some order inclusively, then the solutions of /31/ will be approximated with corresponding precision with respect to the parameter ε. As an averaged system of n-th order of approximation we take the system of the form

$$\dot{\bar{x}} = \varepsilon \bar{X}_1(\bar{x}) + \varepsilon^2 A_2(\bar{x}) + \ldots + \varepsilon^n A_n(\bar{x}),$$
$$\dot{\bar{y}} = Y_0(\bar{x}, \bar{y}, t) + \varepsilon B_1(\bar{x}) + \ldots + \varepsilon^{n-1} B_{n-1}(\bar{x}). \qquad /41/$$

The transformation

$$\bar{x} + \varepsilon u_1(\bar{x}, \bar{y}, t) + \ldots + \varepsilon^{n-1} u_{n-1}(\bar{x}, y, t),$$
$$+ \varepsilon v_1(\bar{x}, \bar{y}, t) + \ldots + \varepsilon^{n-2} v_{n-2}(x, \bar{y}, t). \qquad /42/$$

corresponds to this approximation with the required precision. If we insert the solutions of /41/ in /42/ then the expressions in /42/ will approximate the solutions x and y of the system /31/ with errors of the orders ε^n and ε^{n-1} respectively on the interval $t \sim \frac{1}{\varepsilon}$, thus being ra-

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pid variables are approximated with different precision. The theorems giving grounds for the above mentioned construction of asymptotic of the solutions of /31/ are proven.

These results deal with a large but not infinite interval of time $t \sim \frac{1}{\varepsilon}$. As some simple examples show the averaging is not applicable to the infinite interval (t_0, ∞) without some additional restrictions. To extend the method of averaging to the infinite interval of time we introduce some stability conditions for the solutions of /37/. V.M.Volosov shows that under such additional conditions the general method of averaging may be extended to the interval of type (t_0, ∞).

§ 7. The asymptotic of the integrals of some disturbed systems

For some equations presenting a certain interest for the investigation, the assumption that the average values /36/ do not depend on the initial values /35/ is not fulfilled. In such cases there exists a change of variables which reduces these systems to the case considered in § 6. It was said in § 6 in this respect that the assumption of independence of /36/ on y_0, t_0 does not restrict the universality of the problem.

We shall discuss the results of V.M.Volosov concerning the case when $\overline{F}$ depends on y_0, t_0. We write the system /31/ in the form /36/ and consider it as a result of disturbing the system /35/ by means of small terms $\varepsilon Y_1 \ldots$ and slowly varying parameters x. As the average value of a certain function $F(x, y, t)$ /if it exists/ we take the limit /36/ but we admit now that $\overline{F}$ may depend on y_0, t_0 i.e. on the choice of a trajectory of the system /35/ along which the averaging is performed. To describe the above mentioned dependence we suppose that the general solution of /35/ in the domain under consideration has the form

$$y = y(c_1, \ldots, c_m, t, x),$$

where $c_1, \ldots, c_m$ are arbitrary constants. Let the average values /36/ depend on some constants, for definitness we assume that the average values depend on $c_1, \ldots, c_k$ and do not depend on $c_{k+1}, \ldots, c_m$ $(k < m)$.

$$\lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} F(x, y(c_1, \ldots, c_m, t, x), t)\, dt \equiv \overline{F}(c_1, \ldots, c_k, x_1, \ldots, x_n). \qquad /43/$$

Let the arbitrary constants $c_1, \ldots, c_k$ have the corresponding system of the independent integrals of /35/ of type

$$c_i = \Phi_i(x, y, t) = \text{const}, \quad (i = 1, 2, \ldots, k) \qquad /44/$$

and we also suppose that the trajectories of the system have the parametrical representation on the integral surfaces /44/ of the form

$$y = y(c_1, \ldots, c_k, z_1, \ldots, z_{m-k}, t), \qquad /45/$$

where $z_1, \ldots, z_{m-k}$ are independent parameters. Let us now introduce the values $c_1, \ldots, c_k$ and $z_1, \ldots, z_{m-k}$ as new variables into the system /38/ according to the formulas /44/, /45/. Then for c_i, x_j, z_j we get a system of equations belonging to the type /31/ in which $x_1, \ldots, x_n$ and $c_1, \ldots, c_k$ play the role of slowly changing variables and $z_1, \ldots, z_{m-k}$ are rapidly varying functions. The average values of type /43/ for the transformed system will no longer depend on z_0, t_0. That is why it is possible to apply the results of § 6. Thus the averaging method is extended to more general systems. The introduction of the variables c_i, z_j corresponds to variation of constant

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$C_1, \dots, C_k$ and parameters $z_1, \dots, z_{m-k}$ in the solution /45/ of the system /35/. As a result we express the solutions of the disturbed system /38/ in the form /45/ in which the functions C_i and z_j are to be found from the corresponding averaged systems.

In the system /38/ we can formally place the slow and rapid variables x, y and also argument t, in the joint group of variables

$$\mathcal{J} = \{\mathcal{J}_1, \dots, \mathcal{J}_{n+m+1}\} = \{x_1, \dots, x_n, y_1, \dots, y_m, t\}$$

and consider /38/ in the form

$$\dot{\mathcal{J}} = S(\mathcal{J}, \varepsilon) = S_0(\mathcal{J}) + \varepsilon S_1(\mathcal{J}) + \varepsilon^2 \dots, \qquad /46/$$

where $S_0, S_1, \dots$ are vector-functions. The vector S_0 has for example the components corresponding to the variables $x_1, \dots, x_n$ equal to zero, and for argument t the corresponding component of S_0 is equal to 1. The system /46/ belongs to type /31/ but in this form of writting the slow and rapid variables do not differ. It turns out that there exist such cases when averaging is inapplicable to the system taken in the form /46/ whereas if the variables are divided into the groups of slow and rapid variables, averaging can be applied. This can be accounted for by the following fact: from the averaged system of n-th approximation /41/ the variables x and y by means of transformation /42/ are defined with different orders of precision and this is a general fact /with the exception of some special particular cases/. If we take a system in the form /46/ and apply averaging to it, it means that we wry to calculate x, y simaltenuously with equal order of precision which may lead to some difficulties. Thus, the separation of variables into groups of slow and rapid variables extends the sphere where the averaging method can be [illegible].

The asymptotic methods and some problems dealing with averaging were treated in the works of [illegible] Mitropolsky [illegible], O.B.Lihova [9], [illegible] Feschenko [10], [illegible] Kontryagin [14], A.M.Molchanov [11], D.V.Anosov [1] and some other authors.

§ 8. The system with rotating and oscillating coordinates

It is a particular case of the system /31/ when the solutions of the degenerate system /31/ consist of the so-called rotating and oscillating variables. We suppose that the system /35/ has the solutions $y^{(1)} = \{y_1, \dots, y_s\}$ and $y^{(2)} = \{y_{s+1}, \dots, y_m\}$ $(s \le m)$ which are characterized by the following properties: the functions $y^{(1)}$ depend on t periodically and the functions $y^{(2)}$ at every interval of time equal to the period of $y^{(1)}$ get the increments equal to the constants $T_1, \dots, T_{m-s}$. For instance, the equation of physical pendulum oscillations belongs to this class of systems. Such systems under general assumptions make it possible to apply the method of averaging. These problems were considered in the papers of V.M.Volosov [6].

The particular case of the discussed system is that one when the solutions of /35/ are fully periodical. Such systems were also investigated by V.M.Volosov [6]. The results of V.M.Volosov concerning the periodical case were generalized by D.V.Anosov [1] who showed that under more general assumptions the solutions of /31/ are close to the solutions of the system of the first approximation with respect to the measure of initial values, if the surface /44/ are bounded and closed and /35/ has an integral invariant. Under these general assumptions the solutions may not be close to each other in the usual sense of the word.

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§ 9. System similar to Hamilton's ones

In the particular case the undisturbed system may be of Hamilton's tape. Then the system /38/ may be taken in the form

$$\dot{p} = -\frac{\partial H(p,q,x)}{\partial q} - \varepsilon f^{(q)}(p,q,x,\varepsilon),$$
$$\dot{q} = \frac{\partial H(p,q,x)}{\partial p} - \varepsilon f^{(p)}(p,q,x,\varepsilon),$$
$$\dot{x} = \varepsilon X(p,q,x,\varepsilon), \qquad /47/$$

where $p = \{p_1, \dots, p_n\}$, $q = \{q_1, \dots, q_n\}$, $x = \{x_1, \dots, x_n\}$.

In his papers for such systems V.M.Volosov deduces the equation of energy:

$$\dot{E} = \varepsilon \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \left(f_p^{(p)} + f_q^{(q)} + \frac{\partial H}{\partial x} X \right) dt \qquad /48/$$

The /47/ belonging to the type of the systems mentioned in § 8, the following equation for integral of action was deduced in the papers of V.M.Volosov:

$$\dot{J} = \varepsilon \int_{(L)} f_p^{(p)} dp + f_q^{(q)} dq + \varepsilon \int_{t_0}^{} \frac{\partial H}{\partial x} (X - \bar{X}) dt. \qquad /49/$$

The physical meaning of the formula /48/ is that the rapidity of the change of energy is equal to average power of disturbing forces $\varepsilon f^{(p)}$, $\varepsilon f^{(q)}$ and forces εX which change the parameters x. In the case of one degree of freedom when there are fully periodical solutions the equation of energy was also given in /16/.

The physical meaning of the formula /49/ is that the rapidity of change of integral of action J is equal to the work of the disturbing forces $\varepsilon f^{(p)}, \varepsilon f^{(q)}$ on the strip of an undisturbed trajectory which corresponds to one period oscillating variables, added to virtual work connected with the change of the parameters with the velocity $\varepsilon(X - \bar{X})$, i.e. with the velocity equal to the deviation of the velocity ξx

the change of

parameters x from its average value $\varepsilon \bar{X}$. This particular virtue of integral of action differs it from other integrals of motion, for instance from integral of energy. Energy according to /48/ changes in accordance with the velocity of the parameters x, but the integral of action according to the formula /49/ "responds" only to the deviation from the average velocity of the parameters - to "vibration". In /17/ it is known that this phenomenon is responsible for the fact that under certain conditions integral of action is an adiabatical invariant.

§ 10. Some problems of the theory of oscillations

The methods discussed in §§ 6, 7, 8, 9 were applied by V.M.Volosov for the investigation of some systems of the equations of non-linear oscillations. For instance, one of the systems investigated by V.M.Volosov may be written in the form

$$\frac{d}{dt}\left[p_i(x)\dot{q}_i \right] + Q(x,t) = \varepsilon f(\dot{q}, q, x, \varepsilon)$$
$$\dot{x} = \varepsilon X(\dot{q}, q, x, \varepsilon), \quad (x = \{x_1, \dots, x_n\}). \qquad /50/$$

If /50/ contains only one parameter $x_1 = \tau = \varepsilon t$ ($\tau = \varepsilon t$ is usually called slow time), it is reduced to the form

$$\ddot{q} + Q_1(\varepsilon t, q) = \varepsilon f_1(\varepsilon t, \dot{q}, q, \varepsilon). \qquad /51/$$

The equations /51/ were for the first time considered by N.M. Krylov and N.N.Bogolyubov /12/. For the systems /50/ and certain other systems of more general type V.M.Volosov /6/ deduced the formulas of first and higher approximations, the peculiarity of which is that one needn't solve the degenerate equation beforehand to use them. For these systems the amplitude curves describing the extremes of oscillations were constructed. The method /6/ are characterized by the

containing slow changing parameters

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...t that higher approximations may b easily found /for ...em the finite formulas are deduced/ if the equations of first approximations are solved. Besides the amplitude curves the approximations for the period of oscillation /50/ and for the velocity $\dot{q}$ are constructed as well as the explicit finite formulas for the approximations of the solution of /50/ and its derivative. As a supplement certain problems of physics and technics are also discussed.

These results were also applied by V.M.Volosov /67 for the calculation of periodical solutions of some systems of autooscillation.

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ИССЛЕДОВАНИЕ СИНГУЛЯРНО ВОЗМУЩЕННЫХ СИСТЕМ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

We investigate the following system of differential equations:

$$\frac{dx}{dt} = f(x, z, t, \frac{t}{\mu}), \quad \mu \frac{dz}{dt} = F(x, z, t), \tag{1}$$

where x and f are m-dimensional vectors, z and F are n-dimensional vectors, μ - a small positive parameter.

With the argument $\frac{t}{\mu}$ absent, system (1) can be rewritten in the form

$$\frac{dx}{dt} = f(x, z, t), \quad \mu \frac{dz}{dt} = F(x, z, t) \tag{2}$$

which has been considered in detail in the works of A.N.Tikhonov [1], I.S. Gradsntein [2], A.B.Vasilieva [3], V.M.Volosov [4], L.Flatto and N.Levinson [5], Y.Hirasawa [6], J.Levin [7], U.Cho-Tsiun [8].

Now, if neither $\frac{t}{\mu}$ nor t is contained explicitly in the right sides of equations (1), we obtain the autonomous system

$$\frac{dx}{dt} = f(x, z), \quad \mu \frac{dz}{dt} = F(x, z), \tag{3}$$

which has been investigated from different points of view by L.S.Pontriagnin [9], E.F.Mishchenko [10], D.V.Anosov [11], K.O.Friedrichs and W.R.Wa-

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sow [12].

With the argument x absent in the right-hand sides of (1), we arrive at a system of m equations

$$\frac{dx}{dt} = f\left(x, t, \frac{t}{\mu}\right)$$

which by the transformation of the argument $t = \mu\tau$ is reduced to a system of m equations in the standard form

$$\frac{dx}{d\tau} = \mu f(x, \mu\tau, \tau), \qquad (4)$$

the latter system having been investigated in detail in a monograph of N.N.Bogoliubov and Yu.A.Mitropolsky [13].

The argument $\frac{t}{\mu}$ contained in system (1) is apt to cause rapidly oscillating terms to appear, thus bringing about "tremblings" in the solutions of the system.

In what follows, both the more general system (1) and its special cases (2), (3), (4) will be investigated.

Along with system (1) we shall also consider the system

$$\frac{d\bar{x}}{dt} = f_0(\bar{x}, \bar{z}, t), \quad F(\bar{x}, \bar{z}, t) = 0, \qquad (1')$$

averaged with respect to the argument $\frac{t}{\mu}$ and degenerate ($\mu = 0$), where

$$f_0(x, z, t) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x, z, t, \vartheta)\, d\vartheta. \qquad (4)$$

Corresponding to systems (2), (3) will be the degenerate systems

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \bar{z}, t), \quad F(\bar{x}, \bar{z}, t) = 0, \qquad (2')$$

$$\frac{d\bar{x}}{dt} = f(\bar{x}, \bar{t}), \quad F(\bar{x}, \bar{z}) = 0, \qquad (3')$$

and to system (4) the system

$$\frac{d\bar{x}}{d\tau} = \mu f_0(\bar{x}) \qquad (4')$$

averaged with respect to τ.

The difference in the orders of the differential systems for $\mu \neq 0$ and for $\mu = 0$ causes additional difficulties to arise in establishing the qualitative and quantitative correspondence between the solutions $x(t, \mu)$, $z(t, \mu)$ of the systems (1), (2), (3) and the solutions $\bar{x}(t)$, $\bar{z}(t)$ of the corresponding degenerate systems (1'), (2'), (3').

The fairly great number of problems in physics and engineering which are reduced to the investigation of systems (1) to (4) emphasizes the practical importance of the above problem. Among these are the problems of oscillations in electrical networks with a series of capacitances and inductances with some very small capacitances and inductances present, the problem of the mechanical vibrations of elastic bounded masses with some of them very small compared to the others, a number of problems of Van der Pol type for very great values of the parameter as stated in a monograph by A.A.Andronov, A.A.Vitt and S.E.Khaikin [14]. At present, a number of cases concerned with the control of systems is reduced to the above problem.

In this connection the following questions arise:

1. If there exists the solution $\bar{x}(t)$, $\bar{z}(t)$ for the initial conditions x_0, z_0 of the degenerate systems (1'), (2'), (3'), then what

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are the conditions for the existence of the solution $x(t,\mu)$, $z(t,\mu)$ of the systems (1), (2), (3)?

2. If there exists a solution of some one of the systems (1), (2), (3), then will it be unique?

3. Does the solution $x(t,\mu)$, $z(t,\mu)$ for $\mu \to 0$ tend to the solution $\bar{x}(t)$, $\bar{z}(t)$ for each of the above systems? Will this tendency be uniform with regard to t ?

4. If some one of the systems (1'), (2'),(3') has a periodic solution in t of period T, then under what conditions will systems (1), (2), (3) admit of periodic solutions ?

5. If some one of systems (1), (2), (3) admits of a periodic solution, will it be unique ?

6. Will systems (1), (2) (3) admit of almost periodic solutions if the corresponding degenerate systems have almost periodic solutions ?

7. For periodic and almost periodic as well as non-periodic solutions the question of their stability arises?

8. Given the solution $\bar{x}(t)$, $\bar{z}(t)$ of the degenerate system, construct the asymptotic solution for the corresponding non-degenerate system?

Some of these questions have been solved by the author by using the method of averaging and the method of integral manifolds.

Note that if, for instance, being given the manifold $z=\varphi(x,t)$ of system (1'), we succeeded in constructing the integral manifold

$$z = \varphi(x,t) + \psi(x,t,\mu),$$

such that $\psi(x,t,\mu) \to 0$ as $\mu \to 0$, then system (1) could be replaced by the system

$$\frac{dx}{dt} = f(x, \varphi(x,t) + \psi(x,t,\mu), t, \frac{t}{\mu}),$$

i.e. the order of the system being considered would be lowered.

The relation

$$z = \varphi(x,t) + \psi(x,t,\mu)$$

will be said to be a local integral manifold of system (1), if in case it holds at some moment t_0 it will also hold at some t as long as $x(t)$ remains within some given region.

In what follows it will be always assumed that equation $F(x,z,t)=0$ has an isolated solution $z=\varphi(x,t)$ and that

$$x \in G, \ |z-\varphi(x,t)| \le \rho, \ 0 \le t \le \mu\tau \le T, \ 0 < \mu < \mu^*$$

the vectors f, F and φ are sufficiently smooth.

We shall denote by F_z a square n- dimensional matrix with elements $\frac{\partial F_i}{\partial z_j}$ $(i,j=1,2,\ldots,n)$ and similar notations will be used for f_x, φ_t, etc.

Of great importance for what follows will be the matrix

$$\mathcal{A}(x,t) = F_z(x, \varphi(x,t)+y, t)\big|_{y=0}$$

and its characteristic equation

$$Det \| \rho E - \mathcal{A} \| = 0. \qquad (5)$$

We shall now state some lemmas to be used in the following text.

1/ For some cases in the following the systems (1), (2), (3) will be considered on an infinite interval for t.

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Lemma I

Let $A(t)$ be a continuous n-dimensional square matrix ($a \le t \le b$) and let the real parts of all its eigenvalues be less than -2α at $a \le t \le b$ for some $\alpha > 0$. Also, let $\rho(t,s,\mu)$ be the solution of system

$$\mu \frac{dX}{dt} = A(t)X, \quad X(s) = E$$

at $a \le t \le b$, $a \le s \le b$, where E is a unitary n-dimensional matrix.

Then there exists a constant K such that

$$|\rho(t,s,\mu)| \le K e^{-\frac{\alpha}{\mu}(t-s)} \quad \text{for } a \le s \le t \le b.$$

The proof of this lemma is to be found in an article by L.Flatto and N.Levinson [3].

For more severe limitations superposed on matrix $A(t)$, the above estimate will hold on an infinite interval t.

Lemma 2.

Let the n-dimensional matrix $A(t)$ have n_1 eigenvalues with negative real parts and n_2 eigenvalues with positive real parts for all $t (a \le t \le b)$, $(n = n_1 + n_2)$. Then there exists a real matrix $V(t)$ such that

$$V^{-1}(t) A(t) V(t) = \begin{pmatrix} B(t) & 0 \\ 0 & C(t) \end{pmatrix},$$

where $B(t)$ is a square n_1-dimensional matrix whose eigenvalues have negative parts, and $C(t)$ is a square n_2-dimensional matrix whose eigenvalues have positive real parts.

This lemma is a well known statement of Linear Algebra, matrix $V(t)$ having as many derivatives

as there are in matrix $A(t)$.

Lemma 3

If for the system

$$\frac{dy}{d\tau} = Y(x, y, \mu\tau) + \mu f^*(\tau), \quad Y(x, 0, \mu\tau) = 0,$$

where y and Y are n-dimensional vectors and x is an m-dimensional vector, in the closed region $0 \le t = \mu\tau \le T$, $x \in G$, $|y| \le \rho_1$ to which belongs point (τ_0, x_0, y_0), the conditions are fulfilled that

a/ $\left|\frac{dx}{d\tau}\right| \le \mu M, |f^*(\tau)| \le N$ (M, N are constant numbers);

b/ the roots of the characteristic equation $Det\|\rho E - A\| = 0$ satisfy the condition

$$Re\{\rho_i(x, \mu\tau)\} \le -\alpha < 0$$

/matrix $A = \frac{\partial Y_i}{\partial y_j}\Big|_{y=0}$ /;

c/ the vector Y and its derivatives with respect to all arguments up to second order are continuous, then the following estimate holds for the solution of this system:

$$|y(\tau)| \le \rho_1' e^{-\frac{\alpha}{\delta}(\tau - \tau_0)} + \mu\rho_2,$$

where $\rho_1 + \mu\rho_2 \le \rho_2$, $\rho_1' = K|y(\tau_0)|$, $\rho_2 = \frac{\delta K N}{\alpha}$, $K = const$.

The proof of this lemma is to be found in the author's article [1].

Lemma 4

Let function $X(x,t)$ in some region D satisfy the following conditions:

a/ $|X(x,t)| \le M, |X(x',t) - X(x'',t)| \le \lambda |x' - x''|$

/where M and λ are constants/;

b/ there exists the limit

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$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X(x,\tau)\,d\tau = X_0(x),$$

uniformly with regard to x in D;

c/ the solution $\xi = f(t)$ of equation

$$\frac{df}{dt} = X_0(f)$$

together with its δ-neighbourhood lies in D.

Then to any constants $\delta, \eta, \mathcal{K}, C, C_1$ a $\mu_0 > 0$ can be brought into correspondence such that the solution of the equation

$$\frac{dx}{dt} = \mu X(x,t) + E$$

where $|E| \le \mu^2 C + \mu C_1 e^{-\frac{\mathcal{K}}{2}(t-t_0)}$, which coincides with $\xi = \xi(t)$ for $t = t_0$, differs from $\xi(t)$ less than by η on the whole of the interval $\left[0, \frac{T}{\mu}\right]$, $|x(t) - \xi(t)| < \eta$ for all $\mu < \mu_0$.

The proof of this lemma is to be found in the author's article [16].

We shall first assume that the right-hand sides of system (1) are such that it has a unique solution, and by making use of lemmas 1, 3, 4 we arrive at the following statement.

Theorem 1. If:

a/ the roots $\rho_i(x,t)$ of the characteristic equation (5) satisfy the condition $Re\{\rho_i(x,t)\} \le -\nu < 0$;

b/ at every point of the region above there exists the limit (A);

c/ the solution $x = \bar{x}(t)$ of equation (1') together with its δ-neighbourhood lies in G,

then the solution of system (1) at the initial conditions x_0, z_0 tends towards the solution of system (1') as $\mu \to 0$. The proof of this theorem is given in the author's article [17].

We proceed now to the question of the existence

and uniqueness of a local integral manifold of the most general system (1), which is close to the integral manifold $z = \varphi(x,t)$ of system (1').

We shall assume that in the region above the vectors F, f, and φ are sufficiently smooth, there exists the limit (A), matrix $\mathcal{A}$ has n_1 eigenvalues with negative real parts and n_2 eigenvalues with positive real parts ($n_1 + n_2 = n$).

In this case, the following theorems hold:

Theorem 2. A positive number μ_0 can be pointed out such that for every positive $\mu < \mu_0$ the system of equations (1) has a unique integral manifold, representable by

$$z = \varphi(x,t) + \Psi(x,t,\mu),$$

where $\Psi(x,t,\mu)$ as a function of (x,t) is defined in region $0 \le t \le T$, $x \in G$ and satisfies the inequalities

$$|\Psi(x,t,\mu)| \le D(\mu) < \rho, \quad |\Psi(x',t,\mu) - \Psi(x'',t,\mu)| \le \Delta(\mu)|x' - x''|$$

where $D(\mu) \to 0$, $\Delta(\mu) \to 0$ as $\mu \to 0$.

Theorem 3. If, in addition to the conditions of theorem 2, the integral manifold of system (1') has bounded and uniformly continuous derivatives with respect to x up to $(\kappa+1)$-order, the vector f has bounded and uniformly continuous derivatives with respect to x and z up to κ-order, and the vector F up to $(\kappa+2)$-order, then the integral manifold $z = \varphi + \Psi$ of system (1) will have bounded and uniformly continuous derivatives with respect to x up to κ-order.

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Theorem 4. If, in addition to the conditions of theorems 2 and 3 ($k = 1$), the vector f has in the region above a bounded and uniformly continuous derivative with respect to t, then the integral manifold $z = \gamma + \psi$ of system (1) has a bounded and uniformly continuous first derivative with respect to the parameter μ.

Let us now outline the proof of theorem 2.

By substituting

$$z = \varphi(x,t) + \xi$$

in system (1), we obtain the following system:

$$\frac{dx}{dt} = \Phi(x,\xi,t,\mu), \quad \mu\frac{d\xi}{dt} = \mathcal{U}(x,t)\xi + Q(x,\xi,t,\mu), \quad (6)$$

where

$$\Phi = f(x,\varphi+\xi,t,\frac{\xi}{\mu}), \quad \mathcal{U} = F_z(x,\varphi+\xi,t)\big|_{\xi=0},$$

$$Q = F(x,\varphi+\xi,t) - F_z(x,\varphi,t)\xi - \mu\left(\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial t}\right),$$

the functions Φ and Q satisfying the following conditions:

$$|\Phi| \le \mathcal{L}, \quad |Q| \le M|\xi|^2 + N\mu,$$

$$|\Phi(x',\xi',t,\mu) - \Phi(x'',\xi'',t,\mu)| \le A(|x'-x''| + |\xi'-\xi''|),$$

$$|Q(x',\xi',t,\mu) - Q(x'',\xi'',t,\mu)| \le (B|\xi''| + C\mu)(|x'-x''| + |\xi'-\xi''|),$$

where $\mathcal{L}, M, N, A, B, C$ are constants and $|\xi| = \max\{|\xi'|, |\xi''|\}$.

Consider the class of functions $\Psi(x,t)$ satisfying the conditions

$$|\Psi(x,t)| \le D, \quad |\Psi(x',t) - \Psi(x'',t)| \le \Delta|x'-x''|$$

where D and Δ are fixed constants, and consider also two equations

$$\frac{dx}{dt} = \Phi(x,\Psi(x,t),t,\mu), \quad \frac{dx}{dt} = \Phi(x,\Psi^*(x,t),t,\mu),$$

whose solutions are respectively denoted by

$$x(t,x^\circ) = \Omega(t,t_\circ,x^\circ\psi), \quad x^*(t,x^\circ) = \Omega(t,t_\circ,x^\circ,\Psi^*),$$

where x° and x^* are the respective initial conditions for $t = t_\circ$ and $\tau = t - t_\circ$.

Using S.A.Chaplyghin's theorem on differential inequalities, the following estimate is obtained:

$$|x^*(t,x^*) - x(t,x^*)| \le |x^*-x^\circ|e^{A(1+\Delta)} + \frac{\|\Psi^*-\Psi\|}{1+\Delta}\left(e^{A(1+\Delta)\tau} - 1\right),$$

where

$$\|\Psi^*(x,t) - \Psi(x,t)\| = \sup_{(x,t)}|\Psi^*(x,t) - \Psi(x,t)|.$$

In accord with lemma 2

$$\mathcal{U} = V\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}V^{-1}$$

Let $p(t,s,\mu)$ and $q(t,s,\mu)$ be the solutions of the systems

$$\mu\frac{dp}{dt} = B(x,t)p, \quad p(s) = E_1,$$

$$\mu\frac{dq}{dt} = C(x,t)q, \quad q(s) = E_2,$$

where E_1 and E_2 are square unitary matrices of dimensions n_1 and n_2, respectively ($n_1+n_2=n$).

We define the matrix $U(t,s,\mu)$ in the following manner

$$U(t,s,\mu) = V\begin{pmatrix} p(t,s,\mu) & 0 \\ 0 & 0 \end{pmatrix} \quad (t > s),$$

$$U(t,s,\mu) = V\begin{pmatrix} 0 & 0 \\ 0 & q(t,s,\mu) \end{pmatrix} \quad (t < s).$$

The matrix U just introduced satisfies the equation

$$\mu\frac{dU}{dt} = \left(\mathcal{U} + \mu V'V^{-1}\right)U \qquad \text{for } t \ne s$$

and the condition of discontinuity

$$U(t,t-0,\mu) - U(t,t+0,\mu) = V(t).$$

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We now make use of lemma 1 and obtain the estimate

$$|U(t,s,\mu)| \leq Ke^{-\frac{\alpha}{2\mu}|t-s|},$$

where K and α are constants.

Considering the transformation

$$S_{t,x}(\Psi) = \frac{1}{\mu}\int_0^T U(t,s+t,\mu)Q^*\{X(s+t,x),\Psi(X(s+t,x),\mu),\mu\}ds,$$

where

$$Q^* = V^{-1}LQ - \mu[V' V^{-1}]Q,$$

and taking the estimates

$$|S_{t,x}(\Psi)| \quad \text{and} \quad |S_{t,x}(\Psi) - S_{t,x^*}(\Psi)|,$$

we arrive at the conclusion that the functions $S\Psi$ belong to class (D,Δ) and that there exists a unique solution of the equation

$$\Psi = S\Psi,$$

which determines the integral manifold for the system of differential equations (6), denoted by

$$\Psi = \Psi(x,t,\mu),$$

the function $\Psi(x,t,\mu)$ possessing the properties as indicated in theorem 2.

Theorems 2, 3, 4 hold also for systems (2)and (3), bearing in mind that if the right-hand sides of system (2) are periodic functions of period T, then the integral manifold $z = Y + \Psi$ will also be periodic in t of the same period.

The theorems obtained allow to establish the structure of the solutions of systems (1), (2), (3) given the structure of the solutions of the respective degenerate systems. For instance, if

system (2') possesses a periodic or almost periodic solution, then system (2) will have a unique periodic or respectively an almost periodic solution.

A special case of this statement has been proved by using this method in the author's article [18].

Finally note that the existence and uniqueness of the integral manifold for systems (1),(2), (3) holds also on an infinite interval t, if the note to lemma 1 is taken into consideration.

In this case, the integral manifold enables us to solve different problems in which the stability of solutions of the above systems is concerned.

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ПЕРИОДИЧЕСКИЕ И АСИМПТОТИЧЕСКИЕ ДВИЖЕНИЯ n СЛОЖНЫХ ФИЗИЧЕСКИХ МАЯТНИКОВ, ПОСЛЕДОВАТЕЛЬНО СОЕДИНЕННЫХ И РАСПОЛОЖЕННЫХ В ОДНОЙ ВЕРТИКАЛЬНОЙ ПЛОСКОСТИ

Dans la présente note je vais relater quelques résults de mes travaux de recherches sur les mouvements périodiques et asymtotiques de l' n pendules physiques situés dans un plan vertical au voisinage de l'état d'équilibre.

Soit un système de n pendules constitué de n corps solides tournant autour des axes parallèles de sorte que le premier pendule tourne autour d'un axe fixe O, le deuxième autour d'un axe O_1 du premier, etc. Le centre de gravité du premier corps S_1 est situé sur la droite OO_1, celui du deuxième corps S_2 sur la droite O_1O_2 etc., et le point O_μ, ne se trouve pas entre S_ν et O_ν. Des pendules au nombre de p sont dirigés en bas et q pendules en haut où $p+q=n$. Si l'on pose $\psi_{z_p}=\lambda\psi_{z_p}$, $\psi_{z_q}=\pi+\lambda\psi_{z_q}$, ou z_p et z_q prennent p respectivement q valeurs différentes, les équations du mouvement de ces pendules sont de la forme

$$\varepsilon_\nu B_\nu\sum_{\kappa=1}^{\nu-1}\varepsilon_\kappa B_\kappa\left[\cos\lambda(\psi_\nu-\psi_\kappa)\psi_\kappa''+\lambda\sin\lambda(\psi_\nu-\psi_\kappa)\psi_\kappa'^2\right]+$$
$$+(A_\nu-J_\nu)\psi_\nu''+\varepsilon_\nu B_\nu\sum_{\kappa=\nu+1}^{n}\varepsilon_\kappa B_\kappa\left[\cos\lambda(\psi_\nu-\psi_\kappa)\psi_\kappa''+\right.$$
$$\left.+\lambda\sin\lambda(\psi_\nu-\psi_\kappa)\psi_\kappa'^2\right]=\varepsilon_\nu\left[gB_\nu\psi_\nu+\lambda gB_\nu\sum_{n=2}^{\infty}(-1)^n\frac{\lambda^{2n-1}\psi_\nu^{2n-1}}{(2n-1)!}\right],$$

où $\varepsilon_{z_p}=1$, $\varepsilon_{z_q}=-1$, ψ_ν est l'angle formé par la direction $\overrightarrow{O_{\nu-1}S_\nu}$ avec la position verticale, $a_\nu=\overrightarrow{O_{\nu-1}S_\nu}$, $b_\nu=\overrightarrow{O_{\nu-1}O_\nu}$, J_ν est le moment d'inertie du $\nu^{\text{ème}}$ pendule par rapport à son centre de gravité et

$$A_\nu=a_\nu^2m_\nu+b_\nu^2\sum_{\kappa=\nu+1}^{n}m_\kappa,\quad B_\nu=a_\nu m_\nu+b_\nu\sum_{\kappa=\nu+1}^{n}m_\kappa.$$

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On voit immédiatement des équations /I/ que si le système de pendules à son mouvement passe deux fois par l'état d'équilibre, le mouvement devient périodique, c'est-à-dire la solution des équations /I/ exprimant ce mouvement est périodique. Cette remarque est utilisée pour démontrer l'existence de mouvements périodiques de /I/.

I. Pour $\lambda = 0$ l'équation différentielle /I/ est système linéaire

$$\varepsilon_\nu B_\nu \sum_{\kappa=1}^{\nu-1} c_\kappa B_\kappa \psi_\kappa'' + (A_\nu + J_\nu)\psi_\nu'' + \varepsilon_\nu B_\nu \sum_{\kappa=\nu+1}^{n} \varepsilon_\kappa B_\kappa \psi_\kappa'' = -\varepsilon_\nu g B_\nu \psi_\nu. \qquad /2/$$

Si l'on multiplie les colonnes et les lignes avec $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ l'équation caractéristique du système /2/ obtient la forme

$$|a_{i\kappa} p^2 + b_{i\kappa}| = 0. \qquad /3/$$

où $a_{ii} = A_i + J_i$, $b_{ii} = \varepsilon_i g B_i$, $a_{i\kappa} = a_{\kappa i} = B_\kappa B_i$ $(i > \kappa)$, $b_{i\kappa} = 0$ $(\kappa \neq i)$. Si l'équation caractéristique /3/ possede une paire de racines conjugués purement imaginaires $\pm i\rho_\kappa$ et si toute autre racine n'est pas multiple de $i\rho_\kappa$ et en outre les racines différent entre elles, on peut demontrer qu'il existe une telle famille de mouvements périodiques où tous les pendules restent dans le voisinage de la position verticale et dont la période est approximativement égale à $\frac{2\pi}{\rho_\kappa}$.

Comme démonstration de ceci nous examinons spécialement l'intégrale particulière de /2/

$$\psi_\nu = \lambda_{\nu\kappa} \sin \rho_\kappa t, \quad \psi_\nu' = \lambda_{\nu\kappa} \rho_\kappa \cos \rho_\kappa t \qquad /4/$$

correspondant aux valeurs initiales

$$\psi_\nu(0) = 0, \quad \psi_\nu'(0) = \lambda_{\nu\kappa} \rho_\kappa.$$

L'intégrale de /I/ aux valeurs initiales modifiées

$$\psi_\nu(0) = 0, \quad \psi_\nu'(0) = \lambda_{\nu\kappa} \rho_\kappa + \sum_{\mu=1}^{n}{}' L_{\nu\mu} \beta_{\nu\mu} \alpha_\mu \quad (\alpha_\kappa = 0)$$

est de la forme

$$\psi_\nu(t) = \psi_\nu(t, \alpha_1, \ldots, \alpha_{\kappa-1}, \alpha_{\kappa+1}, \ldots, \alpha_n, \lambda),$$

où les membres droits sont développés en series suivant les puissances croissantes de $\alpha_1, \ldots, \alpha_n, \lambda$.

L'intégrale /4/ a pour période $\frac{2\pi}{\rho_\kappa}$. L'intégrale /5/ aura, pour des valeurs suffisamment petites de $|\lambda|, |\alpha_1|, \ldots, |\alpha_n|$ la période modifiée $\frac{2\pi + 2\delta}{\rho_\kappa}$ si

$$\psi_\nu\left(\frac{\pi+\delta}{\rho_\kappa}, \alpha_1, \ldots, \alpha_{\kappa-1}, \alpha_{\kappa+1}, \ldots, \alpha_n, \lambda\right) = 0. \qquad /6/$$

Si l'on pose $\omega_\mu = \frac{\pi \rho_\mu}{\rho_\kappa}$ les series entières des équations /6/ peuvent être représentées sous la forme

$$-L_{\nu\kappa}\delta + \sum_{\mu=1}^{n}{}' L_{\nu\mu} \sin \omega_\mu \alpha_\mu + f_\nu = 0 \quad (\nu = 1, \ldots, n). \qquad /7/$$

où les membres de f_ν qui ne possèdent pas de λ sont de deuxième puissance.

Le déterminant fonctionnel de /7/ par rapport aux $\delta, \alpha_1, \ldots, \alpha_{\kappa-1}, \alpha_{\kappa+1}, \ldots, \alpha_n$ est

$$\Phi = -\sin\omega_1 \ldots \sin\omega_{\kappa-1} \sin\omega_{\kappa+1} \ldots \sin\omega_n \begin{vmatrix} L_{11} \ldots L_{1n} \\ \ldots\ldots\ldots \\ L_{2n} \ldots L_{nn} \end{vmatrix}.$$

Ce déterminant est différent de zéro si ω_μ ne sont pas multiples de π, c'est-à-dire si ρ_μ $(\mu \neq \kappa)$ ne sont pas multiples de ρ_κ. Cette condition est suffisante pour que les équations /7/ admettent par rapport à δ_μ, α_μ une seule solution comme fonction de λ ce qui prouve qu'il existe une famille de mouvements périodiques dont la période est approximativement égale à $\frac{2\pi}{\rho_\kappa}$.

Si le nombre des racines non imaginaires de l'équation caractéristique /3/ est $2h$, alors il y a h

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racines négatives et réelles. On peut s'attendre donc à ce que dans le voisinage de l'état d'equilibre instable il existe des mouvements asymptotiques dépendant de h constantes arbitraires. Nous avons prouvé [1] qu'à certaines conditions une telle famille de mouvements existe c'est-à-dire que $\mathcal{E}'_n$ pendule multiple pour $t \to \infty$ s'approche de l'état d'équilibre instable tout pendule restant toujours d'un même côté de sa verticale et son mouvement étant constamment dirigé vers cette verticale.

2. Reste à établir quelles sont les racines de l'équation caractéristique. Dans un travail [3] nous avons prouvé que l'équation /3/ possède 2ρ racines purement imaginaires, q racines positives et q racines négatives. Ayant en vue cette qualité de l'équation caractéristique et en se basant sur les résultats ci-dessus on peut en conclure qu'en general l' n pendule multiple possède ρ familles de mouvements periodiques et une famille de mouvements purement asymptotiques dependant de q paramètres c'est-à-dire si q pendules sont dirigés en haut, q familles de mouvements périodiques disparaissent et à leur place apparaît une famille de mouvements purement asymptotiques dependant de q paramètres.

Nous avons démontré aussi qu'en particulier l'équation caractéristique /3/ de l' n pendule mathématique multiple ne possède que des racines simples.

De même nous avons prouvé [3] que dans le voisinage de l'état d'équilibre stable ($\rho = n$) il n'y a pas d'autres mouvements périodiques que ceux où le système de pendules passe deux fois par l'état d'équilibre stable à condition que toutes $\rho_1, \ldots, \rho_n$ ne soient pas rationnelles et que pas un des nombres ρ_k / ρ_κ ne soit pas entier.

3. A présent on peut poser la question si l' n pendule mathématique possède des mouvements périodiques et dans le cas critique pour lequel une ou plusieurs racines sont multiples de la racine donnée. Dans un autre travail [3] nous avons demontré que pour cette racine il existe en general autant de familles de mouvements périodiques qu'est le nombre des solutions réelles d'un système algébrique de quatrième degré de $\rho - 1$ equations avec $\rho - 1$ inconnues lequel exprime les conditions de periodicite. Ici $\rho - 1$ est le nombre des racines qui sont multiples de la racine donnée.

Nous allons donner quelques moments de la démonstration.

Admettons que $\rho - 1$ racines sont multiples de ρ_1 par exemple de, par exemple

$$\rho_k = \kappa_k \rho_1 \qquad (k = 2, \ldots, \rho)$$

où $\kappa_2, \ldots, \kappa_\rho$ sont des nombres entiers positifs. Pour de telles racines le déterminant fonctionnel est égal à zéro et la démonstration pour l'existence de solutions périodiques de la racine cesse d'etre en vigueur. Dans ce cas l'examen de la question exige un traitement non linéaire.

Nous avons démontré que pour la racine ρ_1 il existe aussi des mouvement périodiques.

Pour cette demonstration on utilise aussi la remarque faite précédemment /plus haut/ d'aprés laquelle le mouvement est périodique si l' n pendule mathématique passe á son oscillation deux fois par l'état d'équilibre.

Dans le but on examine l'intégrale particuliére du systéme /2/

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$$\psi_{\nu 0}(t) = L_{\nu 1} \sin \rho_1 t + \sum_{\mu=2}^{p} L_{\nu\mu} N_\mu \sin \kappa_\mu \rho_1 t,$$

$$(\nu = 1, \ldots, n) \qquad /8/$$

dont la période est $\frac{2\pi}{\rho_1}$, et les valeurs initiales sont

$$\psi_{\nu 0}(0) = 0, \quad \psi'_{\nu 0} = \rho_1 L_{\nu 1} + \rho_1 \sum_{\mu=2}^{p} \kappa_\mu L_{\nu\mu} N_\mu. \qquad /9/$$

où $N_2, \ldots, N_p$ sont des constantes inconnues qui seront déterminées de telle façon que la solution recherchée du système /I/ soit périodique.

L'intégrale de /I/ à valeurs initiales modifiées

$$\psi_\nu(0) = 0,$$

$$\psi'_\nu(0) = \rho_1 L_{\nu 1} + \rho_1 \sum_{\mu=2}^{p} L_{\nu\mu}(N_\mu + \alpha_\mu) + \sum_{\mu=p+1}^{n} L_{\nu\mu} \rho_\mu \alpha_\mu \qquad /10/$$

peut être développé, pour des valeurs de λ et de α_μ suffisamment petites, dans la forme suivante

$$\psi_\nu(t) = \psi_{\nu 0}(t) + \sum_{\mu=2}^{n} P_{\nu\mu}(t) \alpha_\mu + \lambda^2 \left[Q_\nu(t) + \sum_{\mu=2}^{n} R_{\nu\mu}(t) \alpha_\mu + \ldots \right]. \qquad /11/$$

Les coefficients $P_{\nu\mu}$, Q_ν et $R_{\nu\mu}$ représentent des solutions d'un système d'équations différentielles non homogènes, solutions qui peuvent être trouvées par de longs développements.

En exprimant la condition de périodicité de l'intégrale /II/ et après certains calculs et transformations, on arrive au système par rapport aux paramètres inconnus N_μ:

$$\sum_{i_2, \ldots, i_p = 0}^{4} Q_{m, i_2, \ldots, i_p} N_2^{i_2} \ldots N_p^{i_p} = 0$$

$$(i_2 + \ldots + i_p \leq 4, \quad m = 1, \ldots, p-1). \qquad /12/$$

Ce système a au plus 4^{p-1} solutions réelles pour $N_2, \ldots, N_p$. Un examen détaillé ultérieur des solutions s'avère assez difficile, les coefficients $Q_{m, i_2, \ldots, i_p}$ étant des expressions compliquées et le nombre des équations étant $p-1$. Pourtant on peut tirer les conclusions suivantes:

Si le système /I2/ admet S solutions réelles $N_{2s}, \ldots, N_{ps}$ où $s \leq 4^{p-1}$ il existe, pour l'n pendule multiple dans le voisinage de l'état d'équilibre, S familles de mouvements périodiques pour la racine ρ_1 à période modifiée $\frac{2\pi + 2\delta}{\rho_1}$ qui s'expriment par des équations approximatives

$$\psi_{\nu s} \approx \lambda \left[L_{\nu 1} \sin \rho_1 t + \sum_{\mu=2}^{p} L_{\nu\mu} N_{\mu s} \sin \kappa_\mu \rho_1 t \right]. \qquad /13/$$

Bien entendu il est possible qu'une partie ou tous les nombres κ_μ soient égaux à I.

En particulier si une seule racine de l'équation caractéristique est multiple de la racine donnée, par exemple $\rho_2 = \kappa \rho_1$, le système /I2/ se réduit à une équation de troisième degrée par rapport au paramètre unique N:

$$P(N) \equiv \rho_0 N^3 + \rho_1 N^2 + \rho_2 N + \rho_3 = 0. \qquad /14/$$

Après certains calculs on démontre que le coefficient ρ_3 pour $\kappa \neq 1, 3$ est égal à zéro et que pour $\kappa = 1, 3$ il est différent de zéro. On peut en tirer les conclusions suivantes:

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1. Pour $\varkappa \neq 1, 3$ l'équation cubique /I7/ possède une racine zéro parce que $\rho_3 = 0$. Au moyen d'un calcul direct on établit que pour cette racine $\frac{d\rho}{d\mathcal{N}} \neq 0$. On peut en conclure que pour $\varkappa \neq 1, 3$ il existe une famille de mouvements périodiques pour l'n pendule multiple dans le voisinage de l'état d'équilibre à période modifiée $\frac{2\pi + 2\delta}{\rho_1}$. Ces mouvements peuvent être exprimés par les équations approximatives

$$\varphi_\nu(t) \approx \lambda L_{\nu 1} \sin \rho_1 t \qquad /I5/$$

Si les autres racines de l'équation /I4/ sont réelles et $\frac{d\rho}{d\mathcal{N}} \neq 0$, il existe pour la racine $\pm\rho_1$ encore deux familles de mouvements périodiques pour l'n pendule multiple qui s'expriment par les équations approximatives

$$\varphi_\nu(t) \approx \lambda \left(L_{\nu 1} \sin \rho_1 t + L_{\nu 2} \mathcal{N} \sin \varkappa \rho_1 t \right). \qquad /I6/$$

2. Pour $\varkappa = 3$ le membre libre de l'équation cubique est en général différent de zéro. Ceci prouve que dans ce cas l'équation cubique /I4/ possède au moins une racine réelle différente de zéro. Par conséquent si cette racine est simple, il existe une famille de mouvements périodiques pour l'n pendule multiple dans le voisinage de l'état d'équilibre à période modifiée $\frac{2\pi + 2\delta}{\rho_1}$ dont les équations approximatives sont de la forme /I6/. Si les autres racines de l'équation cubique sont aussi réelles et en outre simples, il existe encore deux familles de mouvements périodiques pour l'n pendule multiple dont les équations approximatives sont aussi de la forme /I6/.

3. Pour $\varkappa = 1$ l'équation caractéristique possède une racine double $\pm\rho_2 = \pm\rho_1$. Le membre libre de l'équation cutique /I4/ etant en general différent de zéro cette équation possède au moins une racine reelle différente de zéro. Par suite si cette racine est simple, il existe une famille de mouvements périodiques de l'n pendule multiple dans le voisinage de l'état d'équilibre à période modifiée $\frac{2\pi + 2\delta}{\rho_1}$ dont les équations approximatives sont de la forme

$$\varphi_\nu(t) \approx \lambda \left[L_{\nu 1} \sin \rho_1 t + L_{\nu 2} \mathcal{N} \sin \rho_1 t \right]. \qquad /I7/$$

Si les autres racines de l'équation cubique sont réelles et simples, il y aura encore deux familles de mouvements périodiques dont les équations approximatives sont de la forme /I7/.

Tous les résultats indiqués jusqu'ici sont aussi valables pour un système de pendules physiques situés dans un plan vertical qui tourne uniformément autour d'un axe situé en dehors du plan des pendules.

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Л. ПУСТ

ПЕРЕХОД ЧЕРЕЗ ОБЛАСТЬ РЕЗОНАНСА У КОЛЕБАТЕЛЬНЫХ МЕХАНИЧЕСКИХ СИСТЕМ С УЧЕТОМ ВЛИЯНИЯ ВИБРАТОРА

When passing through a resonant zone we gradually change the frequency of the exciting force and determine the course of the amplitude a and the phase angle φ with respect to time. We usually make the assumption that the relation of the frequency of the exciting force with respect to time - or in the case of a vibrator with a low output the relation of the energy input with respect to time - is given.

As we know from literature, the resulting relation of amplitude with respect to time in a transient state shows a considerable variation from the conditions when the system is in steady state. Thus, for instance, the peak of the resonance curve is lowered, it is shifted towards higher (or lower) frequencies and there is also a marked change in the shape of the resonance curve, which may even show further maxima. The dependence of the phase angle on time differs considerably from the curves of the steady state, as the phase angle usually rises rapidly and reaches values which may be several times higher than those of the periodic state.

In systems with more degrees of freedom, particularly where the resonance zones are near

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each other, then when passing through these zones some of the peaks may be suppressed by others, new maxima may appear, so that in the experimental determination of the resonance curves we get quite a distorted picture.

In this paper it is shown the method of solving the transient phenomena when passing through resonance zones in vibratory systems which contain a non-linear member which may be a non-linear spring and a damper or the non-linearity may be given by the characteristics of the vibrator, or both.

Contrary to the usual methods of solution in which the course of frequency $\omega(t)$ is given and we seek $a(t)$, $\psi(t)$ etc. we shall start out with the demand that the course of the amplitude or of the phase angle will deviate as little as possible from the values in the wave shape of stationary resonance curves; that is we choose, for instance, $a(\omega)$ and seek the course of the other unknown quantities such as the phase angle ψ, frequency ω and the value by which we regulate the energy input into the vibrator, etc.

The calculation of the problem stated above, explains some properties of the transient phenomena from a different point of view than it was known heretofore. This method of approach has a further advantage in that the numerical solution is simpler, as for instance in a system with one degree of freedom and considering the

properties of the vibrator, it suffices to solve only one differential equation, instead of the necessity of solving three differential equations. At the same time we get the direct dependence of unknown quantities at the interdependence of the quantities we seek from frequency ω so we can plot them into the co-ordinate system in which we have plotted the curves of stationary periodic vibrations.

Let us assume a mechanical system (Fig.1) which is vibrated by means of a spring with coefficient of rigidity c_1 and a cam with radius r, driven by an electric motor whose characteristics are $M(\dot{\varphi},\alpha)$.

The differential equations of the motion are:

$$m\ddot{x}+k\dot{x}+cx+\varepsilon f(x,\dot{x})=c_1 r\cos\varphi,$$

$$-J\ddot{\varphi}+M(\dot{\varphi},\alpha)+c_1 r(-x+r\cos\varphi)\sin\varphi=0, \qquad (1)$$

where:

- J is the moment of inertia of the vibrator with the motor,
- φ is the angle of displacement of the rotor,
- α is the parameter determining the energy input into the system.

When taking the momentary angular velocities at $\frac{d\varphi}{dt}=\vartheta$ the equations will assume the following form

$$m\left(\vartheta\frac{d\vartheta}{d\varphi}\frac{dx}{d\varphi}+\vartheta^2\frac{d^2x}{d\varphi^2}\right)+k\vartheta\frac{dx}{d\varphi}+cx+\varepsilon f\left(x,\vartheta\frac{dx}{d\varphi}\right)=c_1 r\cos\varphi,$$

$$-\vartheta\frac{d\vartheta}{d\varphi}+\varepsilon\rho M(\vartheta,\alpha)+\varepsilon\rho c_1 r(-x+r\cos\varphi)\sin\varphi=0, \qquad (2)$$

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where $\frac{1}{J} = \varepsilon\rho$.

If we apply the method of Krylov and Bogoljubov we can, in the vicinity of the resonance curves, that is in the ε-th range of the resonance curve, write

$$\begin{aligned} &[c - \omega^2 m]a_0 - c_1 r\cos\psi_0 = 0, \\ &k\omega a_0 - c_1 r\sin\psi_0 = 0, \\ &\omega = const. \end{aligned} \tag{3}$$

The solutions of the equations (2) can be expressed formally in the form of asymptotic series

$$\begin{aligned} x &= a\cos(\varphi - \psi) + \varepsilon\mu_1(a,\psi,\omega,\varphi) + \dots \\ \vartheta &= \omega + \varepsilon\omega_1(a,\psi,\omega,\varphi) + \dots \end{aligned} \tag{4}$$

where the amplitude a, phase angle ψ and average angular velocity ω are given by the relations:

$$\begin{aligned} \frac{da}{d\varphi} &= \varepsilon A_1(a,\psi,\omega) + \varepsilon^2 A_2(a,\psi,\omega) + \dots, \\ \frac{d\psi}{d\varphi} &= \varepsilon B_1(a,\psi,\omega) + \varepsilon^2 B_2(a,\psi,\omega) + \dots, \\ \frac{d\omega}{d\varphi} &= \varepsilon D_1(a,\psi,\omega) + \varepsilon^2 D_2(a,\psi,\omega) + \dots \end{aligned} \tag{5}$$

After substituting (4) and (5) into equation (2) and disregarding all terms higher than ε^1 whereby we substitute the non-linear function $\varepsilon f(x,\vartheta,\frac{dx}{d\varphi})$ by the expression

$$\varepsilon C_1(a)x + \varepsilon k_1\omega\frac{dx}{d\varphi} \tag{6}$$

we can write

$$\begin{aligned} \frac{da}{d\varphi} &= \frac{1}{2\omega^2 m}\left[m\omega\varepsilon D_1 a + (k + \varepsilon k_1)a\omega - c_1 r\sin\psi\right], \\ \frac{d\psi}{d\varphi} &= -\frac{1}{2\omega^2 ma}\left[(c - \varepsilon C_1(a))a - m\omega^2 a - c_1 r\cos\psi\right], \\ \frac{d\omega}{d\varphi} &= \varepsilon D_1 = \frac{\varepsilon\rho}{\omega} M(\omega,\alpha) - \frac{\varepsilon\rho c_1 r a}{2\omega}\sin\psi \end{aligned} \tag{7}$$

An analysis of the results of these equations for a given course of velocity α with respect to the angle φ or time t but particularly for the given course of the parameter α on the angle $\varphi(or\,t)$ is very difficult, as the results can be arrived at only by a numerical solution of the system of three differential equations.

Such solution, without considering the properties of the vibrator was arrived at, for instance, by Ju.A.Mitropolski and the solution including the properties of the vibrator by V.O. Kononenko with his collaborators, brought forth some interesting results which have already been mentioned before.

If we examine more closely the reasons for the occurence of further maxima, which are caused by the fact that the phase angle ψ in the transient states does not follow the curve of steady periodic vibrations $\psi_0(\omega)$ but after passing resonance begins to rise steeply and changes not within the range $0 \div 2\pi$ but within the range $0 \div 2\pi\ell$ where $\ell >> 1$. The terms of equation (7) multiplied by $\sin\psi$ or $\cos\psi$ change the sign, even several times, in course of the transient state, which is reflected not only in the variation of the amplitude a but also in the angular velocity ω.

Let us now determine such the input of energy into the vibrator (defined here by the parameter α) so as to receive the desired shape of the resonance curve.

From the first two equations (7) we can, with

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the help of the third equation, eliminate the variable φ and substitute it by the independent variable ω.

Having introduced the expression

$$\frac{d}{d\varphi} = \frac{d\omega}{d\varphi}\frac{d}{d\omega} = \varepsilon D_1 \frac{d}{d\omega} \tag{8}$$

and assuming that $\varepsilon D_1 \neq 0$ we have

$$\varepsilon D_1\left(\frac{da}{d\omega} + \frac{a}{2\omega}\right) = -\frac{1}{2\omega^2 m}\left[(k+\varepsilon k_e)a\omega - c_1 r\sin\psi\right],$$
$$\varepsilon D_1 \frac{d\psi}{d\omega} = \frac{-1}{2\omega^2 m a}\left[(c+\varepsilon c_e)a - m\omega^2 a - c_1 r\cos\psi\right] \tag{9}$$

These relations describe directly the dependence of the amplitude and the phase angle on the frequency ω, that is in the co-ordinates in which we usually also express the resonance curves or periodic excited vibrations. Assuming again that the right part of the equation (9) is not equal to zero we receive, by eliminating εD_1, the basic relation:

$$\frac{da}{d\omega} + \frac{a}{2\omega} = \frac{d\psi}{d\omega}\, a\, \frac{(k+\varepsilon k_e)a\omega - c_1 r\sin\psi}{(c+\varepsilon c_e - m\omega^2)a - c_1 r\cos\psi}. \tag{10}$$

For a given course of $\psi(\omega)$ we can ascertain by the solution of only one differential equation the amplitude resonance curve $a(\omega)$ and after substituting into any one of the equations (9) determine directly the value $\varepsilon D_1 = \frac{d\omega}{d\varphi}$:

$$\varepsilon D_1 = \frac{-(k+\varepsilon k_e)a\omega + c_1 r\sin\psi}{m\omega\left[2\omega\frac{da}{d\omega} + a\right]}; \quad \varepsilon D_1 = \frac{-(c+\varepsilon c_e - m\omega^2)a + c_1 r\cos\psi}{m 2\omega^2 a \frac{d\psi}{d\omega}}. \tag{11}$$

Then from the third equation (7) we can simply determine the required relationship of the moment $M(\omega,\alpha)$, respectively the energy input α at which the resonance curves will acquire the desired properties.

$$M(\omega,\alpha) = \frac{rc_1 a}{2}\sin\psi + \frac{rc_1\cos\psi - (c+\varepsilon c_e - m\omega^2)a}{m 2\omega a \frac{d\psi}{d\omega}}\,\varepsilon\rho. \tag{11a}$$

If we know $M(\omega),\alpha(\omega)$ we receive, due to the relation $\varphi = \int_{\omega_0}^{\omega} \frac{d\omega}{\varepsilon D_1(\omega)}$ and $t = \int_{\omega_0}^{\omega} \frac{d\omega}{\omega\varepsilon D_1(\omega)}$ the functions $M(\varphi),\alpha(\varphi)$ resp. $M(t),\alpha(t)$.

When arriving at the relationship (10) we made certain assumptions which need not hold true for the entire range of values of a,ψ,ω. When coming to the solution of real cases let us analyze the relations shown in equation (10). We transform the left part and get

$$\frac{d(a^2\omega)}{d\omega} = 2\omega a^2 \frac{d\psi}{d\omega}\,\frac{c_1 r\sin\psi - (k+\varepsilon k_e)a\omega}{c_1 r\cos\psi - (c+\varepsilon c_e - m\omega^2)a}. \tag{12}$$

The value $2\omega a^2$ is always positive and $\frac{d\psi}{d\omega}$ is dependent on the course of the function $\psi(\omega)$. For the sake of simplicity let us assume this function to be monotone $\frac{d\psi}{d\omega} > 0$. Then the value of the derivative, that is if the integral curve will have an increase or a decrease characteristics, will be determined by the sign of the numerator and denominator on the right part of the equation (12).

Upon fulfilling the condition

$$c_1 r\sin\psi - (k+\varepsilon k_e)a\omega = 0$$

then (with the exception of the singular points) we'll have $\frac{d(a^2\omega)}{d\omega} = 0$.

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1/ Curves A , B have several peak values;
2/ At the intersections A , B further singular points appear II, III /focal type/;
3/ At these singular points the integral curves assume higher values.
4/ Between the singular points the integral curves have steeper characteristic than in the case analyzed before $(0<\psi<2\pi)$ which is given by a higher value of $\frac{d\psi}{d\omega}$.
5/ At the intersection of the curve B with the axis ω further singular points IV, V appear (saddle type).

It is apparent from Fig.3 that the tangent $\frac{da}{d\omega}$ behind the region of resonance changes its sign and further maxima appear. The position of these maxima is given by the decreasing part of the curve A , the position of the minima by the increasing part.

If we wish, when passing the region of resonance, to have the curve of amplitude $a(\omega)$ as near as possible to the curve of steady periodic vibrations, we can, on the basis of the results so far obtained, state that:

1/ The singular point I will be as nigh as possible, that is the curve $\psi(\omega)$ will intersect $\psi_0(\omega)$ very near the angle $\psi_0=90°$; The curve C limiting the maximum height of the first resonance peak of the curves $a(\omega)$, then approximately reaches the amplitude of the peak of the curve of steady vibrations;
2/ The point $\psi=90°$ will not be shifted too far in the direction of the axis ω from the point

$\psi_0=90°$,so that the displacement of the peak of the transient resonance curve from the peak of the curve of steady vibrations will be as small as possible.
3/ The angle ψ will not at any time during the transient state exceed the value of $360°$, as with angles of the higher value the amplitude resonance curves begin to rise after passing the first resonance peak, thus giving rise to another peak.

An example of such a curve $\psi(\omega)$ is shown in Fig.4b. Fig.4a shows the course of amplitude resonance curve $a(\omega)$. Part c in Fig.4 shows the rate of change of the average angular velocity εD, , the course of the requisite moment $M(\omega,\alpha)$ and the dependence of parameter α on ω for the relationship $M(\omega,\alpha)=40\alpha(\omega-\alpha)$, $\varepsilon\rho=1$.

In a similar way further Fig.5 shows the analysis of transient phenomena in systems which contain besides a vibrator of non-linear properties also a certain non-linearity in the system itself. The system which we shall now analyse has the same values as before, but besides that it contains non-linear element:

$$\varepsilon f(x,\dot{x})=0{,}53x^4 ; \qquad \varepsilon G(a)=0{,}4a^2 .$$

Besides the transient curve with steadily rising angular velocity ω (Fig.5) there is a solution on the next Fig.6 of the case where during a part of the transient state the angular velocity decreases. In this region of the diagrams $\omega^x<\omega<\omega^{xx}$ the tangents of the integral curves are multivalued. The other characteristics

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of the curves are similar as in the cases we analyzed before. Even here, through suitable energy input (shown at the bottom of Fig.6) we can reach the position where the resonance curves of the transient state lie near the curves of the steady state. As it can be seen in Fig.5,6, the moment can in some interval be even negative. In these figures the curves A , B show the geometrical loci of vertical and horizontal tangents, points I, II the singular points of the saddle and focal type.

The described method of analyzing transient phenomena allows examination in relatively simple ways of the nutual dependence of the variable amplitude a phase angle φ angular velocity ω angular acceleration εD, moment of the vibrator M and the energy input α and so to find directives how to determine the energy input into the system or how to pass and with what speed the regions of resonance so that the amplitude and phase resonance curves would have the desired properties.

It is supposed to extend the use of this method on the analysis of transient phenomena in systems with two degrees of freedom.

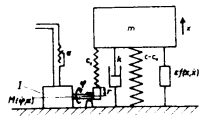

Fig 1

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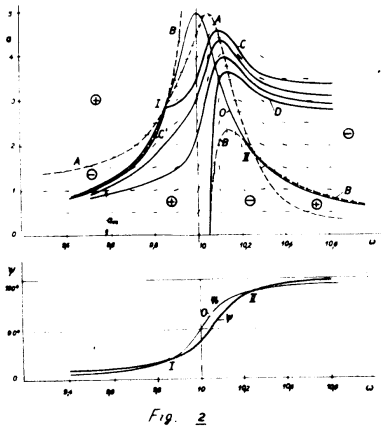

Fig. 2

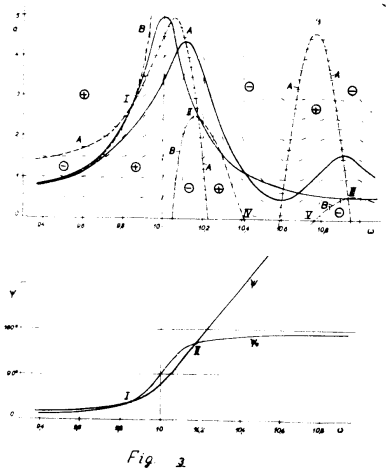

Fig. 3

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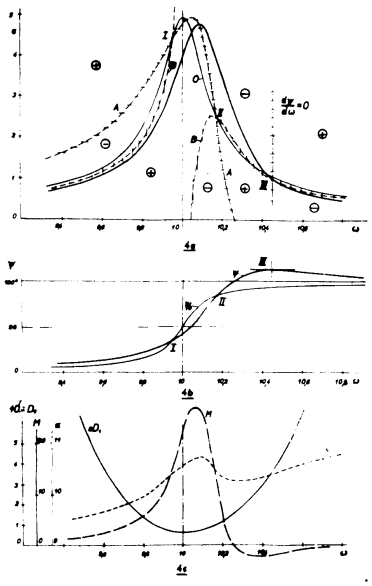

Fig 4

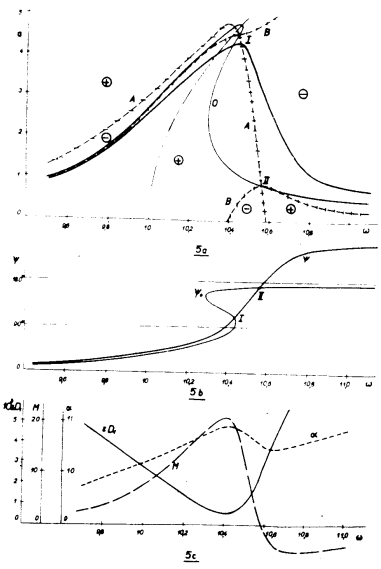

Fig 5

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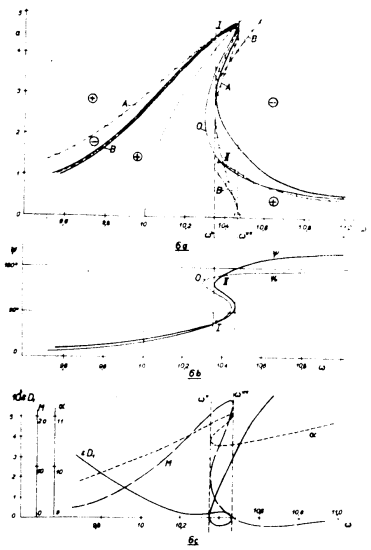

Fig 6

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А. ТОНДЛЬ

О РЕШЕНИИ НЕКОТОРЫХ ЗАДАЧ КВАЗИЛИНЕЙНЫХ СИСТЕМ

The paper presents a description of several methods for studying vibrations of quasi-linear systems with particular respect paid to the suitability of their application in engineering practice. With the exception of the third method which was elaborated just recently, the procedures outlined in the paper have been used for solving various problems from the field of rotor dynamics.

In view of the limited scope the paper is primarily devoted to the elucidation of the principles underlying the methods under description.

I. Method for determining the intervals of instability of quasi-harmonic systems

The problem of the stability of periodic vibration of quasi-linear systems or the problem of stability of quasi-harmonic systems leads to the study of the stability of the solutions of differential equations with periodic variable coefficients. In the general case they can be expressed in the form

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$$\ddot{y}_s + \sum_{k=1}^{n} a_{sk} y_k = \varepsilon \sum_{k=1}^{n} (q_{sk} \dot{y}_k + p_{sk} y_k) \qquad (s = 1,2,3,\ldots,n) \qquad (1)$$

where a_{sk} are constants $q_{sk} = q_{sk}(t)$, $p_{sk} = p_{sk}(t)$ are periodic continuous time functions with a period $2\pi/\omega$ (or constants) which can be expanded into Fourier series, and ε is the small parameter. Our task is to determine the widths of intervals ω when the motion is unstable.

By substituting

$$\omega t = \tau \qquad (2)$$

we can write the system of equations (1) in the form

$$\ddot{y}_s + \sum_{k=1}^{n} \frac{a_{sk}}{\omega^2} y_k = \varepsilon \sum_{k=1}^{n} \left(\frac{q_{sk}}{\omega} \dot{y}_k + \frac{p_{sk}}{\omega^2} y_k \right), \qquad (s = 1,2,3,\ldots,n) \qquad (3)$$

so that functions q_{sk}, p_{sk} now have a period 2π.

Let us assume that the roots of the characteristic equation of the abbreviated system (1) (for $\varepsilon = 0$)

$$\ddot{y}_s + \sum_{k=1}^{n} a_{sk} y_k = 0 \qquad (s = 1,2,3,\ldots,n) \qquad (4)$$

are all purely imaginary and simple $\pm i\Omega_s$ $(s = 1,2,3,\ldots,n;\ \Omega_k \neq \Omega_j;\ \Omega_s^2 > 0,\ i = \sqrt{-1})$.

It is known for the differential equations with periodically variable coefficients of type (1) that the intervals of unstable solutions can be expected in the vicinity of those values of $\omega = \omega_0$ for which such an equality holds

/see, for instance, [1, 2, 3] :

$$\omega_0 = \frac{\Omega_j \pm \Omega_r}{N} \qquad \begin{matrix} (j, r = 1,2,\ldots,n) \\ (N = 1,2,\ldots) \end{matrix} \qquad (5)$$

It was proved for certain systems (e.g. canonical) that such intervals of instability are unique.

We shall now outline the procedure used in the determination of these intervals. A more detailed treatment is contained in (4).

Let us introduce the notation

$$\lambda_s = i \frac{\Omega_s}{\omega} , \qquad (6)$$

then

$$\lambda_j \pm \lambda_k = iN \qquad (N \text{ is an integer}) \qquad (7)$$

corresponds to equation (5).

Let us now transform system (3) to quasi-normal, i.e. so that its abbreviated system be converted to normal coordinates. We then obtain the system

$$\ddot{x}_s - \lambda_s^2 x_s = \varepsilon \sum_{k=1}^{n} (Q_{sk} \dot{x}_k + P_{sk} x_k), \qquad (s = 1,2,3,\ldots,n) \qquad (8)$$

where $Q_{sk} = Q_{sk}(\tau)$, $P_{sk} = P_{sk}(\tau)$ are again periodic continuous functions with a period 2π, which can be expanded into Fourier series. Substituting

$$x_s + \lambda_s x_s = \xi_s , \quad \dot{x}_s - \lambda_s x = \rho_s \qquad *) \qquad (9)$$

*) Substitution (9) can also be used in the form

$$\xi_s + \eta_s = e_s , \quad -\frac{1}{\lambda_s}\dot{x}_s + x_s = \rho_s . \qquad (9a)$$

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we convert system (8) to the form

$$\xi_s' = \lambda_s \xi_s + \frac{1}{2}\varepsilon \sum_{k=1}^{n} [Q_{sk}(\xi_k + \rho_k) + \frac{1}{\lambda_k} P_{sk}(\xi_k - \rho_k)],$$
$$\rho_s' = \lambda_s \rho_s + \frac{1}{2}\varepsilon \sum_{k=1}^{n} [Q_{sk}(\xi_k + \rho_k) + \frac{1}{\lambda_k} P_{sk}(\xi_k - \rho_k)] \qquad (10)$$
$$(s = 1, 2, 3, \ldots, n)$$

On the basis of Floquet's theorem we can write the solution in the form

$$\xi_s = u_s e^{(\lambda + i\varepsilon\varkappa(\varepsilon))\tau}, \quad \rho_s = v_s e^{(\lambda + i\varepsilon\varkappa(\varepsilon))\tau}, \qquad (11)$$

where u_s and v_s are periodic functions in τ with a period 2π and λ is one of the roots of the characteristic equations of the abbreviated system.

The condition of the instability of the solution is that an imaginary part of $\varkappa$ be negative.

Substituting (11) into system (10) we obtain

$$u_s' = (\lambda_s - \lambda)u_s - i\varepsilon\varkappa u_s + \frac{1}{2}\varepsilon U_s,$$
$$v_s' = (\lambda_s - \lambda)v_s - i\varepsilon\varkappa v_s + \frac{1}{2}\varepsilon V_s, \qquad (s = 1, 2, 3, \ldots, n) \qquad (12)$$

where

$$U_s = \sum_{k=1}^{n} [Q_{sk}(u_k + v_k) + \frac{1}{\lambda_k} P_{sk}(u_k - v_k)],$$

If we now desire to determine the interval of instability in the vicinity of value

$$\lambda_j = i\nu_p$$

we insert in equations (12) [illegible] for ν

and introduce the substitution

$$\lambda_j + \lambda_r = iN + \varepsilon i\alpha, \qquad (13)$$

where α is a real function of ε. We then obtain the system

$$u_r' = \varepsilon(-i\varkappa u_r + \frac{1}{2}U_r),$$
$$v_j' = -iNv_j + \varepsilon[-i(\varkappa + \alpha')v_j + \frac{1}{2}V_j],$$
$$u_s' = (\lambda_s - \lambda_r)u_s + \varepsilon(-i\varkappa u_s + \frac{1}{2}U_s), \qquad (s \neq r)$$
$$v_s' = -(\lambda_s + \lambda_r)v_s + \varepsilon(-i\varkappa v_s + \frac{1}{2}V_s) \qquad (s \neq j) \qquad (14)$$

If we seek the periodic solution u_s, v_s with a period 2π by means of the modified Poincare's method, we obtain from the condition of periodicity a solution with any arbitrary accuracy (according to the number of approximation) of coefficient $\varkappa$ as a function of α. Coefficient $\varkappa$ with a negative imaginary part will correspond to the interval of unstable solutions. From this condition we determine α and the limits of the unstable interval.

Thus we can successively limit all intervals of instability. The advantage of the above procedure is that instead of proceeding tediously as with other methods we divide the solution into several calculations and, if we are satisfied with the first approximation, we obtain simple and clear results.

This method has been used by the author in the solution of several problems of engineering practice as for instance when investigating the stability of motion of a rotor with unsymmet-

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rical shaft mounted on an elastically supported mass foundation (see [5]) or of a rotor if both bending and torsional rigidities of the rotor are considered or analysing the effect of the distribution of the rotor unbalance [6].

II. An approximate solution of the stability of linear systems

The above described method can be modified to also include the approximate solution of the stability of vibrations of linear systems with constant parameters. Although in this instance the solution leading to Hourwitz's conditions presents no theoretical difficulties, a general solutions for systems with several degrees of freedom is practically not feasible.

In many cases the system can be written in or rearranged to the form

$$\ddot{y}_s + \sum_{k=1}^{n} a_{sk} y_k = \varepsilon \sum_{k=1}^{n} (b_{sk} \dot{y}_k + c_{sk} y_k), \qquad (s=1,2,3,\dots,n) \qquad (15)$$

where a_{sk} , b_{sk} , c_{sk} are constants and ε the small parameter. The subsequent solution will particularly be advantageous if the abbreviated system (15) is decomposed into several independent systems so that the roots of the characteristic equation can easily be determined. As in the former procedure let us transform system (15) so that the abbreviated system be converted into normal coordinates. Using again substitution (9) we obtain system (15) in the form

$$\begin{aligned} \dot{u}_s &= \lambda_s u_s + \tfrac{1}{2}\varepsilon F_s , \\ \dot{v}_s &= -\lambda_s v_s + \tfrac{1}{2}\varepsilon F_s , \qquad (s=1,2,3,\dots,n) \\ F_s &= \sum_{k=1}^{n} \left[\rho_{sk}(u_k + v_k) + \frac{1}{\lambda_k} q_{sk}(u_k - v_k) \right], \end{aligned} \qquad (16)$$

where coefficients ρ_{sk} , q_{sk} are, however, constants.

Since system (16) is linear we can seek the solution in the form

$$u_s = U_s e^{\nu t} , \quad v_s = V_s e^{\nu t} , \qquad (17)$$

where U_s , V_s are constants which are-like the characteristic exponent; ν - functions of the small parameter ε . The exponent can be written in the form

$$\nu = \lambda + \varepsilon \mathcal{H}(\varepsilon) , \qquad (18)$$

where λ is the root of the characteristic equation of the abbreviated system.

If we seek the solution of equation (17) in the form

$$\begin{aligned} u_s &= (U_{s0} + \varepsilon U_{s1} + \varepsilon^2 U_{s2} + \dots) e^{(\lambda_j + \varepsilon \chi_{j1} + \varepsilon^2 \chi_{j2} + \dots)t} \\ v_s &= (V_{s0} + \varepsilon V_{s1} + \varepsilon^2 V_{s2} + \dots) e^{(\lambda_j + \varepsilon \chi_{j1} + \varepsilon^2 \chi_{j2} + \dots)t} \end{aligned} \qquad (19)$$

where λ_j is one of the roots of the characteristic equation of the abbreviated system (15) then by substituting this solution into (16), we can by means of the Poincaré's method determine with any arbitrary degree of accuracy to the individual roots λ_j the corresponding exponents ν . The condition of stability is satisfied if the real parts of all these exponents are negative, i.e.

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$$Re(\gamma_j) < 0 \qquad (j = 1,2,3,\ldots,n) \tag{20}$$

For a more detailed explanation and example of the procedure see [7].

III. A method for determining periodic and non-periodic resonances of quasi-linear systems

On a system with two degrees of freedom we shall outline the procedure for studying the resonances of quasi-linear systems, which can be used with advantage in practical determination of non-periodic resonances.

Let us consider a quasi-linear system the motion of which is described by differential equations in the form

$$\ddot{x}_s + \sum_{k=1}^{n} a_{sk} x_k + \varepsilon f_s(x_1, \ldots, \dot{x}_n, x_1, \ldots, x_n) = P_s \cos(\omega t + \psi_s), \quad (s = 1,2,3,\ldots,n) \tag{21}$$

where a_{sk}, P_s and ψ_s are constants, ε is the small parameter and f_s the analytic functions of deflections x_k and velocities $\dot{x}_k$.

System (21) can be transformed into a quasinormal system

$$\dot{y}_s + \Omega_s^2 y_s = Q_s \cos(\omega t + \psi_s) + \varepsilon \Phi_s(y_1, \ldots, y_n), \quad (s = 1,2,3,\ldots,n) \tag{22}$$

where Ω_s are the natural frequencies of the abbreviated system.

It is known (see e.g. [8]) that such system has resonances for values of ω lying in the vicinity of $\frac{N}{M}\Omega$ (N, M are integers); if $N = M = 1$, the resonances are called principal,

if $M = 1$, $N = 2,3,\ldots$, subharmonic, if $N = 1$, $M = 2,3,\ldots$, ultraharmonic and if $N \neq M$; $M, N = 1$ subultraharmonic. Furthermore non-periodic (quasi-periodic) resonances can exist for those values of ω (in case of several exciting components with mutually different frequencies - their linear combination $\sum_{k=1}^{n} M_k \omega_k$, where M_k are integers) which are equal to the linear combination of natural frequencies $\sum_{k=1}^{n} N_k \Omega_k$ where N_k are integers (see [1]).

On the above mentioned system with two degrees of freedom we shall outline the procedure applicable to the solution of periodic and especially non-periodic resonances.

Let us have a two-mass system (see schematic of Fig.1) containing a linear spring of rigidity c_1 as well as a non-linear spring with a restoring force $c_2 x_1 + \alpha f(x_2)$, where α is the small parameter and $f(x_2)$ the analytic function. Let us suppose that an exciting force with frequency ω acts upon the first mass. The motion of this system is described by differential equations

$$\begin{aligned} m_1 \ddot{x}_1 + c_1(x_1 - x_2) &= P \cos \omega t, \\ m_2 \ddot{x}_2 + c_1(x_2 - x_1) + c_2 x_2 + \alpha f(x_2) &= 0. \end{aligned} \tag{23}$$

By means of transformation

$$\begin{aligned} x_1 &= y_1 + y_2, \\ x_2 &= a_1 y_1 + a_2 y_2, \end{aligned} \tag{24}$$

we convert system (23) to quasi-normal

$$\ddot{y}_1 + \Omega_1^2 y_1 = Q_1 \cos \omega t + \varepsilon f(a_1 y_1 + a_2 y_2),$$

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and for $\alpha > 0$ (hard spring) toward higher values of ω (see diagram in Fig.3).

The advantage of the procedure outlined above lies in the fact that simplification (38) is the same even in the case when function $f(x_2)$ is a general polynomial of coordinates x_2. This method can of course also be used in the instance when the non-linearity applies to more than one flexible member.

IV. Investigating the stability and self-excited vibration of one system

The study of the stability of motion of a rotor with a single disk mounted on a massless vertical shaft when considering the effect of internal damping material damping of the shaft and dry friction between the shaft and disk hub) and external damping leads to the solution of the following type of equation

$$\ddot{u} + \Omega^2 u + \varkappa(|u|)\dot{u} + \delta(|u|, i\dot{u} - i\omega u)\frac{\dot{u} - i\omega u}{|\dot{u} - i\omega u|} = 0, \quad (42)$$

where u is the vector of the deflection of the disk centre of gravity, $\Omega = \sqrt{\frac{c}{m}}$ is the natural frequency of the undamped rotor motion (c - shaft rigidity, m - disk mass), $\varkappa(|u|)$ is the monotonously growing function of the absolute value of velocity $|\dot{u}|$, $\delta(|u|, |\dot{u} - i\omega u|)$ is the function of internal damping, and ω is the constant angular velocity of the rotor rotation.

It can easily be proved that the external damping alone always causes damping of natural vibration. Under certain conditions, however,

the internal damping causes instability of vibration. When both campings are in equilibrium a simple stationary solution of the following form is available:

$$u = U e^{i\lambda t}, \quad (43)$$

where U is a generally complex constant and λ is real constant. The conditions for the stationary solution of equation (42) is to satisfy the following equations:

$$\Omega^2 - \lambda^2 = 0, \quad (44a)$$

$$\lambda\, \varkappa(R|\lambda|) + \delta(R, R|\lambda - \omega|)\frac{1}{R}\frac{\lambda - \omega}{|\lambda - \omega|} = 0, \quad (44b)$$

where $R = |U|$ is the amplitude of the stationary vibration.

It follows from the first equation and with respect to the second one that $\lambda = \Omega$ (For $\lambda = -\Omega$ condition (44b) is not satisfied in any case). Neither is the second condition satisfied for $\omega < \Omega$; it can be fulfilled only for $\omega > \Omega$ if

$$\varkappa(R\Omega) = \frac{1}{R\Omega}\delta(R, R|\omega - \Omega|) = M(R). \quad (45)$$

If we know the curves of functions $\varkappa(R)$ and $M(R)$ we can determine not only the amplitude of stationary vibration but also the stability of vibration for any arbitrary random deflection R. If for certain R, $\varkappa(R)$ is larger than $M(R)$ damping of vibration occurs (excess of external damping), if $\varkappa(R)$ is smaller than $M(R)$ an increase in the amplitude of vibration takes place.

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The condition of stability of the stationary solution R_s is then

$$\varkappa(R_s, \varepsilon) < M(R_s - \varepsilon),$$
$$\varkappa(R_s + \varepsilon) > M(R_s + \varepsilon), \qquad (46)$$

where $\varepsilon > 0$.

A more detailed exposition is contained in [9] and [10].

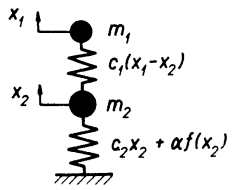

Fig. 1

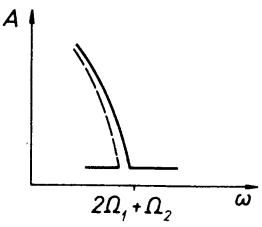

Fig. 2

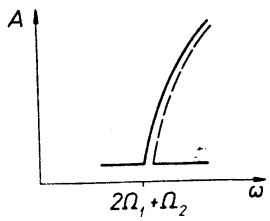

Fig. 3

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З. ОСИНСКИЙ

КОЛЕБАНИЯ СИСТЕМЫ С ОДНОЙ СТЕПЕНЬЮ СВОБОДЫ С НЕЛИНЕЙНЫМ ВНУТРЕННИМ ТРЕНИЕМ И РЕЛАКСАЦИЕЙ

1. Introduction

We shall consider vibrations of a system with one degree of freedom. We assume that the elastic element is non-linear and besides of elastic properties it is characterized with non-linear internal friction and relaxation. We assume also that the change in the state of strain and stress of an element is [illegible] three [illegible] ance [illegible] the internal friction resistance [illegible] the stress relaxation resistance R. [illegible] posed that the resistances [illegible] R, T can be [illegible] sumed in the form of functions of stress strain and strain rate [illegible]

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relaxation properties of the elastic element. In the case of homogeneous strain ε means strain at any point of the elastic element. In the case of non-homogeneous strain ε means the strain of the elastic element of the reduced system. The form of the equation of motion is the same.

Our purpose is to investigate whether equations (1,2) can be applied to the description of the vibrational motion of a system with non-linear elasticity, damping and relaxation.

Hitherto the vibrations of the mechanical systems have been described by means of the equations of the second order. Here we are dealing with an equation of the third order.

2. Solution of the Equation of Free Vibrations

In the case of free vibrations of a system, we can transform the equations (1,2) to the form of the following equation of the third order:

$$\dddot{\varepsilon} - \varkappa\ddot{\varepsilon} + \alpha\dot{\varepsilon} + \omega^2\varkappa\varepsilon = -\mu\beta\varepsilon^3 - \mu\gamma\dot{\varepsilon}^3 - \mu\delta\dot{\varepsilon}^3, \qquad (2,1)$$

where

$$\varkappa = \frac{1}{b}, \quad \alpha = \frac{c}{bm}, \quad \omega^2\varkappa = \frac{a}{m}\frac{1}{b},$$
$$\mu\beta = \frac{d}{bm}, \quad \mu\gamma = \frac{e}{bm}, \quad \mu\delta = \frac{f}{b}m^2. \qquad (2,2)$$

We assume that the coefficients $\varkappa$, α and ω are positive, β, γ and δ positive or negative. We assume that the characteristic equation for the linear case (2,1) for $\mu = 0$

$$\varkappa^3 + \varkappa^2\varkappa + \varkappa\alpha + \omega^2\varkappa = 0$$

has one real and positive root, and two complex

roots with negative real parts. This assumption is of course correct for mechanical systems characterized with dissipation of energy. We shall seek the solution of equation (2,1) by means of the Kryloff - Bogoliuboff asymptotic method /2/ in the form:

$$\begin{aligned} \varepsilon &= a + b\cos\psi + \mu u_1(a,b,\psi) + \mu^2 u_2(a,b,\psi) + \\ \frac{da}{dt} &= -\xi a + \mu A_1(a) + \mu^2 A_2(a) + \\ \frac{db}{dt} &= \xi b + \mu B_1(b) + \mu^2 B_2(b) + \\ \frac{d\psi}{dt} &= \eta + \mu C_1(b) + \mu^2 C_2(b) + \end{aligned} \qquad (2,3)$$

Substituting this solution into equation (2,1) and comparing the parts with the same order of μ, we get a system of non-homogeneous linear equations. By the successive solution of these equations we obtain functions $u_n(a,b,\psi)$

In any approximation we have one differential equation for four unknown functions u_n, A_n, B_n and C_n, therefore we need assume three additional conditions. We assume as follows:

1) Functions u_n cannot contain free components depending on a only.
2) Functions u_n cannot contain components with $\sin\psi$ depending on b only.
3) Functions u_n cannot contain components with $\cos\psi$ depending on b only.

Functions u_n will thus contain higher harmonics only and perhaps some free components and first harmonics depending on a and b simultaneously. Using these conditions we obtain the following system of equations:

$$\xi^2 A_1 + \xi^2 a\frac{\partial A_1}{\partial a} + \xi^2 a^2\frac{\partial^2 A_1}{\partial a^2} + \varkappa\xi A_1 + \varkappa\xi a\frac{\partial A_1}{\partial a} + \alpha A_1$$

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$$= -\beta a^3 + \gamma \xi^3 a^3 + \delta \xi^6 a^3$$

$$\xi^2 B_1 + 2\xi^2 \beta \frac{\partial B_1}{\partial \beta} + \xi^2 \beta^2 \frac{\partial^2 B_1}{\partial \beta^2} + 6\xi \beta p C_1 - 3B_1 p^2 + 3\beta^2 \frac{\partial C_1}{\partial \beta} \xi p - \varkappa \xi B_1 -$$

$$- \varkappa \xi \beta \frac{\partial C_1}{\partial \beta} - 2\varkappa p C_1 \beta + \alpha B_1 = -\frac{3}{4} \beta \beta^3 + \frac{3}{4} \gamma \xi^3 \beta^3 + \frac{3}{4} \delta \xi^6 \beta^3 + \frac{3}{40} \xi^9 \beta^3 -$$

$$- \frac{18}{4} \delta \xi^3 p^2 \beta^3 - 60 \xi^3 p^6 \beta^3 - \frac{3}{4} \delta \xi p^6 \beta^3$$

$$- \xi^2 \beta^2 \frac{\partial C_1}{\partial \beta} + 3\beta p^2 C_1 + 2\varkappa \xi \beta C_1 - 2\varkappa \xi B_1 + \varkappa \beta^2 \xi \frac{\partial C_1}{\partial \beta} - \varkappa p^2 C_1 = \quad (2,4)$$

$$= \frac{3}{4} \xi^2 \beta^2 p + \frac{3}{2} \xi \beta^2 p^3 + \frac{3}{4} \delta \xi^6 \beta^2 p + \frac{3}{4} \delta \xi^{10} p^3 \beta^2 + 60 \delta \xi^6 p^5 \beta^2 +$$

$$+ \frac{18}{4} \delta \xi^3 p^3 \beta^2 - \frac{3}{4} \delta p^6 \beta^3$$

After the solution of these equations we find functions A_1, B_1 and C_1. Substituting these functions into equations (2,3) we obtain in the first approximation:

$$\frac{da}{dt} = -\varepsilon a + \mu E a^3,$$

$$\frac{d\beta}{dt} = \varepsilon \beta + \mu E \beta^3,$$

$$\frac{d\gamma}{dt} = \beta - \mu \Gamma \beta^3,$$

where

$$\Sigma = \frac{\beta V_\alpha X + \beta (V_\beta X + W_\beta y) + \delta (V_\alpha X + W_\beta y)}{\xi^2 + \varkappa \xi - \alpha},$$

$$E = \frac{\beta V_\alpha y + \beta (V_\beta X + W_\beta y) + \delta (V_\alpha X + W_\beta y)}{X^2 + y^2},$$

$$V_\alpha = -\frac{3}{4} \beta \qquad V_\beta = \frac{3}{4} \xi^2 + \varkappa \xi^2,$$

$$W_\beta = \frac{3}{4} \xi^9 - \frac{18}{4} \xi^3 p^2 - \frac{3\varkappa}{4} \xi^3 p^2 - \frac{3}{4} \xi p^6,$$

$$W_\gamma = \frac{3}{4} \xi^8 p + \frac{3}{4} p^3, \qquad (2,6)$$

$$W_\delta = \frac{3}{2} \xi^8 p - \frac{18}{4} \xi^4 p^3 + 15 \xi^6 p^3 + \frac{15}{4} \xi^2 p^5 - \frac{3}{4} p^9.$$

Solving the system of equations (2,5) we obtain the solution of the equations (2,3) in the first approximation:

$$\gamma = \frac{a_0 e^{-\varepsilon t}}{\sqrt{1 + \mu \frac{D a_0^2}{\xi} (e^{-2\varepsilon t} - 1)}} \cos(pt + \theta), \qquad (2,7)$$

where a_0 and θ are represented in the form:

$$a = \frac{a_0 e^{-\varepsilon t}}{\sqrt{1 + \mu \frac{D a_0^2}{\xi} (e^{-2\varepsilon t} - 1)}} \qquad (2,8)$$

$$\beta = \frac{\beta_0 e^{-\varepsilon t}}{\sqrt{1 + \mu \frac{E \beta_0^2}{\xi} (e^{-2\varepsilon t} - 1)}}, \qquad (2,9)$$

$$\psi = pt - \frac{\mu F \beta_0^2}{\xi \sqrt{\left(1 - \frac{\mu E \beta_0^2}{\xi}\right) \frac{\mu E \beta_0^2}{\xi}}} \operatorname{arctg} \frac{e^{-\varepsilon t}}{\sqrt{\frac{\xi}{\mu E \beta_0^2} - 1}} + \theta. \qquad (2,10)$$

The integral constants can be defined from the initial conditions:

$$\varepsilon(0) = \varepsilon_0,$$

$$\dot{\varepsilon}(0) = \dot{\varepsilon}_0,$$

$$\ddot{\varepsilon}(0) = \ddot{\varepsilon}_0 = -\frac{\beta_0}{m}.$$

3. Analysis of the Solution of Free Vibrations

The solution which we have obtained above describes vibrational motion with a functional curvelinear null-axis (Fig.1) of the form (2,8). This displacement of the null-axis is observed on the vibrograms which are obtained in the investigation of internal damping in metalls, if the stress relaxation phenomenon is observed /3/ /4/. The time after which the displacement $a(t)$ will assume the value e ... smaller than the initial value a_0, depends on the initial value a_0 and is expressed in the form:

$$\tau = \frac{1}{\xi} \ln \sqrt{\frac{e^2 \xi - \mu D a_0^2}{\xi - \mu D a_0^2}}. \qquad (3,1)$$

When the laws of damping, relaxation and elastic force are linear ($\mu = 0$), time τ has a constant value not depending on the initial con-

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ditions.

In the investigation of vibrational motion which is expressed by means of the function we can get the formula for the logarithmic decrement:

$$\delta = \ln \frac{a_n}{b_{n+m}} = \frac{\xi\pi}{\eta} + \frac{1}{2m} \ln \frac{1 + \frac{\mu}{\xi} E b_0^2 \left(e^{-\frac{2\xi(n+m)}{\eta}} - 1\right)}{1 + \frac{\mu}{\xi} E b_0^2 \left(e^{-\frac{2\xi n}{\eta}} - 1\right)} . \qquad (3,2)$$

We can observe that logarithmic decrement depends on the initial amplitude. When $E<0$, the decrement increases as the amplitude increases; when $E>0$, the decrement decreases as the amplitude increases. When $E=0$, the decrement does not depend on the amplitude (Fig.2). The sign of E depends on relations among the constants $\xi, \eta, \gamma, \beta, \delta$. Having considered certain number of numerical examples we can ascertain the influence of the non-linear components on the decrement. In case of linear internal friction and relaxation $(\gamma=0, \delta=0)$, with hard elastic resistance $(\beta>0)$, δ decreases as the amplitude increases, with soft elastic resistance $(\beta<0)$, increases as the amplitude increases (Fig.3). In the case of linear elastic resistance and relaxation $(\gamma=0, \delta=0)$ and with hard internal friction $(\gamma>0)$ δ increases according as the amplitude increases, with soft internal friction δ decreases as the amplitude increases (Fig.4). The influence of the relaxations resistance is of the same character as that of elastic resistance. In all cases when b tends to zero δ approaches the value $\frac{\xi\pi}{\eta}$ independent both of the non-linear parameters and the

amplitude.

Fig.5 represents the results of the experiment which was made by the author. It was investigation of free torsional vibrations of disc on a brass wire of the diameter 1,492 mm and length 2480 mm. Frequency of vibrations was 0,473 1/sek. The same figure shows the curve which is obtained from formula (3,2) for the parameters

$$\varkappa = 1, \quad \alpha = 0{,}249, \quad \omega = 0{,}473, \quad \mu\beta = 2{,}14 \cdot 10^{-3},$$
$$\mu\gamma = 0, \quad \mu\delta = 0$$

We can observe that the agreement is very good.

4. Solution in the Case of Forced Vibrations

We assume now that the system is under the action of the exciting force of small amplitude. The motion of such a system can be described by the equations:

$$m\ddot{\varepsilon} + \sigma = \mu F \sin \nu t$$
$$\sigma + b\dot{\sigma} + f\dot{\sigma}^3 = a\varepsilon + d\varepsilon^3 + c\dot{\varepsilon} + e\dot{\varepsilon}^3 \qquad (4,1)$$

These equations can be reduced to the third order equation:

$$\ddot{\varepsilon} + \varkappa\dot{\varepsilon} + \alpha\dot{\varepsilon} + \omega^2\varkappa\varepsilon = -\mu\bar{\beta}\varepsilon^3 - \mu\bar{\gamma}\dot{\varepsilon}^3 - \mu\bar{\delta}\ddot{\varepsilon}^3 + \mu\bar{q}\sin\tau + \qquad (4,2)$$
$$+ \mu\bar{q}\,[illegible] + 3\mu^2 bq \frac{1}{\varkappa}(b)^2\cos\tau - 3\mu^3 b^2 q^2 \frac{1}{\varkappa^2} b\cos^2\tau +$$
$$+ \mu^3 [illegible]$$

where

$$\nu t = \tau, \quad \frac{1}{\varkappa} = \varkappa, \quad [illegible] \qquad (4,3)$$
$$[illegible] \quad \frac{F}{bm\nu^2} = \mu\bar{q}$$

We assume that the difference between α and

$$\alpha = \frac{\mu^2}{\nu^2} + \mu\bar{\gamma} . \qquad (4,4)$$

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This means the damping is small. We want to obtain a resonance-solution and therefore we shall assume

$$\frac{\omega^2}{\nu^2} = \lambda + \mu\psi. \tag{4,5}$$

We assume μ to be the small parameter, its physical sense being given by formula (4,4). After a transformation:

$$\sqrt{\frac{\theta}{3\bar{\kappa}}}\, x = \dot{\varepsilon} \tag{4,6}$$

equation (4,2) can be rewritten in the form

$$\ddot{x} + \varkappa x + x + \varkappa x = -\mu\theta(1+x^2)\dot{x} - \mu\psi\dot{x} - \mu\psi x -$$
$$-\mu\beta\left(\int_0^\tau x\,d\tau\right)^2 x - \mu\delta x^2\ddot{x} + \mu\bar{q}\cos\tau - \mu\bar{q}\frac{1}{\varkappa}\sin\tau + \mu^2 \ldots \tag{4,7}$$

where

$$\beta = \frac{\beta}{\bar{\kappa}}\theta, \quad \delta = \bar{\delta}\frac{\theta}{\bar{\kappa}}. \tag{4,8}$$

We shall seek a solution of equation (4,7) by the small parameter method /5/ in the form:

$$x(\tau) = x_0(\tau) + \mu x_1(\tau) + \mu^2(\tau) + \ldots. \tag{4,9}$$

Introducing the solution into the equation (4,7) we obtain the following system of equations:

$$\ddot{x}_0 + \varkappa x_0 + \dot{x}_0 + \varkappa x_0 = 0,$$
$$\ddot{x}_1 + \varkappa x_1 + \dot{x}_1 + \varkappa x_1 = -\theta(1+x_0^2)\dot{x}_0 - \psi\dot{x}_0 - \psi x_0 - \tag{4,10}$$
$$-\beta\left(\int_0^\tau x_0\,d\tau\right)^2 x_0 - \delta x_0^2\ddot{x}_0 + \bar{q}\cos\tau - \bar{q}\frac{1}{\varkappa}\sin\tau,$$

(4,11)

Equation (4,10) has the three particular solutions, according to the three roots of its characteristic equation. The one root is real negative $r_1 = -\varkappa$ and two are imaginery $r_{2,3} = \pm i$.

The solution which is connected with the first root is exponentially damped motion and the next two are periodic. This is a resonance case. Seeking the periodic solution of equation (4,7) we neglect the damped solution. The solution in the null approximation can be written in the form:

$$x_0 = M_0\sin\tau + N_0\cos\tau. \tag{4,12}$$

Substituting this in equation (4,11) and using the periodicity conditions we come to an algebraic equation from which we can find the values of M_0 and N_0.

$$M_0(-\psi + A^2 y) + N_0(x + A^2 z) = -\bar{q}\frac{1}{\varkappa},$$
$$M_0(-x - A^2 z) + N_0(-\psi + A^2 y) = -\bar{q}, \tag{4,13}$$

where $a = A^2 = M_0^2 + N_0^2$ (the square of the amplitude of the vibrations in the null approximation)

$$y = -\frac{1}{4}\beta, \quad X = \theta + \psi, \quad Z = \frac{1}{4}\theta - \frac{1}{4}\delta. \tag{4,14}$$

5 A Resonance Curve

After soving equation (4,13) we obtain an equation of the resonance curve:

$$a\left[(-\psi + a y)^2 + (X + a Z)^2\right] = q^2 \tag{5,1}$$

where

$$q = \bar{q}\sqrt{1 + \frac{1}{\varkappa^2}}. \tag{5,2}$$

This equation can be solved by the graphical method. We draw a curve:

$$f_1(z) = \left[(-\psi + z y)^2 + (X + z Z)^2\right] z \tag{5,3}$$

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and find the intersection point of this curve and a straight line:

$$J_2(z) = q^2 = Const. \qquad (5,4)$$

Thus we can find one (Fig.6) to three points (Fig.7).

In order to find the points at which the tangent to resonance line is vertical we must differentiate equation (5,1) with respect to "a" and we assume $\frac{d\psi}{da}=0$ Thus we get the equation:

$$a^2(3Y^2+3Z^2)+a(-4\psi Y+4XZ)+\psi^2+X^2=0. \qquad (5,5)$$

From this equation we obtain the points at which $\frac{d\psi}{dt}=0$.

The curve $a=\varphi(\psi)$ determines the field of the unstable solutions.

The existence of unstable solutions can be investigated by means of observing the sign of discriminant Δ of equation (5,5):

$$\Delta = 16(-\psi Y+XZ)^2-12(Y^2+Z^2)(\psi^2+X^2). \qquad (5,6)$$

If $\Delta<0$, then for a given ψ there are no unstable solutions. If $\Delta>0$, then there is one unstable solution. The case $\Delta=0$ is a particular case where we have only one point at which [illegible]

To find the maximum point of the resonance curve we differentiate equation (5,1) with respect to [illegible] and assume [illegible] We obtain values [illegible] corresponding to the maximum points of the resonance curve.

[illegible]

The place of the maximum point depends on the amplitude. For $Y-Z>0$ the maximal point is displaced in the direction of larger ψ when the amplitude increases. For $Y-Z<0$ the maximal point is displaced in the direction of smaller ψ when the amplitude increases. For $Y-Z=0$ the maximal point has no displacement when the amplitude increases (Fig.9,10). The displacement of the maximal point is a square function of the amplitude. The displacement of the null point of the maximal points curve with respect to the initial point $\psi=0$ of the axis depends on the parameter θ. The displacement of the maximal points depends on the parameters $\theta, \bar{\beta}, \bar{\gamma}, \bar{\delta}$. That we can observe from the formula:

$$Y-Z=\frac{\theta}{4}\left(-\frac{\bar{\beta}}{\bar{\gamma}}-\theta+\frac{\bar{\delta}}{\bar{\gamma}}\right).$$

The maximal value of the amplitude can be obtained from the formula:

$$\frac{a_m}{2}[a_m(Y+Z)+\theta]^2=q^2.$$

This solution is valid for hard internal friction $\bar{\gamma}>0$. If we want to find a solution for soft internal friction we may assume that the parameter $\bar{\gamma}$ is positive, but, instead of the component $\theta(1+X^2)\dot{X}$, we must include in the equation (4,7) the component $\theta(1-X^2)\dot{X}$. In this case the above solution is correct if we replace the parameter

$$Z=\frac{1}{4}\theta-\frac{1}{4}\delta by, \quad Z^*=-\frac{1}{4}\theta-\frac{1}{4}\delta.$$

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The parameters $\bar{\beta}$ and $\bar{\delta}$ can be positive or negative in both cases.

We assume for example a relation between the non-linear parameters

$$\bar{\beta}=\bar{\delta}=-\bar{\gamma} \quad (\bar{\gamma}>0), \quad \theta<0.$$

And assuming

$$\psi=\lambda\theta$$

we can draw the resonance curves as functions $a=a(\lambda)$ (Fig.11). The positive values of parameter λ corresponds to the values of the exciting forces frequency $\nu>\omega$. The maximal points curves are displaced in the direction of smaller frequencies ν. For all the values of the exciting forces amplitude there is only one stable solution.

As a second example we assume:

$$\bar{\beta}=-4\bar{\gamma}, \quad \bar{\delta}=5\bar{\gamma}, \quad \bar{\gamma}>0, \quad \theta<0.$$

The resonance curves are represented in fig.12. The resonance curve maximal point is displaced in the direction of larger frequencies ν. For the larger values of the exciting forces amplitude there are three periodic solutions. The shaded field is the unstable solutions field.

To explain the influence of non-linear internal friction compared with that of linear friction one more example is given. We assume, for the same $\bar{\beta}=\bar{\delta}=-|\bar{\gamma}|$ and θ and Q, three values of $\bar{\gamma}$: positive, negative and null $\bar{\gamma}>0, \bar{\gamma}=0, \bar{\gamma}<0$. For these values we obtain three resonance curves (Fig.13). We can observe

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Summary

Vibrations of One Degree of Freedom of a System with Non-linear Internal Friction and Relaxation

In this paper it is considered a mechanical system with non-linear internal friction and relaxation. The motion of this system can be described by means of a differential equation of the third order (in the case of a system with one degree of freedom):

$$\dddot{\xi} + \mu\ddot{\xi} + \alpha\dot{\xi} + \omega^2\xi = F(t). \qquad (1)$$

In the free vibrations case the solution is obtained by means of the small parameter method. The author has investigated the influence of non-linear components describing the elastic resistance, internal friction resistance and relaxation resistance on the motion of the system and in particular on the displacement of the null axis and on the logaritamic decrement.

A solution is obtained in the case of the forced resonance vibrations. The resonance curve and the problem of the number of stable and unstable periodic solutions have been investigated.

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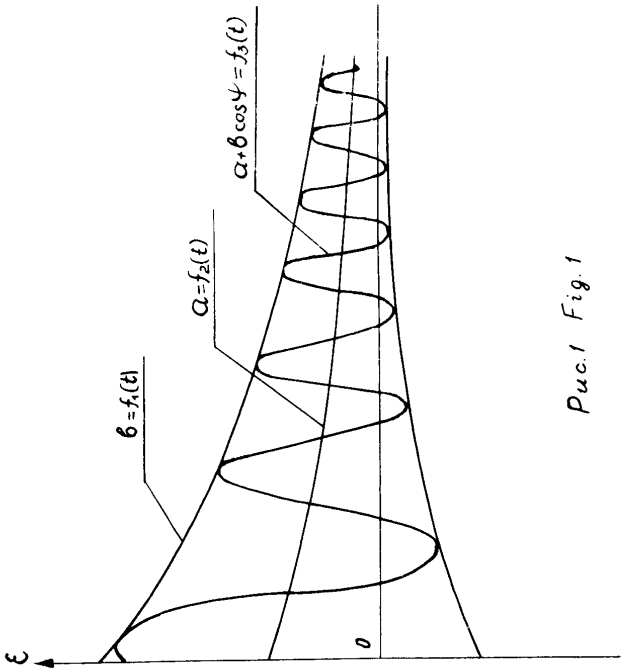

Рис. 1 Fig. 1

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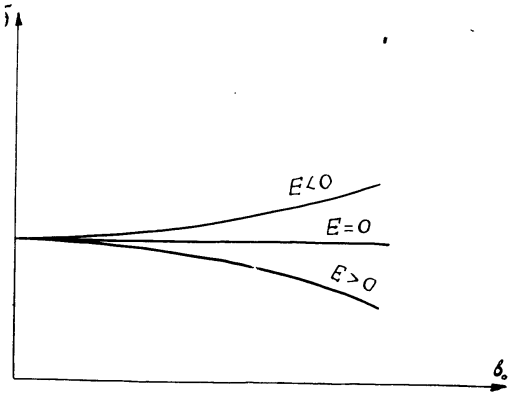

Рис.2 Fig.2

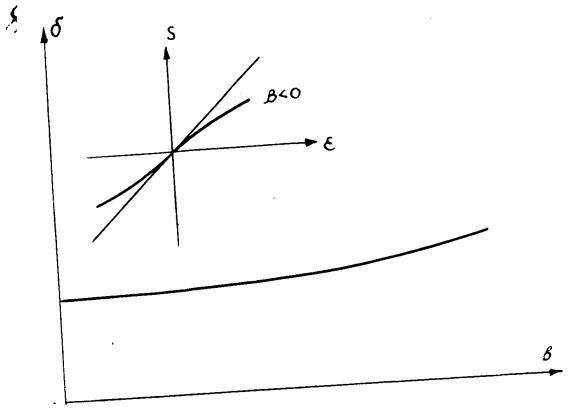

Рис.3а Fig.3а

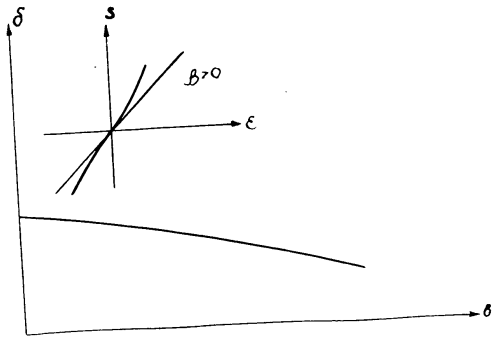

Рис.3б Fig.3б

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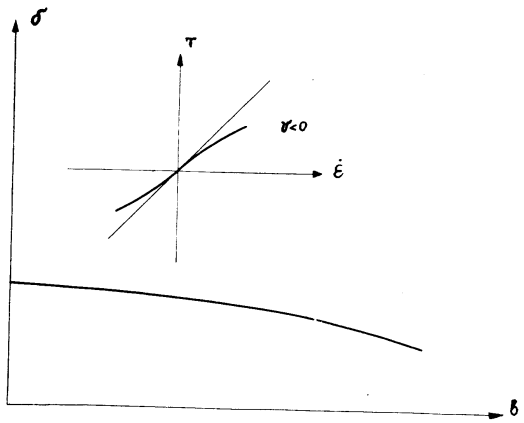
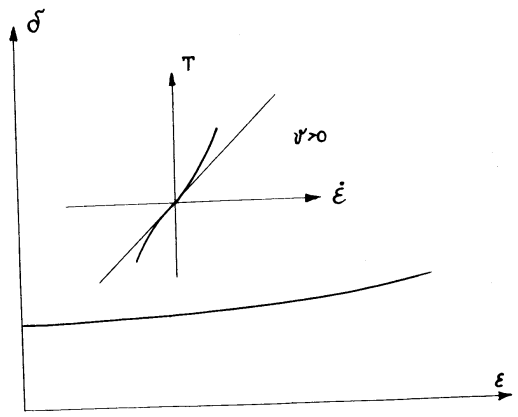


Puc.4a Fig. 4a



Puc. 4б Fig. 4б

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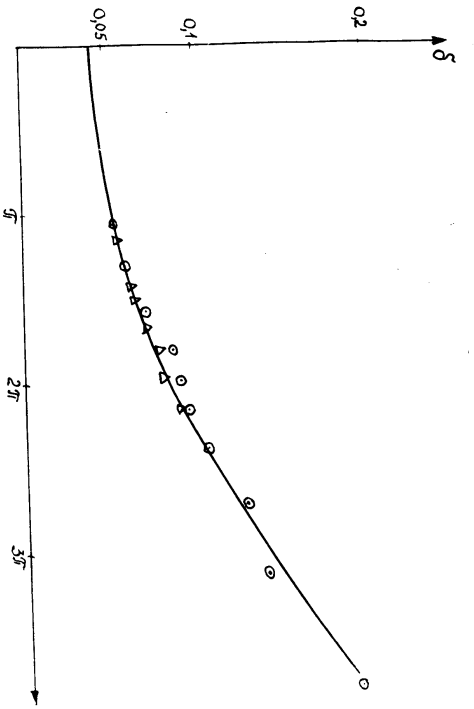

Рис.5 Fig.5

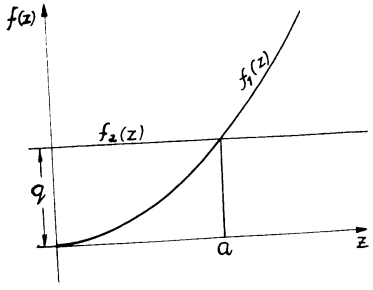

Рис.6 Fig.6

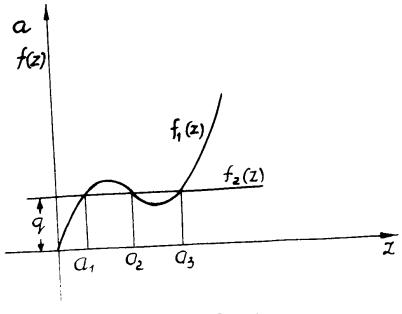

Рис.7 Fig.7

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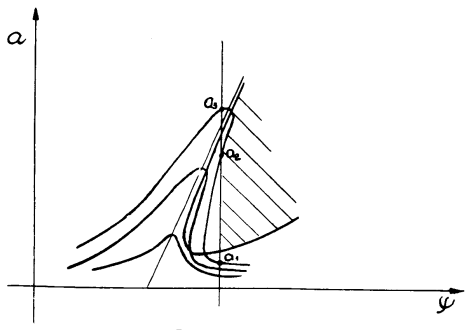
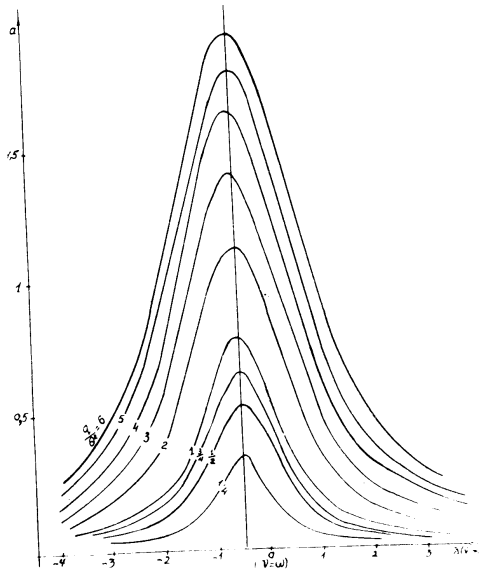


Рис. 8 Fig. 8



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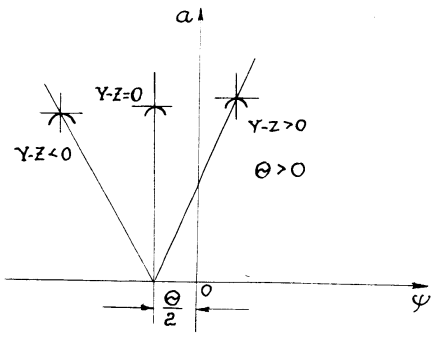

Рис. 9 Fig. 9

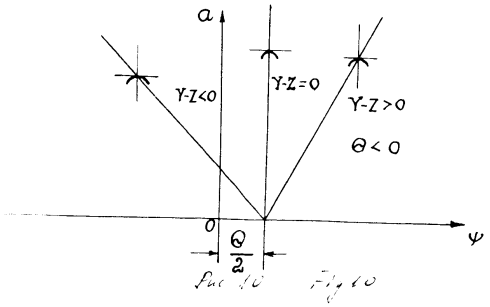

Рис. 10 Fig. 10

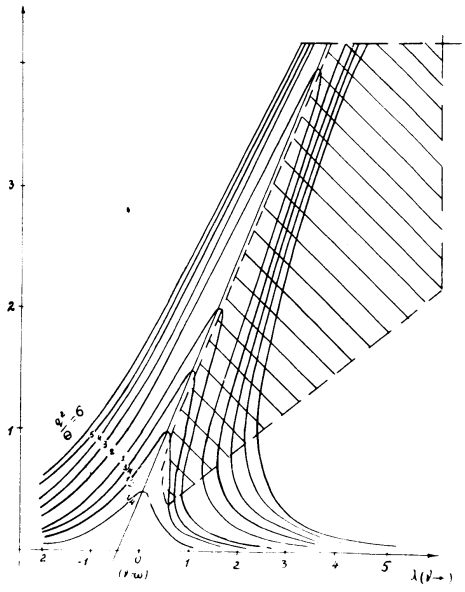

Рис. 12 Fig. 12

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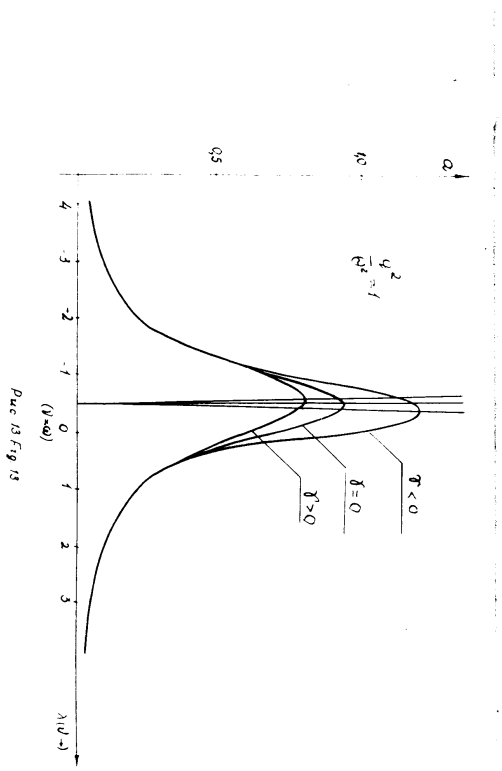

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ОБЛАСТЬ ОГРАНИЧЕННОСТИ ДВИЖЕНИЯ СИЛЬНО НЕЛИНЕЙНЫХ НЕАВТОНОМНЫХ МЕХАНИЧЕСКИХ СИСТЕМ С ЧАСТИЧНО ОТРИЦАТЕЛЬНЫМ ДЕМПФИРОВАНИЕМ

Let us consider the strongly non-linear mechanical vibration system with finite number of degrees of freedom, modelled in the generalized way, /1/, /2/, /3/, /4/, as "the chain" of lumped masses coupled one to one with the strongly non-linear: spring, positive damper and the negative damper, so that the motion of this system can be described in the configuration space by the vector equation:

$$\ddot{q} + F(q, \dot{q}, t) = 0 \qquad /1/$$

with the initial conditions

$$q(t_0) = q^0, \quad \dot{q}(t_0) = \dot{q}^0;$$

$$q^{02} + \dot{q}^{02} \neq 0; \qquad /2/$$

where $q\{q_1, \ldots, q_n\}$, $\dot{q}\{\dot{q}_1, \ldots, \dot{q}_n\}$ are the adequate n-vectors in this configuration space, and the operator F can be described as the sum of the operators: characteristic and forcing, /5/, /6/, /2/, /3/:

$$F(q, \dot{q}, t) = \tilde{F}(q, \dot{q}) - G(q, \dot{q}, t). \qquad /3/$$

and

$$\tilde{F} = \tilde{F}(\tilde{F}_1, \ldots, \tilde{F}_n); \quad \tilde{F}_i = \tilde{F}_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

$$G = G(G_1, \ldots, G_n); \quad G_i = G_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t).$$

Using the following transformation to the coordinates of the phase-space:

$$x_i = \begin{cases} q_i & \text{for } i = 1, \ldots, n \\ \dot{q}_{i-n} & \text{for } i = n+1, \ldots, 2n, \end{cases} \qquad /4/$$

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$$f_i = \begin{cases} \dot{q}_i & \text{for} \quad i=1, \ldots, n, \\ -F_i & \text{for} \quad i=n+1, \ldots, 2n, \end{cases} \qquad /4/$$

we obtain the vector equation

$$x = f(x,t) \qquad /5/$$

where $x = x\{x_1, \ldots, x_{2n}\}$, $f = f\{f_1, \ldots, f_{2n}\}$, and $f_i = f_i(x_1, \ldots, x_{2n}, t)$, with the adequate initial conditions

$$x(t_0) = x^0, \quad |x| \geq 0. \qquad /6/$$

We shall use the both forms /I/ or /5/, according to the conveniences. As to the functions F_i or f_i we shall assume they satisfy the necessary and sufficient conditions for the existence, uniqueness and prolongability of all the solutions /I/ or /5/ at any $t \in [t_0, \infty)$. In particular we shall assume F_i or f_i well defined and continuous in some open domain Ω:

$$\Omega : (-\infty < q_i < \infty, -\infty < \dot{q}_i < \infty, t \in [t_0, \infty)), i=1, \ldots, n$$

or

$$\Omega : (-\infty < x_i < \infty, \ t \in [t_0, \infty)), \quad i=1, \ldots, 2n$$

Moreover, in connection with the physical conditions of mechanical systems [3], [I], we assume that the characteristic operator $\tilde{F}$ can be described in the form:

$$\tilde{F}(q, \dot{q}) = \Phi^p(|q|, \dot{q}) + \Phi^n(|q|, \dot{q}) + \Psi(q) \qquad /7/$$

where $\Phi^p(|q|, \dot{q})$ is representing the positive damping characteristics of the total energy of the system, i.e. the internal and external positive damping forces;

$\Phi^n(|q|, \dot{q})$ - is representing the partially negative damping characteristics in the moving structures or the

action of the external energy source on the internal forces of the model, /4/.

$\Psi(q)$ - is representing the spring characteristics of the system, i.e. the potential forces of the system.

As it was postulated and proved many times [3], [2], [I], we shall assume as to these operators the following conditions, in relation to the physical sense of the mechanical motion phenomenon:

$$\Phi^p \cdot \dot{q} \geq 0 \quad \text{for arbitrary } \dot{q}, q \qquad /8/$$

$$\Phi^p \equiv 0, \quad \text{for } \dot{q} = 0, \text{ and for arbitrary } |q| < \infty. \qquad /9/$$

$$\frac{\partial \Phi^p}{\partial \dot{q}} \geq 0, \quad \frac{\partial \Phi^p}{\partial |q|} > 0, \qquad /10/$$

i.e. the function Φ^p is the monotonic function of velocity and strongly monotonic function of the modula of generalized deformations.

Furthermore for $|\dot{q}| \in [G, \infty)$ and arbitrary $|q| < \infty$ /where $G(Q_1, \ldots, Q_n)$ is some positive constant and depends on the character of partially negative damping and on the power of the external source of the auto-vibrations/: we have

$$\Phi^n \cdot \dot{q} \geq 0 \qquad /11/$$

$$\frac{\partial \Phi^n}{\partial \dot{q}} \geq 0, \quad \frac{\partial \Phi^n}{\partial |q|} > 0. \qquad /12/$$

These conditions may not be satisfied for $|\dot{q}| \in [0, Q]$, when the shape of Φ^n can be arbitrary, for arbitrary $|q| < \infty$, unless the modulus of Φ^n is bounded:

$$|\Phi^n| < M^*(Q)$$

where M^* is some positive constant. In particular Φ^n may have the oscillatory property or in general be of arbitrary signum in this domain. This is the "partially nega-

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...ve" damping. Moreover, for any $|q| < \infty$ we shall have

$$\Psi \cdot \dot{q} \geq 0 \quad , \qquad /13/$$

and

$$\Psi(0) \equiv 0 \quad . \qquad /14/$$

As to the forcing forces, we shall assume:

$$G(q, \dot{q}, t) \not\equiv 0 \quad \text{for} \quad t \neq 0 \quad . \qquad /15/$$

Furthermore, according to the physical conditions the power of the external forcing sources is limited:

$$G(q, \dot{q}, t)\, \dot{q} < M \qquad /16/$$

where M is some positive constant.

We shall consider in the phase space the domain of the motion where the total energy of the system $H(q, \dot{q}) = H(x)$ ... monotonic function of the radius-vector in the phase space or the monotonic function of $|x|$.

We consider in this paper the problems of the boundedness and of the ultimate boundedness "in the large" of the paths of our mechanical system, and the problem of the existence of a finite domain of this boundedness too, the size of which depends on the positive damping characteristics in the system.

The problem of the boundedness of the solutions of /I/ or /5/ was examined many times. For the most part the conclusions gave however the criterion of the boundedness as sufficient in the technical sense, i.e. they gave the necessary and sufficient conditions for the solutions of /I/ or /5/ to be bounded in the large. The problem is however to prove for the most general but definite mechanical system that the above mentioned sufficient and necessary conditi-

ons are satisfied, i.e. that the motion of the definite generalized model of the mechanical system is really bounded "in the large". Thus this paper is the continuation of a number of previous works of the authors on some particular cases of the systems /I/ or /5/. From the same point of view the problem of boundedness was examined by B.Manfredi, [1], assuming, however, the linear Ψ characteristics and $G(q, \dot{q}, t) = G(t)$. In the paper [2] the authors applied the B.P.Demidowich theorem [8], to the mechanical systems and proved the boundedness for the particular system $G(q, \dot{q}, t) = G(t)$, with an arbitrary nonlinear characteristic operator $F(q, \dot{q})$. The general case of the type /I/ or /5/, however assuming only the positive dissipation of the total energy of the system, was considered by the authors in the paper [9]. Assuming finite the impulse of the external generation of the motion, i.e. assuming finite the integral of the forcing term in the equations this assumption does somewhat restrict the generality of the problem, we have succeeded to find a very simple proof of the boundedness of the motion [10]. We shall name the motion of the considered mechanical model "uniformly bounded in the large", if the solutions of /5/ will have the following property: for every particular solution $x(t, x^\circ, t_\circ) \in \Omega$ starting from $(t_\circ, x^\circ) \in \Omega_\alpha$, $\Omega_\alpha : (|x^\circ| < \alpha,\ t_\circ \in [0, \infty))$, there exist such a positive number $\beta(\alpha)$, independent from $t_\circ$ that $|x(t, t_\circ, x^\circ)| < \beta$. We shall name the motion of our model "ultimately bounded in the large", if all the solutions of /5/ with the adequate initial conditions will have the following property: for all the particular integrals $x = x(t, t_\circ, x^\circ)$ starting from $(t_\circ, x^\circ) \in \Omega_\alpha$, there exists such a common positive constant β , and for

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every particular integral there exists such a positive constant $T(x^0)$, independent from t_0, that for $t \in [t_0+T, \infty)$ there is already $|x(t, t_0, x^0)| < \beta$, with arbitrary but finite t_0.

Theorem I. The motion of the considered generalized model of the mechanical system under the assumed conditions is uniformly bounded in the large.

Proof: We take in Ω the plane $|\dot{q}| = 0$ and we denote it by A^*: $(E_{q,\dot{q},t} \in \Omega, |\dot{q}| = 0)$. Let furthermore $\omega = \Omega - A^*$, or B^*: $(E_{q,\dot{q},t} \in \Omega, |\dot{q}| \neq 0)$. We shall consider some sufficiently large, finite domain $\bar{\Omega}_0 \subset \Omega$, $\bar{\Omega}_0$: $(|q|, |\dot{q}| < R_0, t \in [t_0, \infty))$ where R_0 - some positive constant. Let moreover $C_\Omega \bar{\Omega}_0$ be the complementary domain of $\bar{\Omega}_0$ to Ω. We denote

$$A = C_\Omega \bar{\Omega}_0 \cap A^* \subset C_\Omega \bar{\Omega}_0$$

$$B = C_\Omega \bar{\Omega}_0 \cap B^* \subset C_\Omega \bar{\Omega}_0$$

and, of course $A \cup B = C_\Omega \bar{\Omega}_0$.

We shall use the Yoshisava theorem [II], which in our notation can be expressed in such way:

I. Let $\Phi(x,t)$ be a function satisfying in $C_\Omega \bar{\Omega}_0$ the following conditions:

1° $\Phi \to 0$ uniformly with t, for $\rho^2 \to \infty$;

2° for every positive number $R > 0$, there exists a positive constant $G(R)$ such that $\rho^2 = R^2$ implies $\Phi \geq G(R)$;

3° Φ is satisfying the Lipschitz conditions locally with respect to x and

$$\rho^2 = \sum_{i=1}^{2n} x_i^2 = x^2. \qquad /17/$$

About such function $\Phi(x,t)$ we shall say, it satisfies the $\mathcal{Y}$ - conditions.

2. If the right-hand-sides of the equations /5/ are such that there exists a function $\Phi(x,t)$ satisfying the $\mathcal{Y}$ -conditions and moreover:

$$\lim_{h \to 0} \frac{1}{h} \{\Phi(t+h, x+hf) - \Phi(t,x)\} \geq 0 \qquad /18/$$

in the domain $C_\Omega \bar{\Omega}_0(R_0)$, then every solution $x(t, t_0, x^0)$ of the system /5/ is uniformly bounded in the large, or more exactly: there exists such positive constants α and $\beta(\alpha)$ that if $|x^0| < \beta$, we have $|x| < \beta$, for every $t \in [t_0, \infty)$.

We shall prove that for the motion of our system this function exists and is of the form $\Phi = e^{-H(x)}$. The function $H(\rho)$ is an exactly monotonic function from the assumption, therefore for $\rho^2 \to \infty$, we have $H(\rho) \to \infty$, or $e^{-H} \to 0$. The same property of the function $H(x)$ implies the second of the $\mathcal{Y}$ - conditions. The third condition $\Phi \in L\rho$ is automatically satisfied because $H(x) \in C^2$ at least. Let us consider now the condition /18/. We denote $\delta x = hf$ and $\eta = x + \delta x$. The value of $\Phi(t+h, x+\delta x) = \Phi(t,u)$ under constant (t,x) is due only to the increament h, we can put it as $\tilde{\Phi}(t,x,h)$. Hence the condition /18/ can be rewritten:

$$\left[\frac{\partial \tilde{\Phi}}{\partial h}\right]_{h=0} \geq 0 \qquad /19/$$

or

$$\sum_{i}^{2n} \left[\frac{\partial \tilde{\Phi}}{\partial u_i} f_i\right]_{h=0} \geq 0; \qquad /20/$$

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and further

$$\sum_{i}^{2n}\left[\frac{\partial\tilde{\Phi}}{\partial\dot{u}_i}\right]_0 f_i = e^{H(t)}\sum_{i}^{2n}\frac{\partial H}{\partial x_i}f_i = -e^{-H}\cdot\frac{dH}{dt} \geq 0. \quad /21/$$

Taking into account the composition of the characteristic operator $\tilde{F}$ we can write the equation /I/ in the form:

$$\ddot{q} = -\Phi^p(q,\dot{q}) - \Phi^n(q,\dot{q}) - \Psi(q) + G(q,\dot{q},t).$$

Multiplying the both sides by $d\dot{q}\{d\dot{q}_1, \ldots, d\dot{q}_n\}$ we obtain:

$$d\left[\tfrac{1}{2}\dot{q}^2\right] = -\Psi(q)dq - \Phi^p(q,\dot{q})\dot{q}dt - \Phi^n(q,\dot{q})\dot{q}dt + G(q,\dot{q},t)\dot{q}dt.$$

Therefore

$$\frac{dH}{dt} = -\Phi^p(q,\dot{q})\dot{q} - \Phi^n(q,\dot{q})\dot{q} + G(q,\dot{q},t)\dot{q}. \quad /22/$$

To satisfy the conditiom /2I/ in the considered domain there should be satisfied in this domain the condition

$$\frac{dH}{dt} \leq 0 \quad /23/$$

what we have to show.

To this end we devide the previously defined domain $B \subset C_\Omega \bar{\Omega}_\circ$ into two subsets: according to /I0/ the functiom. Φ^p is momotonic increasing with $\dot{q}$ for arbitrary q, t in B, and according to /I2/ we can say the same about the functiom Φ^n for $|\dot{q}| \geq Q$, moreover the functiom $|\Phi^n| < M^*$ is bounded for $|\dot{q}| < Q$, hence $|\Phi^n\dot{q}| < 0$, is bounded too, where C is some positive constant, dependent on M^* and Q. Therefore we can

take in B first of all such a domain $B_1 \in B$, that for arbitrary q, t in B_1 accoroding to /I6/: $G\dot{q} < M$, and we shall find always such a number $\varepsilon > 0$, that for $|\dot{q}| \geq \varepsilon$ again we shall have

$$\Phi^p\dot{q} + \Phi^n\dot{q} \geq M(\varepsilon) + \lambda(|q|,\varepsilon) \text{ under } |\dot{q}| \geq Q \quad /24/$$

and $\Phi^p\dot{q} \geq C(\varepsilon) + M(\varepsilon) + \lambda(|q|,\varepsilon)$ under $|\dot{q}| < Q$,

where

$$M(\varepsilon) = \sup G\dot{q} = M + \tilde{f}(\varepsilon)$$

$$C(\varepsilon) = \sup \Phi^n\dot{q} = C + \tilde{\tilde{f}}(\varepsilon), \text{ for } |\dot{q}| < Q;$$

and $\tilde{f}(\varepsilon)$, $\tilde{\tilde{f}}(\varepsilon)$ monotonic, non-negative functions of ε. Hence in virtue of /22/ and /24/ the condition /23/ is satisfied in our domain B_1.

According to /I0/ the function Φ^p is monotonically increasing with $|q|$ for arbitrary $\dot{q}$, t in B. Further, according to /I2/, Φ^n is also increasing monotonically with $|q|$, under $|\dot{q}| \geq Q$, and is bounded under $|\dot{q}| < Q$, thus as in the previous case we have $|\Phi^n q| < C$.

Now we can find again a subset $B_2 \subset B$ so that there exists a finite number $\nu > 0$ and under $|q| \geq \nu > 0$ and arbitrary $|q|, t$ in B_2 we have

$$\Phi^p\dot{q} + \Phi^n\dot{q} \geq M(\nu) + \lambda(\nu,|\dot{q}|) \text{ for } |\dot{q}| \geq Q$$

$$\Phi^p\dot{q} \geq C(\nu) + M(\nu) + \lambda(\nu,|\dot{q}|) \text{ for } |\dot{q}| < Q \quad /25/$$

where $M(\nu)$, $C(\nu)$ are as previously but to change the parameter ε into ν.

Thus the condition /23/ under /22/ and /25/ will be satisfied in B_2 too.

Consequently it is proved, that there exists domains in B where the limited power of the external forcing and also the

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limited power of the auto-vibrations source is majorized by the power of positive damping.

Now we can find values of $R_o > 0$ so that in appropriate $\bar{\Omega}_o \{|q|, |\dot{q}| < R_o \quad t \in [t_o, \infty)\}$ it will be $B_1 \cup B_2 = B$, $B \subset C_\Omega \Omega_o$. Then in virtue of the above proved the condition /23/ is satisfied in the whole region B, and so that

$$\frac{dH}{dt} \leq \lambda(|q|, |\dot{q}|) < 0. \qquad /26/$$

Hence we conclude it can not be $\frac{dH}{dt} = 0$ in B. Let us consider $A = C_\Omega \bar{\Omega}_o - B$; we have always $\frac{dH}{dt} = 0$ in this domain as it is seen putting $|\dot{q}| = 0$ in /22/. Since $A \cup B = C_\Omega \bar{\Omega}_o$ we can say, that the condition 23' is satisfied in the whole $C_\Omega \bar{\Omega}_o$, hence the condition /18/ is satisfied too, and according to the above mentioned theorem of Yoshizawa the motion of our model is in the domain Ω_o uniformly bounded in the sense of assumed definition.

Theorem II: The motion of the consider generalised model with our assumptions is ultimately bounded in the large.

Proof: We profit again from the Yoshizawa theorem, [12], which in our conditions can be formulated as follows: "If there exists along the solutions /5/ a funotion $\Phi(x, t)$, satisfying the $\mathcal{Y}$-conditions well defined and continuous on the domain $C_\Omega \bar{\Omega}_o$ of its arguments and if

$$\lim_{h \to 0} \frac{1}{h}\{\Phi(t+h, x+hf) - \Phi(t, x)\} \geq \lambda^2(|x|) > 0 \qquad /27/$$

/where λ^2 is an arbitrarily small but finite number/ in this domain, then the solutions of /5/ are ultimately bounded in the large in the sense of the assumed definition". We shall prove that the same function $\Phi = e^{-f'(x)}$ satisfies these conditions in $C_\Omega \bar{\Omega}_o$. By the similar considerations as above we can replace the condition /27/ by the following

$$-e^{-H} \frac{dH}{dt} \geq \lambda^2(|q|, |\dot{q}|) > 0 \qquad /28/$$

which must be satisfied, if

$$\frac{dH}{dt} \leq -\lambda(|q|, |\dot{q}|) < 0, \qquad /29/$$

where λ is uniquely determined by λ^2 and may be arbitrarily small but finite number too.

We take as previous:

$$C_\Omega \bar{\Omega}_o = A \cup B. \qquad /30/$$

In the previous proof we have substantiated the condition /26/ in the whole region B, for some chosen $\bar{\Omega}_o$ by using adequate R_o. It is clear moreover, that the last number dependends on the characteristic function Φ^e. According to the Yoshizawa-criterion our motion is ultimately bounded in the large in $C_{B^*} B$ in the sense of assumed definition. It is easy to prove that in the domain A^* of zero-velocities /the locus of instantaneous rest-states/ there is no motion and that the paths are crossing A^* along the normal lines to this hyperplane. Hence it follows according to /30/ that, if the motion is ultimately bounded in the large in the domain $C_{B^*} B$ then it must be ultimately bounded in the whole $C_\Omega(A \cup B) = \bar{\Omega}_o$ too.

Theorem III: For the paths of our model, i.e. for the solutions of /I/ or /5/ starting from the finite domain of the initial conditions $(t_o, x^o) \in \Omega_\alpha$, there exis-

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ts such a common, sufficiently large, finite, closed domain of the boundedness $\bar{\Omega}_o \subset \Omega$, that all of the solutions will be in this domain after a finite period time, i.e. for every $t \in [t_o + T_o, \infty)$, where $T_o = T_o(x^o)$ is independent from the initial value t_o.

Proof: It follows from the ultimate boundedness of motion in $\bar{\Omega}_o$ that for any particular solution /5/ these exists such a number $T_o(x^o)$, independent from t_o, that for $t \in [t_o + T_o, \infty)$ the considered solution will be in $\bar{\Omega}_o$. In virtue of the continuity of the solutions in respect to the initial conditions, in particular, in respect to $x^o \in \Omega_\alpha$, $\Omega_\alpha : (E_{x^o,t_o}/x^o| < \alpha,\ t_o \in [0,\infty))$ where $\bar{\Omega}_o$ is a finite, connected domain, there must be such a domain $\bar{\Omega}_o$, that any solution will be in it after a finite time.

The conclusions:

In the proofs of our theorem we did not choose the domain $\bar{\Omega}_o$ arbitrarily, our choice was dependent on the domain B, i.e. on the both domains with the conditions /24/ and /25/ satisfied, i.e. on the value of the power of positive damping $\Phi^p \dot{q}$, i.e. on majorization of the power of external generating forces by the power of positive damping, and the last, is dependent on the velocity of the monotonic increasing of Φ^p with $\dot{q}$, as it is readily seen from /10/. In terms it is dependent on the type of nonlinearity of damping characteristics. The interpretation of these relations is given by the authors in several previous works: [5], [6], [3], [4]. The more "hard" functions of their arguments are the positive damping characteristics $\Phi_i^p(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ the earlier comes the majorization of the power of generating forces / $G\dot{q}$ and $\Phi^n \dot{q}$ / by the power of damping and the domain $\bar{\Omega}_o$ is smaller, in agreement with the physical sense of the considered phenomena.

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С. ЗЕМБА

НЕКОТОРЫЕ ПРОБЛЕМЫ ВАРШАВСКОЙ ГРУППЫ «ТЕОРИИ НЕЛИНЕЙНЫХ КОЛЕБАНИЙ» ПАН ЗА ПОСЛЕДНЕЕ ПЯТИЛЕТИЕ

1. The introduction. Our group consisting of few persons is working in the theory of the mechanical vibrations from several years.

From the origin of our work we have the tendency to the technical applications in the theory of the nonlinear vibrations, but first of all we have taken into account the qualitative investigations which should precede, in our opinion, the quantitative investigations. Just now we make the tests to go to these quantitative investigations with some concrete problems by using the analogue computers. Moreover, in the year 1958 we have adapted some quantitative method (so called "deltha method") to the regarding of mechanical systems with several degrees of freedom but this work was lying on the margin of our interest. The general object of our investigations there are the vibrations of the strongly nonlinear systems, because, in view of our assumption, for many actual problems standing before the nonlinear vibrations theory, there is not sufficient the assumption of the little nonlinearity or the little vibrations amplitude. We

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mean, what is now natural, that the other qualitative way is necessary in the investigations. Hence the set of works enlarging some criteria of the motion character from the arbitrary little neighbourhood of the isolated singular point in the phase space into the finite, bounded, sufficiently large regions of this space. Therefore there were following also the several works, in particular, according to systems with the single degree of freedom, taking into account the several extremum of the potential in our systems.

It is assumed to name the investigations of the shape of motion of the structural system data as the analysis of the motion. The reversed process, i.e. the process basing on the separating such a particular parameters of the system that the system will realize a priori supposed form of the motion, we shall name as the synthesis, similarly to the definition used by the automatic theory, as it was many times proposed /16/, /20/, /25/. We can mention that the process of the synthesis, in some sense, is the basic process in the broad considered dynamical theory of structures. Besides, it would be necessary to take into consideration two types of the way, from analysis to the synthesis.

The first, from the analysis of the structural system to modelling and defining the number and the form of the equations of motion (according to the number degrees of freedom). We name this way in our works as the structural analysis and synthesis or the analysis and synthesis of the

first order.

The second, from the analysis of motion of the regarded model data with the concrete characteristics to the synthesis based on the separating parameters (constant or variable) of these characteristics, that the regarded model will realize the desired form of the motion. Let us name, as in our works, this process as a kinetic analysis and synthesis, or the analysis and synthesis of the second order. Besides there arise the problems coupled with the necessity to take into account some appendix conditions. As one of the examples we may take an optimalization of the mechanical systems.

On this background, it seems to be logical the following selection of the investigation in the theory of the vibrations of the mechanical lumped systems.

1° - The structural investigations, i.e. the research of the proper-transposing from the technical real systems of structures to the investigations of the adequate models and inversely. All by using the investigation of model, of characteristics of the system, of the generation of the motion etc.

2° - The investigation of the motion of the assumed model by the research of the shape of the phase-space-trajectories or, in general, of the integral curves of motion in the above referred analytical and synthetical sense.

3° - The research of the influence of the change of the separate parameters of the system

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on the motion, for example the optimalization or "contuning" of the mechanical system.

We shall discuss next the several our problems in view of the mentioned selection.

2. The structural investigations. The vibration theory of the lumped system made use and now is making use with the large mathematical apparatus (the ordinary differential equations, the dynamical topology, functional analysis) for the research of the motion phenomena, from the one side, and from the second side applying to this theory the technical and physical assumptions according to the adequate moving constructures or according to the used materials (for example: rheology).

In the theory of vibrations of the electrical or automatic-control systems there exists the set of works connecting in the sufficiently general manner the both sides of vibrations theory by using mentioned structural analysis and synthesis. However for the systems with the specific mechanical character such the works are concerning only very numerous particular cases. According to these remarks the building more general and united theory of the characteristics is very important for the vibration theory of the lumped systems, on the background of the general structural investigations.

In the paper,/21/, with J.M.Skowronski we have introduced some sufficiently general model of the structure system, which is taken as the elasto-plastic structure, allowing on the arbitrary great deformations. The model has a form of the finite set of the material points in the real three-dimensional Euclidean space, coupled one to one with the strongly non-linear damper and spring. The set of points is representing the mass-selection. The forces acting in the construction are represented by the coupling elements. In any element of this coupling the positive dissipation forces are represented by a damper, and the potential restoring forces by the parallel taken spring. The last selection is not unique, but possible and frequently profitable. Furthermore we have defined for such a taken model, the potential of the restoring forces and the unitable function of the energy dissipation. Besides we have discussed the character of the energy dissipation in the free motion of this model showing in the same time the structural adequateness of ones to the construction.

Let us denote this type of model as the model in the Carthesian coordinates, or shortly as the Carthesian model. In the paper, /47/, is shown the possibility to remove the motion investigations on the another model in the generalized coordinates, or shortly, on the generalized model, taken as the chain of the discrete masses coupled one to one by using the external and internal characteristics, besides, the number of the moving masses is adequate strictly to the number of [illegible] degrees of freedom. The last number determines [illegible]

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of the lumped masses. During the transition from the Carthesian to the generalized model, the mentioned selection of the forces into potential and dissipative forces remains unchanged in the unique manner. The influence of the positive energy dissipation is the same in the both models. By assuming some simplifications (for example by leaving the set of the external and internal couplings), we may obtain the simple chain-model, often used in the technical vibrations theory. The last remark justifies the name "generalized" of our model. The idea of this model was introduced a little earlier in the paper, /19/. In the subsequent work, /38/, J.M.Skowronski has introduced some complementary remarks to this generalized model taking into account the possibility of the partially reversing of the energy stream (the partially internal energy comulation), by the influence of the negative damping as an effect of the action of the external sources. The physical basis of such an interpretation of this influence is given in the paper, /48/, by E. Giergiel. In the mentioned paper, /38/, there are introduced the equations of the taken model with the partially negative damping with remarking the physical and especially mechanical assumptions for the characteristics. Now we have conceptions to enlarge so taken model by respecting the memory of the construction or of the material, and in the another direction by respecting the especially case of the system with infinity number of the lumped masses.

On the background so taken model in the previously reported sence, there is a set of papers devoted to the research of the characteristics of the mechanical systems. This research direction was initiated in 1957 by S.Ziemba with the series of considerations about the characteristics in the systems with the single degree of freedom, particularly about the characteristics of the positive damping, and theirs influence on the free motion of this system. In the paper, /6/, there are given the basic remarks to the further investigations on the dry friction influence on the motion. The problems of the internal damping on the basis of the internal microdeformations were considered by me in the paper, /23/. It seems to be important to study the dependency of the damping characteristic from the generalized velocities in the viscous sense, and from the generalized coordinates in the Coulomb- and general sense too, /7/, /8/, /15/. In the mentioned papers I have investigated more exactly the technical possible types of these characteristics, the shape of the characteristics functions due to the physical conditions in the sufficiently little and also sufficiently large finite neighbourhood of the equilibrium position of the system, introducing the definitions of the soft, hard and linear types of the nonlinearity. The influence of the positive damping on the motion by the exactly analysis of the physical base of adequate characteristics was regarded by W.Rogusz, /49/, too. The form of the spring-charac-

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teristics of the single-degree of freedom-systems was investigated by R.Gutowski,/50/, and J.M.Skowronski,/3/,/13/.

These results were generalized and substantiated by J.M.Skowronski into the systems with several degrees of freedom,/20/,/24/, /31/, /39/, especially broad and ultimately in the work,/47/. In the report /39/, and in the paper,/38/, written by the same author the problem is enlarged by the analysis of the partially damping characteristics. The only material couples of the structural models were investigated at the same time by Z.Osinski in the sense of the rheological models of these coupling elements which are adequate to the characteristics. In the work /51/ there is proposed some rheological model of the visco-elastic body respecting the possibility of the little nonlinearity.

I should mention here also about the several papers concerning the conditions of the motion-generation. The analysis of the technical assumption of the power-boundedness of the external energy sources was given in my work,/22/,written common with J.M.Skowronski. In the problem: to formulate the technical real assumptions for the right-hand sides of equations according to the types of the external forcing there are interested W.Bogusz,/33/, J.M.Skowronski,/38/. The latter has proved in the paper,/47/, that the motion generated with the singular impulse during the finite, sufficiently little period of time, can be taken as the free motion. This is such a period, that the system would not go out the mentioned neighbourhood of the basic energy level.

3. The research of motion of the introduced models. Let us regard first of all the analytical problems. The general tendency of the whole work of our group may be illustrated in the following manner: our purpose, first of all, was to prove which of the necessary and sufficient criteria of the shape of the integral motion curves, formulated in the theory of the ordinary differential equations or in the generalized dynamical systems theory, are satisfied by our concrete, but mechanically general, model. Having qualitatively analysed character of the motion, we have thought next about the exactly determining the particular integrals of motion by using the grapho-analytical methods adapted to our systems. Independently of this a few works was devoted of necessity to the sensu stricto mathematical problems connected for example with the necessary and sufficient criterion of the phase space trajectories behaviour for the cases especially important from the technical point of view. The reported part of the works concerning the analysis of motion was initiated by the several papers about the research of the systems with the single degree of freedom. This was of course pertained to the structure of the phase-plane in the particular cases data. S.Ziemba was considering successive in papers /7/,/8/,/12/, /15/ the cases:

$$\ddot{x} + \varphi(\dot{x}) + cx = 0 \qquad (1)$$

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$$\ddot{x} + \varphi(x,\dot{x}) + \psi(x) = 0 \qquad (3)$$

$$\ddot{x} + f(x,\dot{x}) = 0 \qquad (5)$$

with adequate initial conditions and adequate physically and mechanically substantiated assumptions

$$\psi(x)\cdot x > 0\,,\quad \varphi(x,\dot{x})\dot{x} \geq 0\,,\quad \varphi(-x,\dot{x}) = \varphi(x,\dot{x})$$
$$\psi(0) = 0\,,\quad \varphi(x,0) = 0\,,\ \ldots\ \text{e.c.}$$

where the functions of the characteristics are taken in such a manner that there are satisfied all the conditions necessary for the existence, uniqueness and prolongability for the solutions of (1), (2), (3) for the entire axis t. These functions are moreover analytic in some domain of its arguments. The decomposing into the uneven series was stopped on the 3-power.

The basic problem in these papers was the problem of the oscillatory property of the motion assuming the positive dissipation of energy. There were regarded also the free vibrations only. Therefore we have assumed that all of the phase trajectories are the O^+ curves. The investigations are realized in the arbitrary large but bounded neighbourhood of the equilibrium stady state position, passing many times in the special cases beyond the first maximum of the potential energy of the system. The criteria of the oscillatory property of the solutions (2) were investigated by G.Sonsone with the assumption $\varphi(x,\dot{x}) = [A + B\psi(x)]\dot{x}$ and the "hardness" of this function according to $\dot{x}$, besides for the linear spring characteristic. This is generalized by

...iemba,/3/, on the base for the arbitrary non-linear $\varphi(x,\dot{x})$, showing that in the arbitrary but bounded nei bourhood of the stady state equilibrium position the criteria of the oscillatory property are the same as for the adequate comparative linear system. For the linear damping and arbitrary nonlinear spring characteristics this problem was regarded by K.Gutowski and J.M.Skowronski in the works /50/, /3/, /13/. Some united interpretation of this criterion is given in the paper /41/, by Z.Osinski too. It seems that the original and interesting method of investigations of the behaviour of the phase-plane-trajectories is given by W.Bogusz,/35/, by introducing the research of the change of the hodograph of the tangent vector to a motion curve in the plane $(\dot{x}, \ddot{x})$. The last author presented also the set of the particular cases of applying the T.Wazewski retract method for the single-degree of freedom systems with external forcing /55/.

We may suppose too, that to the works of the systems with the single degree of freedom belongs the work /43/ by Z.Osinski. This is some enlarging of the Bogoliubow remarks on the small parameter vibrations:

$$\ddot{x} + \ell x = \mu f(x,\dot{x},t)\,,$$

where the linear damping as the exterior of the functions included sub a small parameter. Concerning the plane systems nobody of us was interested in the so called parametric vibrations. The conclusions of G.Sansone and S.Ziemba were generalized by J.M.Skowronski,/31/, into a general

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case of the free vibrations of systems with several degrees of freedom with arbitrary strong nonlinearity all of the characteristics with arbitrary nonlinear coupling. For some particular cases of the system with several degrees of freedom the problem of the oscillatory property of motion was regarded in the papers /24/, /47/, too. More strictly the number of the zero-points for particular phase-space-trajectories was regarded by J.Szadkowski,/53/. In this work there were also regarded the zero-points for the finite form (integral) of the motion.
The critical remarks about the definition of the oscillatory property of motion introduced many times in the literature and the practical criterion of estimation of the oscillatory property for a general system are given in my work, /30/, written with J.M.Skowronski.

Let us regard the general form of the motion equation of our model:

$$\ddot{q}+F(q,\dot{q},t)=0 \qquad (4)$$

with adequate initial conditions, where $q\{q_1,\dots,q_n\}$, $\dot{q}\{\dot{q}_1,\dots,\dot{q}_n\}$ the n-dimensional vectors of the generalized coordinates and velocities of our system, and F some strongly nonlinear operator in the configuration space of the system.
For many problems by using some transformation of variables we may pass to the phase-space, where the motion may be described by the another vectorial equation $\dot{x}=f(x,t)$ (5)

where again $x\{x_1,\dots,x_m\}$, $f\{f_1,\dots,f_m\}$ the adequate vectors. We have taken and many times substantiated, /21/, /19/, /47/, /39/, /8/, the following separation

$$F(q,\dot{q},t)=\Phi^{p}(q,\dot{q})+\Phi^{n}(q,\dot{q})+\Psi(q)-G(q,\dot{q},t) \qquad (6)$$

where $\Phi^{p}(q,\dot{q})$ - the characteristic operator of positive damping,
$\Phi^{n}(q,\dot{q})$ - the characteristic operator of partially negative damping
$\Psi(q)$ - the characteristic operator of restoring forces
$G(q,\dot{q},t)$ - the operator of the external forcing forces.

In the works belonging to the analysis we have regarded the various particular cases of (4) or (5), and in the many papers the most general case. On the functions $F_1,\dots,F_n$ or $f_1,\dots,f_m$, we have assumed the normally assumed mechanical conditions which were reported shortly in the introductions to the adequate papers and in the several especially separated works about the structural analysis,/7/, /47/, /39/, which was already mentioned.

1° - For the automonic systems with positive damping ($G=0$, $\Phi^{n}=0$) as it was mentioned in the work /31/ there are given the oscillatory property criteria, and in the paper /30/ there is given a method of estimation of the zero points number for solutions with the oscillatory property.

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2° - For the autonomic systems with partially negative damping ($G=0, \Phi'\neq 0, \Phi'\neq 0, \Psi=0$) i.e. with the negative damping "in the little", after assuming the existence, uniqueness and entire prolongability of solutions for any $t\in[t_0,\infty)$ there are given the following research: J.Szadkowski,/37/, /54/, has proved the boundedness in the large of these solutions, the existence of some domain of boundedness in the $2n$-dimensional phase space, further the existence of some limit domain in such a space, i.e. such a limit domain in which all of the solutions will be find for $t\to\infty$, and the existence at least one periodical solution too. This last problem is also given in the work /33/ by W.Bogusz. W.Bogusz has enlarged moreover his own hodograph method of the tangent vector to the integral motion curve in the space $(q, \dot{q})$ into case of the $2n$-dimensional phase space. There are now some suggestions in the direction of proving the existence the unique limit regime for these systems with assuming our model conditions but these tests are not finished to the publication.

3° - The composition of the problem of the existence uniqueness and prolongability of solutions for any $t\in[t_0,\infty)$ for the nonautonomic systems (the general case according to (5) is given in the paper /34/ by W.Bogusz. Further, however on these problems there are working in principle till now S.Ziemba and J.M.Skowronski only. In the paper /19/, for a special case $\Phi'=0, G=G(t)$ tney have proved by using the W.P.Demidowich criteria that the motion of our model must be bounded "in the large".

In the work /22/ by using the criteria of T.Yoshizawa it is proved the analogous boundedness in the case $\Phi'=0, G=G(q,\dot{q},t)\neq 0$ i.e. in the more general case. For the same case but with confining assumptions: there exists the finite integral from forcing term G, we have obtained the very simple proof of the boundedness of motion /29/. The further work of the same authors /55/ is showing that the motion of our general model is bounded, ultimately bounded, and there exists one finite closed domain of boundedness in the most general case $\Phi'\neq 0, \Phi''\neq 0, G\neq 0, \Psi=0$ In the work /27/ it is proved by these authors in the particular case $\Phi''=0, \Phi'\neq 0, G=0, \Psi\neq 0$ the existence of the finite closed limit domain of regarded motion, i.e. domain into which all of the integrals must enter after a finite time. In the paper /38/, and also in the work /56/, presented now on our symposium by J.M.Skowronski there is proved in the other way supplementary for the most general case of the forcing vibrations $\Phi'\neq 0, \Phi''\neq 0, G\neq 0, \Psi\neq 0$ the boundedness, ultimately boundedness, the existence of the boundedness domain and next the existence of the finite limit domain for the integral curves of motion into which all of these curves

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must enter at some finite time. Further the author investigates the trensient process of motion to this limit regime showing the existence and localization of the convergence domains of the integral curves of motion, the interior structure of this mentioned limit domain, the existence in this interior of the at least only convergence domain in the large, and also the existence one and only one limit energy level of our model in any convergence domain, i.e. the energy level in which the system will be find for $t \to \infty$. These investigations are carried out by using the behaviour of the surfaces of the while energy states for the model, i.e., so to say, in the metrical sense for every $t \in [t_0, \infty)$, i.e. further in the sense of Kolmogorow "mass of motion". It must be remarked that all of these conclusions are in relation to some finite domain of the phase-space, namely to the sufficiently large but bounded neighbourhood of the basic energy level of the total energy of the system. It is also the conception to enlarge these conclusions on the parametric vibrations, i.e. on such systems in characteristics of which there is the argument t explicite. However this conception is now not prepared for the publication.

Some quantitative method given by Jacobsen (so called "deltha" method) for the system with the single degree of freedom, was adapted in the works /16/, /20/, to the qualitative analysis of our systems with several degrees of freedom of

the type (5) researched in the continuous manner. The first conception to generalize this method to the space-systems in the quantitative sense was given in the work /10/, still 1958. There is a certain method based on the introduction in the motion equations all of the nonlinearities sub some function which, as it is proved, in the mentioned qualitative work /16/, may to determine, as a some component of the n-dimensional vector in the 2n-dimensional phase-space in any while t. Researching the space-hodograph of this vector, we have given the space-behaviour of the integral curve of motion. These investigations were carried out by the estimation of the absolute derivative of the mentioned vector, i.e. tensor having the components due to this described nonlinear function introduced in the equations of motion. Taking the boundedness of motion as known we have shown the existence at least one periodical solution in the systems (5) in which $G(q, \dot{q}, t)$ it was the periodical function of t, /14/.

The proof is deduced in the simple manner direct from the remarks of N.Levinson given for a plane system. This simplicity of proof is in general the reason for publication, because the result was already in 1958 not original.

Some works from the analysis of motion were due to the quantitative problems too. This mentioned grapho-analytical method deltha introduced by Jacobsen and Buland was generalized by us for a system with several degrees of freedom /10/,/28/,

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/17/, and adapted to many particularly technical problems /13/, /28/,

4. The synthetical investigations. The problem of syntnesis of motion of our model appeared first in the work /7/ by S. Ziemba describing the possibility of the selection of damping characteristic $\Psi(x,\dot{x})$ for the case of the single degree of freedom.

For the systems with several degrees of freedom the problem of synthesis is considered by J.M. Skowronski, as a some suggestion to apply the reversion of mentioned delthe method in the qualitative sense /25/. Furthermore this conception is substantiated in the paper /20/, /47/, taking nowever only the special case of the system (5), the autonomic system with positive damping

$$\Phi'' \equiv 0,\ G \equiv 0,\ \Phi' \neq 0,\ \Psi \neq 0.$$

The problem of the synthesis of our general mechanical systems and all the problems of the third part of investigations remain now as our perspectival problems.

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НЕКОТОРЫЕ ВОПРОСЫ В КАЧЕСТВЕННОЙ ТЕОРИИ СИСТЕМ С ЗАПАЗДЫВАНИЕМ

Nous allons considérer quelques questions relatives à l'existence des solution périodiques ou presque-périodiques des systèmes à retardement et aussi des théorèmes de stabilité pour ces systèmes.

1. Nous présenterons d'abord quelques résultats qui se rapportent aux systèmes à petit paramètre.

Soit un système de la forme

$$\dot{x}(t) = X_0[t, x(t), x(t-\tau)] + \varepsilon X_1[t, x(t), x(t-\tau), \varepsilon]. \quad (1)$$

Supposons que X_0 et X_1 sont périodiques en t, avec la même période $\omega > \tau$.

Supposons que le système "générateur"

$$\dot{x}(t) = X_0[t, x(t), x(t-\tau)] \quad (2)$$

admet une solution périodique de période ω, soit $x_0(t)$.

Considérons le système aux variations

$$\dot{y}(t) = X'_{0u}(t, x_0(t), x_0(t-\tau))y(t) + X'_{0v}(t, x_0(t), x_0(t-\tau))y(t-\tau). \quad (3)$$

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Théorème 1. Si le système (3) n'admet pas d'autres solutions périodiques de période ω que la solution banale, alors il existe $\varepsilon_0 > 0$ tel que pour $|\varepsilon| < \varepsilon_0$ le s teme (1) admet une solution périodique $x(t,\varepsilon)$ avec la propriété que $\lim_{\varepsilon \to 0} x(t,\varepsilon) = x_0(t)$ (1).

Pour prouver ce théorème nous allons faire quelques considérations qui seront utiles dans tout ce qui suit.

A toute fonction continue définie sur $[-\tau, 0]$ (il faut souligner que toutes ces fonctions seront des fonctions vectorielles à n composantes) correspond une solution unique $x(t, \varphi, \varepsilon)$ du système (1) qui pour $-\tau \le t \le 0$ coincide avec $\varphi(t)$; (on suppose vérifiée une condition qui assure l'unicité et la dépendence continue des conditions initiales et des parametres).

On définit l'opérateur $U_t^\varepsilon \varphi = x(t+s, \varphi, \varepsilon)$ qui pour t et ε fixés attache à la fonction φ définie sur $[-\tau, 0]$ la fonction $x(t+s, \varphi, \varepsilon)$ définie aussi sur $[-\tau, 0]$. Les

points fixes de l'opérateur $U_\omega^\varepsilon \varphi$ seront les fonctions initiales des solutions périodiques. Soit $F[\varphi, \varepsilon] = \varphi - U_\omega^\varepsilon \varphi$. Par hypothèse le système (2) admet la solution périodique $x_0(t)$ donc l'equation $F[\varphi, \varepsilon] = 0$ est vérifiée pour $\varepsilon = 0$, $\varphi = x_0(s)$. La fonction $F[\varphi, \varepsilon]$ est différentiable au sens de Fréchet par rapport à φ et l'on montre que sa différentielle au point $x_0(s)$ pour $\varepsilon = 0$ est $I - U$, où I est l'opérateur identique et U l'opérateur attaché au systeme (3) de la même manière que U_ω^ε était attaché au système (1). Pour $\omega > \tau$ U est un opérateur complètement continu (compact) et si le système (3) n'admet pas d'autres solutions périodiques de période ω que la solution banale, l'opérateur $I - U$ admet un inverse. On peut appliquer le théorème des fonctions implicites dans l'éspace de Banach et il en resulte qu'il existe $\varepsilon_0 > 0$ tel que pour $|\varepsilon| < \varepsilon_0$ l'équation $F[\varphi, \varepsilon] = 0$ admet une solution qui pour $\varepsilon \to 0$ se réduit à $x_0(s)$. Le

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théorème 1 est ainsi prouvé.

Un autre théorème général qui ne fait plus intervenir le systeme linéaire (3) est le suivant.

Théorème 2. Si la solution périodique $x_0(t)$ du système (2) est uniformément asymptotiquement stable, alors pour $|\varepsilon|$ assez petit le système (1) admet une solution périodique $x(t,\varepsilon)$ avec la propriété que $\lim_{\varepsilon \to 0} x(t,\varepsilon) = x_0(t)$ (uniformément).

On prouve ce théorème en montrant que dans les conditions de l'énoncé l'opérateur U_ω^ε admet un point fixe. On utilise le théorème suivant établi recemment par E.Browder: soit X un espace de Banach, S et S_1 des ensembles ouverts et convexes de X, S_0 fermé et convexe, $S_0 \subset S_1 \subset S$, f une application compacte de S dans X; supposons qu'il existe m naturel tel que f^m est définie sur S_1, $f^j(S_1) \subset S_1$ pour $0 \le j \le m$, $f^m(S_1) \subset S_0$; alors f admet un point fixe dans S_0. Nous allons prendre $f = U_\omega^\varepsilon$; montrons que toutes ces conditions sont vérifiées. Par hypothèse la solution $x_0(t)$ est uniformément asymptotiquement stable; on ne restreint pas la généralité si l'on prend $x_0(t) \equiv 0$, parceque l'on peut arriver à cette situation par un simple changement de variables. La stabilité asymptotique considérée signifie l'existence d'un nombre $\delta_0 > 0$ et de deux fonctions $\delta(\eta)$ et $T(\eta)$ telles que si $\|\varphi\| < \delta(\eta)$ alors $|x(t,$

$\varphi, 0)| < \eta$ pour $t \ge \tau$ et si $\|\varphi\| < \delta_0$, $t > T(\eta)$ alors $|x(t, \varphi, 0)| < \eta$. Soit $\delta_1 = \delta(\eta)$, $0 < \eta < \min[\delta_0, \delta(\frac{1}{2}\delta_1)]$, $T = T[\frac{1}{2}\delta(\frac{1}{2}\eta)]$, m le moindre nombre naturel pour lequel $m\omega > T + \tau$. Soit S la sphère $\|\varphi\| < \delta_1$. Alors $|x(t, \varphi, 0)| < 1$ pour $\varphi \in S$ et $t > \tau$. Si $|\varepsilon|$ est suffisamment petit il s'ensuit que $|x(t, \varphi, \varepsilon)| < 2$ pour $\varphi \in S$, $-\tau \le t \le m\omega$. Il s'ensuit que pour $\varphi \in S$ on a $\|U_\varphi^\varepsilon\| < 2$ et en tenant compte du systeme (1) on en déduit que l'opérateur U_ω^ε est compact sur S.

De l'évaluation obtenue résulte aussi la possibilité de prolonger les solutions avec $\varphi \in S$, donc l'opérateur est défini

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pour tous $\varphi \in S$. Soit S_1 la sphère $\|\varphi\| < \eta$ et S_0 la sphère $\|\varphi\| \leq \frac{3}{4}\delta(\frac{1}{2}\eta)$. Pour $\|\varphi\| < \eta$, $t \geq -\tau$ on a $|x(t,\varphi,0)| < \frac{1}{2}\delta_1$ puisque $\eta < \delta(\frac{1}{2}\delta_1)$; pour $|\varepsilon|$ assez petit il s'ensuit que $|x(t;\varphi,\varepsilon)| < \delta_1$ pour $-\tau \leq t \leq m\omega$. On en déduit que $[U_\omega^\varepsilon]^j(S_1) \subset S$ pour $0 \leq j \leq m$, car $[U_\omega^\varepsilon]^j(\varphi) = x(j\omega + s; \varphi, \varepsilon)$.

Pour $\|\varphi\| \leq \frac{3}{4}\delta(\frac{1}{2}\eta)$, $t \geq -\tau$ on a $|x(t,\varphi,0)| < \frac{1}{2}\eta$ donc pour $|\varepsilon|$ assez petit $|x(t;\varphi,\varepsilon)| \leq \eta$ si $\|\varphi\| \leq \frac{3}{4}\delta(\frac{1}{2}\eta)$, $-\tau \leq t \leq m\omega$, donc $[U_\omega^\varepsilon]^j(S_0) \subset S_1$ pour $0 \leq j \leq m$. De $\eta < \delta_0$ il s'ensuit que pour $\varphi \in S_1$, $t > T$ on a $|x(t;\varphi,0)| < \frac{1}{2}\delta(\frac{1}{2}\eta)$, donc pour $|\varepsilon|$ assez petit $|x(t;\varphi,\varepsilon)| \leq \frac{3}{4}\delta(\frac{1}{2}\eta)$ si $\varphi \in S_1$, $m\omega - \tau \leq t \leq m\omega$, c'est à dire $[U_\omega^\varepsilon]^m(S_1) \subset S_0$. Il faut souligner que dans cette démonstration nous avons utilisé seulement le fait que U_ω^ε est compact sur S et la dépendence continue de la solution des paramètres; ceci montre qu'il est très probable que ce résultat peut s'étendre pour d'autres classes d'equations. Il est aussi probable que l'on peut prouver la stabilité de la solution périodique dont l'existence vient d'être établie.

2. Nous allons maintenant établir quelques résultats qui permettent de mettre en évidence non seulement les solutions périodiques mais aussi les solutions presque-périodiques.

Lemme. Soit le système

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + f(t) \tag{4}$$

ou $A(t), B(t), f(t)$ sont bornées pour $t \geq 0$, Si la solution banale du système homogène correspondant est uniformément asymptotiquement stable, alors toutes les solutions du système (4) sont bornées ([3]).

On prouve ce lemme a l'aide d'une fonctionnelle de Liapounoff pour le système homogène, de la forme

$$V[t,\varphi] = \int_0^\infty \|y(t+u+s,t,\varphi)\|^2 du + \sup_{\sigma \geq 0} \|y(t+\sigma+s;t,\varphi)\|^2.$$

On voit aisément que ce lemme reste valable pour les systèmes linéaires à retardement de la forme générale

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$$\dot{x}(t)=\int_{-\infty}^{0} x(t+s)\,d_s\,\eta(t,s)+f(t) \qquad (5)$$

où $\eta_{ij}(t,s)$ sont définies pour $t\geq 0$, $-\infty<s<\infty$, continues en t uniformément par rapport à s, bornées pour $t\geq 0$, $\eta_{ij}(t,s)\equiv 0$ pour $s\geq 0$; il existe $\tau_{ij}(t)>0$, $V_{ij}(t)>0$ bornées pour $t\geq 0$ telles que $\eta_{ij}(t,s)=\eta_{ij}(t,-\tau_{ij}(t))=0$ pour $s\leq -\tau_{ij}(t)$, $\bigvee_{s=\alpha}^{\beta} \eta_{ij}(t,s)\leq V_{ij}(t)$, où, comme d'habitude $\bigvee_{s=\alpha}^{\beta} f(s)$ signifie la variation totale de la fonction f sur $[\alpha,\beta]$.

De ce lemme il s'ensuit que si A, B, f, respectivement $\eta(t,s)$ sont périodiques en t de période $\omega>\tau$ (respectivement $\omega>\max\sup\tau_{ij}(t)$) alors le système (4) resp. (5) admet une solution périodique unique et cette solution vérifie une inégalité de la forme $|x_0(t)|\leq LM$ où L dépend seulement du système homogène et $M=\sup|f(t)|$. Si A, B, f respectivement $\eta(t,s)$ sont presque-périodiques en t, on prouve l'existence d'une solution presque-périodique unique qui admet une évolution comme plus haut.-

Ces faits permettent d'appliquer un procédé standard d'approximations successives pour prouver l'existence des solutions périodiques ou presque-périodiques du système (1); (les solutions presque-périodiques sont cherchées dans le cas où X_0 et X_1 sont presque-périodiques en t, uniformément par rapport aux autres variables et la solution génératrice $x_0(t)$ est presque-périodique). On pose $y(t)=x(t)-x_0(t)$ et on obtient

$$\dot{y}(t)=A(t)y(t)+B(t)y(t-\tau)+Y_0[t,y(t),y(t-\tau)]+$$
$$+\varepsilon\,Y_1[t,y(t),y(t-\tau),\varepsilon].$$

On prend $Y_0(t)\equiv 0$ et $Y_k(t)$ les solutions périodiques resp. presque-périodiques du système

$$\dot{z}(t)=A(t)z(t)+B(t)z(t-\tau)+Y_0[t,Y_{k-1}(t),Y_{k-1}(t-\tau)]+$$
$$+\varepsilon\,Y_1[t,Y_{k-1}(t),Y_{k-1}(t-\tau),\varepsilon].$$

De la même manière on peut considérer les systèmes quassi-linéaires de la forme

$$\dot{x}(t)=\int_{-\infty}^{0} x(t+s)\,d_s\,\eta(t,s)+\int_{-\infty}^{0} g[t,s,x(t+s)]\,d_s\,\eta_1(t,s)+$$
$$+\varepsilon\int_{-\infty}^{0} f[t,s,x(t+s),\varepsilon]\,d_s\,\eta_2(t,s).$$

Les mêmes considérations permettent d'obtenir des résultats relatifs au système.

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(6) $\dot{x}(t) = X_0(t,x) + \varepsilon X_1[t, x(t), x(t-\tau\varepsilon), \varepsilon]$

où X_0 et X_1 sont périodiques en t avec la période $\omega > \tau$, respectivement presque-périodiques, et le système obtenu pour $\varepsilon = 0$ a la solution générale $x_0(t,h)$ périodique en t de période ω respectivement presque-périodique. On fait le changement de variables $x = x_0(t,z)$ et on obtient $\frac{dz}{dt} = \varepsilon Z[t, z(t), z(t-\varepsilon\tau), \varepsilon]$ où

$$Z(t,u,v,\varepsilon) = \left[\frac{\partial x_0(t,u)}{\partial h}\right]^{-1} X_1[t, x_0(t,u), x_0(t-\varepsilon\tau, v), \varepsilon].$$

Soit $Z_0(u,v,\varepsilon) = \lim_{T\to\infty} \frac{1}{T}\int_0^T Z(t,u,v,\varepsilon)\,dt$ et ζ° une solution de l'equation $Z_0(u,u,0) = 0$. Soit $H_1 = \frac{\partial Z_0}{\partial u}(\zeta^\circ, \zeta^\circ, 0)$, $H_2 = \frac{\partial Z_0}{\partial v}(\zeta^\circ, \zeta^\circ, 0)$.

Théorème 3. Si les valeurs propres de la matrice $H_1 + H_2$ ont les parties réelles négatives, alors il existe $\varepsilon_0 > 0$ tel que pour $0 \le \varepsilon \le \varepsilon_0$ le système (6) admet une solution periodique, resp. presque-périodique unique, $x(t,\varepsilon)$, telle que $\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t, \zeta^\circ)$

([4], [5]).

Après les changements de variables habituels dans la méthode "de la valeur moyenne" tout revient à prouver l'existence d'une solution périodique, resp. presque-périodique, du système

$$\dot{h}(t) = \varepsilon H_1 h(t) + \varepsilon H_2 h(t-\varepsilon\tau) + \varepsilon B_1[h(t), h(t-\varepsilon\tau)] + \varepsilon \Gamma[t, h(t), h(t-\varepsilon\tau), \varepsilon].$$

3. Un progrès essentiel dans l'étude des solutions périodiques des systèmes à petit paramètre dépend de l'étude complète des solutions périodiques des systèmes linéaries (4) resp. (5).

En utilisant les formules générales de représentation des solutions des systèmes linéaires non-homogènes

$$x(t) = X(t,\sigma)x(\sigma) + \int_{\sigma-\tau}^{\sigma} X(t, \tau+\alpha) B(\alpha+\tau)x(\alpha)\,d\alpha + \int_{\sigma}^{t} X(t,\alpha) f(\alpha)\,d\alpha,$$

$$x(t) = x(\sigma)\,X(t,\sigma) + \int_{\sigma-\tau}^{\sigma} x(s)\,ds \int_{\sigma}^{\sigma+\tau} \eta(\alpha, s-\alpha)\,X(t,\alpha)\,d\alpha + \int_{\sigma}^{t} f(\alpha) X(t,\alpha)\,d\alpha$$

où $X(t,\sigma)$ est une matrice solution du système homogène qui vérifie les conditions $X(t,\sigma) = 0$

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pour $\chi(s,s') = E$, on prouve les théorèmes suivants.

Théorème 4. Le système (4) resp. (5) admet une solution périodique de période ω pour toute fonction $f(t)$ périodique de période ω si et seulement si le système homogène correspondant n'admet pas d'autres solutions périodiques de période ω que la solution banale. Cette solution périodique admet une évaluation de la forme $|x(t)| \leq L \sup |f|$ où L ne dépend que du système homogène ([6], [8], [9]).

Théorème 5. Si le système (4) resp. (5) admet une solution bornée, alors il admet une solution pèreiodique ([7], [8]).

Théorème 6. La condition nécessaire et suffisante pour que le système (4) resp. (5) admette des solutions périodiques de période ω est

$$\int_0^\omega y_k(\alpha) f(\alpha)\, d\alpha = 0 \quad \text{resp.} \quad \int_0^\omega f(\alpha) y_k(\alpha)\, d\alpha = 0$$

pour toutes les solutions périodiques $y_k(t)$ de periode ω du systeme adjoint ([9]).

Pour le système (4) le système adjoint est

$$\dot{y}(t) = -y(t)A(t) - y(t+\tau)B(t+\tau)$$

et pour le système (5) adjoint est

$$\frac{d}{dt}\left[y(t) + \int_{-\infty}^{0} \eta(t-\gamma,\gamma)\, y(t-\gamma)\, d\gamma\right] = 0.$$

Le système homogène correspondant à (4) resp. (5) et le système adjoint respectif ont le même nombre de solutions périodiques de période ω linéairement indépendentes.

Les conditions initiales des solutions périodiques du système (4) resp. (5) sont données par les solutions de l'equation $(1-U)\varphi = F$ où pour le système (4)

$$U\varphi = X(\omega+s,0)\varphi(0) + \int_{-\tau}^{0} X(\omega+s,\alpha+\tau)B(\alpha+\tau)\varphi(\alpha)\, d\alpha,$$

$$F = \int_0^{\omega+s} X(\omega+s,\alpha) f(\alpha)\, d\alpha,$$

et pour (5)

$$U\varphi = \varphi(0)X(\omega+s,0) + \int_{-\tau}^{0} \varphi(\beta)\, d\beta \int_{-\tau}^{0} \eta(\alpha+\tau,\beta-\alpha-\tau) X(\omega+s,\alpha+\tau)\, d\alpha,$$

$$F = \int_0^{\omega+s} f(\alpha) X(\omega+s,\alpha)\, d\alpha.$$

une étude analogue à celle que l'on fait dans la théorie des équations intégrales montre que l'équation $\varphi - U\varphi = 0$ et l'équation

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$\bar{\varphi}(s) = \bar{\varphi}(-\tau) X(\omega, s+\tau) + \int_{-\tau}^{0} \bar{\varphi}(\eta) B(\eta+\tau) X(\eta+\omega, s+\tau) d\eta$

ont le même nombre de solutions linéairement indépendentes et que l'équation $\varphi - U\varphi = F$ admet des solutions si et seulement si $\bar{\varphi}(-\tau) F(0) + \int_{-\tau}^{0} \bar{\varphi}(\eta) B(\eta+\tau) F(\eta) d\eta = 0$ pour toutes les solutions del'équation en $\bar{\varphi}$.

Un résultat analogue on obtient dans le cas du système (5) et l'on en déduit les affirmations de l'énoncé.

Considérons maintenant le système

$$\dot{x}(t) = \int_{-\infty}^{0} x(t+s) d_s \eta(t,s) + f(t) + \varepsilon F(t, x(t+s), \varepsilon], \quad (7)$$

où pour t, ε fixés, les composantes du vecteur F sont des fonctionnelles définies sur l'espace des fonctions vectorielles continues sur $[-\tau, 0]$ et

$$|F(t, \varphi_1, \varepsilon) - F(t, \varphi_2, \varepsilon)| \leq L \|\varphi_1 - \varphi_2\|.$$

Supposons que le système homogène correspondant au système qui s'obtient pour $\varepsilon = 0$ admet des solutions périodiques de période ω. Il existe alors un nombre fini de telles solutions linéairement indépendentes $p_j(t)$, $j = 1, 2, \ldots, \varkappa$. Le système adjoint

correspondant admet lui aussi le même nombre fini de solutions périodiques indépendentes $q_j(t)$, $j = 1, 2, \ldots, \varkappa$. Supposant remplies les conditions $\int_0^{\omega} f(t) q_j(t) dt = 0$, $j = 1, 2, \ldots, \varkappa$ on peut montrer qu'il existe une solution périodique et une seule du système obtenu pour $\varepsilon = 0$ qui vérifie l'évaluation $|x(t)| \leq L \sup |f|$, où L ne dépend que du système homogène. Si l'on désigne cette solution par $\rho(t)$, les solutions périodiques du système générateur obtenu pour $\varepsilon = 0$ seront de la forme $\rho(t) + \sum \alpha_j p_j(t)$.

Théorème 7. Soit

$$P_j(\alpha_1, \ldots, \alpha_\varkappa, \varepsilon) = \int_0^{\omega} F[t, \rho(t+s) + \sum_i \alpha_i p_i(t+s), \varepsilon] q_j(t) dt$$

Si $\alpha_1^0, \alpha_2^0, \ldots, \alpha_\varkappa^0$ vérifient les relations

$$P_j(\alpha_1^0, \alpha_2^0, \ldots, \alpha_\varkappa^0, 0) = 0, \quad \frac{\partial(P_1, P_2, \ldots, P_\varkappa)}{\partial(\alpha_1, \alpha_2, \ldots, \alpha_\varkappa)}\Bigg|_{\substack{\alpha_j = \alpha_j^0 \\ \varepsilon = 0}} \neq 0$$

alors pour $|\varepsilon| < \varepsilon_0$ il existe une solution périodique $x(t, \varepsilon)$ de période ω du système (7) telle que

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$$\lim_{\varepsilon \to 0} x(t,\varepsilon) = \rho(t) + \sum_j \alpha_j^{\circ} \rho_j(t) \quad (/10/)$$

La démonstration se fait en essence comme pour les systèmes d'equations différentielles ordinaires suivant le scheme donné par I.G.Malkin.

Le théorème 6 permet aussi d'obtenir un progrès essentiel per rapport au théorème 1.

Supposons que le système (2) admet une famille de solutions périodiques de periode ω, soit $x_o(t,h)$. Considérons le système aux variations correspondant à une solution pour laquelle $h = h_o$. Ce système admet en tout cas les solutions périodiques $\frac{\partial x_o(t,h)}{\partial h}\Big|_{h=h_o}$. Supposons qu'il n'admet point d'autres solutions périodiques lineairement independentes. Soient $q_j(t,h_o)$ les solutions périodiques indépendentes du système adjoint du système aux variations.

Théorème 8. Soit $P_j(h) =$

$$= \int_0^{\omega} q_j(t,h) X_1[t, x_o(t,h), x_o(t-\tau,h), o]\,dt,$$

si h_o vérifie les relations $P_j(h_o) = 0$

$$Jet \frac{\partial}{\partial h_e} \int_0^{\omega} q_j(t,h_o) X_1[t, x_o(t,h), x_o(t-\tau,h), o]\,dt \neq 0$$

pour $h = h_o$

et X_o, X_1 sont analytiques, alors le système (1) admet une solution périodique $x(t,\varepsilon)$ telle que $\lim_{\varepsilon \to 0} x(t,\varepsilon) = x_o(t,h_o)$ ([10]).

Observons que si l'on ne tient pas compte de l'hypothèse de l'analyticité on obtient une généralisation considérable du cas périodique du théorème 3.

Il serait bien désirable de prouver ce théorème sans utiliser l'hypothèse de analyticité.

La démonstration du théorème 8 se fait comme pour les systèmes d'equations ordinaires en construisant la solution périodique sous la forme

$$x(t,\varepsilon) = x_o(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$

et en montrant que dans les hypothèses de l'enoncé on peut déterminer successivement, d'une facon unique les fonctions périodiques $x_j(t)$.

Par les mêmes méthodes on peut aborder

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l'étude des systèmes autonomes a petit parametre.

Considérons le système

(8) $\dot{x}(t) = f[x(t), x(t-\tau), \varepsilon]$.

Supposons que pour $\varepsilon = 0$ le système (8) admet une solution périodique $\rho(t)$ de période $\omega_0 > \tau$. Soit

$$A(t) = f'_u[\rho(t), \rho(t-\tau), 0],\ B(t) = f'_v[\rho(t), \rho(t-\tau), 0].$$

Le système

(9) $\dot{z}(t) = A(t)z(t) + B(t)z(t-\tau)$

est le système aux variations correspondant à la solution $\rho(t)$. Ce systeme a les coefficients périodiques de période ω_0. Il admet en tout cas la solution périodique $\frac{d\rho}{dt}$.

Théorème 9. Si le système (9) n'admet pas de solutions périodiques de période ω_0 indépendentes de la solution $\frac{d\rho}{dt}$ et si $\int_0^{\omega_0} q(t)G(t)dt \neq 0$ où $q(t)$ est la solution périodique de période ω_0 du système adjoint du système (9) et

$$G(t) = f[\rho(t), \rho(t-\tau), 0)] + \tau B(t) f[\rho(t-\tau), \rho(t-2\tau), 0],$$

alors pour $|\varepsilon|$ suffisamment petit le système (8) admet une solution périodique $x(t,\varepsilon)$ de période $\omega(\varepsilon)$ telle que $\lim_{\varepsilon\to 0}\omega(\varepsilon) = \omega_0$, $\lim_{\varepsilon\to 0} x(t,\varepsilon) = \rho(t)$ ([10]).

Pour la démonstration on pose

$\omega(\varepsilon) = \omega_0(1+\varepsilon\alpha(\varepsilon))$, $t = s(1+\varepsilon\alpha(\varepsilon))$, $y(s,\varepsilon) = x(s(1+\varepsilon\alpha(\varepsilon)),\varepsilon)$; si $x(t,\varepsilon)$ est périodique de période $\omega(\varepsilon)$ alors y est périodique de période ω_0 et réciproquement.

$$\frac{d}{ds}y(s,\varepsilon) = f[y(s,\varepsilon), y(s-\Theta,\varepsilon), \varepsilon] + \varepsilon\alpha f[y(s,\varepsilon), y(s-\Theta,\varepsilon), \varepsilon]$$

où $\Theta(\varepsilon) = \frac{\tau}{1+\varepsilon\alpha(\varepsilon)}$. Soit $y(s,\varepsilon) = \rho(s) + \varepsilon z(s,\varepsilon)$. On obtient

$$\dot{z}(s) = A(s)z(s) + B(s)z(s-\Theta) + F(s) + \alpha(\varepsilon)G(s) + \varepsilon H(s, z(s), z(s-\Theta), \varepsilon),$$

$$F(s) = f'_\varepsilon[\rho(s), \rho(s-\tau), 0].$$

On construit successivement les suites $\alpha_n(\varepsilon)$ et $z_n(s,\varepsilon)$, où $z_n(s,\varepsilon)$ est la solution périodique du système.

$$\dot{z}(s) = A(s)z(s) + B(s)z(s-\tau) + F(s) + \alpha_n(\varepsilon)G(s) + B(s)[z_{n-1}(s-\Theta_{n-1},\varepsilon) - z_{n-1}(s-\tau,\varepsilon)] + \varepsilon H_{n-1}(s, z_{n-1}(s,\varepsilon), z_{n-1}(s-\Theta_{n-1},\varepsilon))$$

où $\Theta_{n-1} = \frac{\tau}{1+\varepsilon\alpha_{n-1}}$ et H_{n-1} est ce que devient H si α est remplacée par α_{n-1}; $\alpha_n(\varepsilon)$ est déterminée par la condition que ce système admette une solution périodique; $z_0(s)$ est la solution périodique du système

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$$\dot{z}(s) = A(s)z(s) + B(s)z(s-\tau) + F(s) + \alpha_0 G(s).$$

On prouve la convergence uniforme des suites ainsi construites et on obtient les fonctions $\alpha(\varepsilon)$ et $z(s,\varepsilon)$ à l'aide desquelles on obtient $y(s,\varepsilon)$ et ensuite $x(t,\varepsilon)$.

Beacoup plus compliqué est le cas où lesystème générateur admet une famille de solutions périodiques $\rho(t,c)$ de période $\omega_1(c)$; supposons que $\omega_1(c)$ admet des dérivées partielles continues.

Soit $\omega(c,\varepsilon) = \omega_0(c)[1+\varepsilon\alpha(c,\varepsilon)]$, $t = s[1+\varepsilon\alpha(c,\varepsilon)]$, $y(s,c,\varepsilon) = x[s(1+\varepsilon\alpha),\varepsilon]$.

Comme dans le cas précédant

$$\frac{d}{ds}y(s,c,\varepsilon) = (1+\varepsilon\alpha)f[y(s,c,\varepsilon), y(s-\Theta(c,\varepsilon),c,\varepsilon),\varepsilon],$$

$$\Theta(c,\varepsilon) = \frac{\tau}{1+\varepsilon\alpha(c,\varepsilon)}.$$

Soit $y(s,c,\varepsilon) = \rho(s,c) + \varepsilon z(s,c,\varepsilon)$. On obtient

$$\frac{d}{ds}z(s,c,\varepsilon) = A(s,c)z(s) + B(s,c)z(s-\Theta) + F(s,c) + \alpha(c,\varepsilon)G(s,c) + \varepsilon H[s,z(s),z(s-\Theta),\varepsilon],$$

où nous avons employé le même système de notations que plus haut.

Considérons le système

(10) $\dot{z}(s) = A(s,c)z(s) + B(s,c)z(s-\tau) + F(s,c) + \alpha_0 G(s,c)$

et le système homogene

(11) $\dot{z}(s) = A(s,c)z(s) + B(s,c)z(s-\tau)$.

On vérifie aisément que $\rho(t,c)$ et $\frac{\partial\rho}{\partial c_i}$ sont des solutions de (11). Si $\omega_0(c)$ n'est pas constante, les solutions $\frac{\partial\rho}{\partial c_i}$ ne sont pas périodiques. Considérons la fonctions $\rho\left(\frac{\omega_0(c)}{\omega_0(c^*)}t, c\right)$ qui est périodique avec la période $\omega_0(c^*)$. Sa dérivée par rapport à c au point c^* sera une fonction périodique $q(t,c^*)$ de période $\omega_0(c^*)$. On a

$$q(t,c^*) = \frac{\partial\rho}{\partial c}\left(\frac{\omega_0(c^*)}{\omega_0(c^*)}t, c^*\right) + \frac{t}{\omega(c^*)}\dot{\rho}\left(\frac{\omega_0(c^*)}{\omega_0(c^*)}t, c^*\right)\frac{\partial\omega_0}{\partial c}\Big|_{c^*},$$

donc $\frac{\partial\rho}{\partial c}(t,c^*) = q(t,c^*) - \frac{t}{\omega_0(c^*)}\dot{\rho}(t,c^*)\frac{\partial\omega_0}{\partial c^*}$.

Soit $t\dot{\rho}(t,c^*) = z(t)$, on a

$$\dot{z}(t) = tf[\rho(t,c^*), \rho(t-\tau,c^*), o].$$

On obtient

(12) $\dot{z}(t) = A(t,c^*)z(t) + B(t,c^*)z(t-\tau) + G(t,c^*)$.

Supposons $\frac{\partial\omega_0}{\partial c_i^*} \neq 0$. Alors

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$$\frac{\omega_0(c^*)}{\frac{\partial\omega_0}{\partial c_i^*}} q_i(t,c^*) = t\dot{\rho}(t,c^*) - \frac{\partial\rho}{\partial c_i}(t,c^*)\frac{\omega_0(c^*)}{\frac{\partial\omega_0}{\partial c_i^*}}$$

est une solution de (12) et cette solution est périodique en même temps que $q_i(t,c^*)$ Il s'ensuit que le système (12) admet une solution périodique, donc $G(t,c^*)$ est orthogonale sur les solutions périodiques du système adjoint du système (11) pour $c=c^*$. Il s'ensuit que pour l'existence des solutions périodiques de période $\omega_0(c^*)$ du système (10) il est nécessaire et suffisant que $F(s,c^*)$ soit orthogonale sur les solutions périodiques du système adjoint de (11). Le nombre de ces solutions est égal au nombre des solutions périodiques du système (II) et celles-ci seront en nombre au moins égal au nombre des paramètres c. En fait, une de ces solutions est $\dot{\rho}(t,c^*)$ et les autres sont de la forme

$$\frac{\partial\omega}{\partial c_i}\frac{\partial\rho}{\partial c_l} - \frac{\partial\omega}{\partial c_l}\frac{\partial\rho}{\partial c_i}, \quad c=c^*, \quad i=2,3,\dots,k.$$

Supposons que ce sont les seules solutions indépendentes de période $\omega_0(c^*)$ du système (11). Il s'ensuit que si système (8) admet une solution $x(t,\varepsilon)$ telle que $\lim_{\varepsilon\to 0} x(t,\varepsilon)=\rho(t,c^*)$ alors $P_j(c^*)=0$, $j=1,2,\dots,k$ où

$$P_j(c)=\int_0^{\omega_0(c)}\psi_j(t,c)f_\varepsilon'[\rho(t,c),\rho(t-\tau,c),0]\,dt,$$

$\psi_j(t,c)$ étant les solutions périodiques du système adjoint à (11). Il est bien probable que si $P_j(c^*)=0$ et $\det\frac{\partial P_j}{\partial c}\neq 0$ pour $c=c^*$ alors le système (8) admet une solution $x(t,\varepsilon)$ avec $\lim_{\varepsilon\to 0} x(t,\varepsilon)=\rho(t,c)$. Jusqu'à présent je n'ai pas réussi a prouver cette affirmation.

4. Dans ce qui suit nous allons montrer quelques théorèmes de stabilité relatifs aux systèmes à retardement, théorèmes qui peuvent présenter un intérêt dans l'étude de la stabilité des oscillations.

Théorème 10. Si pour fonction $f(t)$ bornée la solution qui vérifie les conditions initiales nulles du système (4) resp. (5) est bornée, alors la solution banale du système homogène correspondant est uniformément asymp-

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totiquement stable ([11]).

La démonstration suit le schéma suivant. La condition de l'énoncé implique $\left|\int_0^t f(\alpha) X(t,\alpha)\,d\alpha\right| < M_f$ pour toure f bornée et d'ici, comme l'a montré Bellman il s'ensuit que $\int_0^t |X(t,\alpha)|\,d\alpha < C$. On prouve ensuite qu'il existe M telle que $|X(t,\alpha)| < M$ pour $0 \leqslant \alpha \leqslant t$ et ensuite que l'on a même $|X(t,t_0)| \leqslant \frac{M^*}{t-t_0}$ et

$$|x(t;t_0,\varphi)| \leqslant \frac{M^{**}}{t-t_0}\|\varphi\|.$$

Dans cette dernière partie de notre exposé nous allons esquisser la théorie de la stabilité pour les systèmes linéaires à coefficients périodiques.

Soit $x(t,\varphi)$ la solution du système (4) resp. (5) avec $f=0$ définie pour $t \geqslant -\tau$ et qui sur $[-\tau, 0]$ coincide avec φ.

A cause de la périodicite du système, $x(t+\omega;\varphi)$ sera aussi une solution du système. Si $U_t\varphi = x(t+s;\varphi)$, $-\tau \leqslant s \leqslant 0$ on en déduit que $U_{t+\omega}\varphi = U_t U_\omega \varphi$. Il s'ensuit que la condition nécéssaire et suffisante pour que les solutions soient bornées

est $\|U_\omega^N\| \leqslant M$ pour $N=1, 2 \ldots n, \ldots$ Si $\|U_\omega^N\| < K(1-\varepsilon)^N$ alors toutes les solutions tendent vers zéro et a lieu la stabilité asymptotique uniforme. L'hypothèse $\omega > \tau$ permet de déduire que U_ω est compact. Il s'ensuit d'ici que si toutes les solutions sont bornées le spectre de l'opérateur U_ω est situé dans $|\lambda| \leqslant 1$ et les valeurs propres situées sur $|\lambda| = 1$ ont des diviseurs élémentaires simples. Réciproquement, si le spectre de U_ω admet cette propriété, toutes les solutions sont bornées et si le spectre est dans $|\lambda| < 1$ a lieu la stabilité asymptotique uniforme. Les valeurs propres de l'opéreteur U_ω s'appellent les multiplicateurs du système. Si ρ est un multiplicateur il existe une solution du systeme pour laquelle $x(t+\omega) = \rho x(t)$ et réciproquement. Si l'on pose $\lambda = \frac{1}{\omega}\ln\rho$ la solution correspondante a la structure $x(t) = e^{\lambda t} v(t)$.

De ces considérations, en utilisant le théorème 4 on obtient le

Théoreme 11. Si pour toute fonction g

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périodique de période ω et pour tout λ avec $|e^{\lambda\omega}| \geq 1$ le système (4) resp. (5) avec $f(t) = e^{\lambda t} g(t)$ admet une solution de la forme $e^{\lambda t} h(t)$ avec $h(t)$ périodique de période ω, alors la solution banale du système homogène est uniformément asymptotiquement stable.

En passant à l'étude plus détaillée de l'opérateur U_ω remarquons que les valeurs propres de cet opérateur sont données par l'équation $U_\omega \varphi = \rho\varphi$; si l'on pose $\rho = \frac{1}{\lambda}$ cette équation s'écrit dans le cas du système (4)

$$\varphi(s) = \lambda\left[X(\omega+s,0)\varphi(0) + \int_{-\tau}^{0} X(\omega+s,\alpha+\tau)B(\alpha+\tau)\varphi(\alpha)d\alpha\right].$$

Soit

$$L(\alpha,s) = X(\omega+s,\tau)X(\tau,\alpha+\tau),$$

$$M(\alpha,s) = \int_{0}^{\tau} X(\omega+s,\xi+\tau)B(\xi+\tau)X(\xi,\alpha+\tau)d\xi.$$

On a la relation $X(\omega+s,\alpha+\tau) = L(\alpha,s) + M(\alpha,s)$. Si ψ,φ sont deux fonctions matricielles définies sur $[-\tau,0]$ pour lesquelles la multiplication est possible, on définit l'opération

$$\langle \psi, \varphi \rangle = \psi(-\tau)\varphi(0) + \int_{-\tau}^{0} \psi(\xi)B(\xi+\tau)\varphi(\xi)d\xi.$$

On prouve que $\langle M(\sigma,s), L(\alpha,\sigma)\rangle = 0$ et on en déduit que $\sigma_X = \sigma_L \cup \sigma_M$ où $\sigma_X, \sigma_L, \sigma_M$

sont les spectres des opérateurs

$$\langle X(\omega+s,\sigma+\tau),\varphi(\sigma)\rangle, \langle L(\sigma,s),\varphi(\sigma)\rangle,$$

$$\langle M(\sigma,s),\varphi(\sigma)\rangle.$$

On vérifie ensuite que si

$$\tau B < \frac{-1+\sqrt{1+4e^{-2\tau\omega}}}{2},$$

où $V = \sup(|A|+|B|)$, $B = \sup|B|$, alors σ_M se trouve dans $|z|<1$, donc les propriétés de stabilité dépendant seulement du spectre de $\langle L(\sigma,s),\varphi(\sigma)\rangle$, le spectre de cet opérateur est identique à celui de la matrice $\langle X(\tau,\sigma+\tau), X(\omega+\sigma,\tau)\rangle$, qui coincide avec $X(\omega+\tau,\tau)$.

On obtient ainsi le

Théorème I2. Si τB est suffisamment petit et $\omega > 2\tau$, alors les propriétés de stabilité du système (4) (avec $f \not\equiv 0$) ne dépendent que des valeurs propres de la matrice $X(\omega+\tau,\tau)$.

Une énoncé analogue s'obtient aussi pour le système (5).

Nous allons esquisser la preuve pour le cas des systèmes (5).

On a

$$U_\omega\varphi = \varphi(0)X(\omega+s,0) + \int_{-\tau}^{0}\varphi(\beta)d\beta\int_{\beta}^{0} b(\alpha+\tau,\beta-\alpha-\tau)X(\omega+s,\alpha+\tau)d\alpha$$

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Soit $K(\alpha,s)=X(\omega+s,\alpha+\tau)$ on définit l'opération

$$\langle \varphi,\psi\rangle=\varphi(\sigma)\psi(\sigma-\tau)+\int_{\sigma-\tau}^{\sigma}\varphi(\beta)d_\beta\int_{\sigma-\tau}^{\beta}\eta(\xi+\tau,\beta-\xi-\tau)\psi(\xi)d\xi$$

Les multiplicateurs sont les valeurs $\rho=\frac{1}{\lambda}$ pour lesquelles l'equation $\varphi(s)=\lambda\langle\varphi(\sigma),K(\sigma,s)\rangle$ admet des solutions non identiquement nulles. La solution de l'équation $\varphi(s)=\lambda\langle\varphi(\sigma),K(\sigma,s)\rangle+X(s)$ peut être construite sous la forme

$\varphi(s)=\sum_{i=0}^{\infty}\lambda^i\varphi_i(s)$ et l'on prouve que la série converge uniformément pour $|\lambda|<\frac{1}{M(1+\tau V)}$, $M\geq|K(\alpha,s)|$. Il s'ensuit que le spectre de l'opérateur $\langle\varphi(\sigma),K(\sigma,s)\rangle$ se trouve dans $|z|\leq M(1+\tau V)$. Par un calcul direct, en utilisant la formule du changement de l'ordre de l'intégration, on prouve la relation.

$$\langle\langle L(\beta),M(\beta,\sigma)\rangle,N(\sigma)\rangle=\langle L(\beta),\langle M(\beta,\sigma),N(\sigma)\rangle\rangle.$$

A l'aide de cette relation on prouve facilement que si $K(\alpha,s)=L(\alpha,s)+M(\alpha,s)$ et si $\langle L(\alpha,\sigma),M(\sigma,s)\rangle=0$ alors $\sigma_K=\sigma_L\cup\sigma_M$. En utilisant l'intégration par parties et tenant compte de l'équation (5) pour $f\equiv 0$ et de l'equation adjointe correspondante, on établit la formule, valable pour $\omega>2\tau$

$$X(\omega+s,\alpha+\tau)=X(\tau,\alpha+\tau)X(\omega+s,\tau)+$$
$$+\int_0^{\tau}X(\beta,\alpha+\tau)d_\beta\int_{\tau}^{\omega+s}\eta(\xi,\beta-\xi)X(\omega+s,\xi)d\xi.$$

Si l'on désigne

$$L(\alpha,s)=X(\tau,\alpha+\tau)X(\omega+s,\tau)$$

$$M(\alpha,s)=\int_0^{\tau}X(\beta,\alpha+\tau)d_\beta\int_{\tau}^{\omega+s}\eta(\xi,\beta-\xi)X(\omega+s,\xi)d\xi$$

on obtient $K(\alpha,s)=L(\alpha,s)+M(\alpha,s)$.

Par un calcul direct on prouve que $\langle L(\alpha,\sigma),M(\sigma,s)\rangle=0$. En utilisant les évaluations

$$|X(\omega+s,\beta+\tau)|\leq\kappa e^{V\omega},\quad |X(\tau,\alpha+\tau)|\leq\kappa e^{V\tau}$$

(où κ dépend de la manière dont on a défini la norme) on en déduit $|M(\alpha,s)|\leq\kappa^2\tau V e^{V(\tau+\omega)}$; si τ est assez petit pour que $\kappa^2\tau V(1+\tau V)<e^{-V(\tau+\omega)}$ il s'ensuit que $|M(\alpha,s)|<\frac{1}{1+\tau V}$ et σ_M est dans le cercle $|z|<1$. Il s'ensuit que les propriétés de stabilité ne dépendent dans ce cas que du spectre de L.

L'equation qui donne σ_L s'écrit

$$\rho\varphi(s)=\langle\varphi(\sigma),X(\tau,\sigma+\tau)X(\omega+s,\tau)\rangle=$$
$$=\langle\varphi(\sigma),X(\tau,\sigma+\tau)\rangle X(\omega+s,\tau).$$

On cherche la solution sous la forme

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$\varphi(s) = \frac{1}{\rho} a X(\omega+s,\tau)$. On obtient

$$a\left[\rho E - \langle X(\omega+\sigma,\tau)X(\tau,\sigma+\tau)\rangle\right] X(\omega+s,\tau) = 0.$$

On prouve que $\det X(\omega+s,\tau)$ ne peut être identiquement nul, donc le spectre de $\langle A(\sigma), L(\sigma,s)\rangle$ coincide avec celui de la matrice

$$\langle X(\omega+\sigma,\tau), X(\tau,\sigma+\tau)\rangle.$$

On a

$$\int_0^\tau X(t+\omega,\tau)\,d_t Y(t,\tau) = X(\tau+\omega,\tau)Y(\tau,\tau) -$$

$$- X(\omega,\tau)Y(o,\tau) - \int_0^\tau \dot{X}(t+\omega,\tau)Y(t,\tau)\,dt =$$

$$= X(\tau+\omega,\tau) - X(\omega,\tau)Y(o,\tau) -$$

$$- \int_{-\tau}^{o} X(\omega+s,\tau)\,d_s \int_o^\tau \eta(t,s-t)Y(t,\tau)\,dt -$$

$$- \int_{-\tau}^{o} X(\omega+s,\tau)\,d_s \int_s^\tau \eta(t,s-t)Y(t,\tau)\,dt, \text{ car } Y(t,\tau) = E.$$

Il s'ensuit que

$$X(\omega+\tau,\tau) = X(\omega,\tau)Y(o,\tau) +$$

$$+ \int_o^\tau X(\omega+s,\tau)\,d_s\left[Y(s,\tau) + \int_s^\tau \eta(t,s-t)Y(t,\tau)\,dt\right] +$$

$$+ \int_{-\tau}^{o} X(\omega+s,\tau)\,d_s \int_o^\tau \eta(t,s-t)Y(t,\tau)\,dt.$$

En tenant compte du fait que $Y(s,t) = X(t,s)$ vérifie système adjoint, on en déduit que

$$\langle X(\omega+\sigma,\tau), X(\tau,\sigma+\tau)\rangle = X(\omega+\tau,\tau).$$

On prouve ainsi que si τ est suffisamment petit, les propriétés de stabilité ne dépendent que du spectre de la matrice $X(\omega+\tau,\tau)$. Se pose naturellement le problème d'utiliser le théorème 12 pour obtenir des critères effectifs pour la stabilité.

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М. ХУКУХАРА

О РАЗЛОЖЕНИИ РЕШЕНИЯ СИСТЕМЫ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ОКРЕСТНОСТИ ТОЧКИ РАВНОВЕСИЯ

Nous avons déjà étudié /Journ. Fac. Sci., Univ. Tokyo, Sec. I, 6 /1951/, 297-317/ le système de deux équations différentielles ordinaires

$$dx_1/dt = f_1(x_1, x_2), \quad dx_2/dt = f_2(x_1, x_2)$$

sous les hypothèses suivantes:

1° f_1 et f_2 sont des fonctions réelles pour les valeurs réelles et assez petites de x_1, x_2 nulles pour $x_1 = x_2 = 0$ et holomorphes dans un voisinage de $(0, 0)$;

2° Les valeurs propres de la matrice formée des coefficients des termes du premier degré dans les développements de f_1, f_2 sont $\pm i$.

Il y a trois cas à distinguer.

PREMIER CAS. La solution réelle peut se développer en une série entière de

$$\xi = Ce^{it}, \quad \bar{\xi} = \bar{C}e^{-it}$$

convergente pour les valeurs assez petites de ξ, $\bar{\xi}$ où C et $\bar{C}$ sont des constantes conjuguées arbitraires.

DEUXIÈME CAS. La solution réelle peut se développer en une série entière de

$$\xi = C\exp(i(1+\delta C^n\bar{C}^n)t), \quad \bar{\xi} = \bar{C}\exp(i(1+\delta C^n\bar{C}^n)t)$$

convergente pour les valeurs assez petites de ξ, $\bar{\xi}$ où C et $\bar{C}$ sont des constantes conjuguées arbitraires et δ une constante réelle déterminée. La solution admet une période qui dépend de la constante [illegible]

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TROISIÈME CAS. La solution réelle peut s'écrire

$$x_1 + i x_2 = \xi F(\xi, \bar{\xi}),$$
$$\xi = \rho e^{i\varphi}, \quad \bar{\xi} = \rho e^{-i\varphi},$$
$$d\rho/dt = \rho^{2n+1}(a + a'\rho^{2n}),$$
$$d\varphi/dt = 1 + \sum_{j=1}^{n} b_j \rho^{2j},$$

$a\ (\neq 0)$, a' et les b_j sont des constantes réelles déterminées et $F(\xi, \bar{\xi})$ se développe asymptotiquement en une série entière de ξ, $\bar{\xi}$, le terme constant étant égal à 1. Toute solution réelle est asymptote à la solution $x_1 = x_2 = 0$.

Nous voulons étendre ces résultsts à un systeme différentiel ordinaire

$$\frac{dx_j}{dt} = f_j(x_1, \ldots, x_n), \quad (j = 1, 2, \ldots, n)$$

en supposant remplies les conditions suivantes:

/1/ Les f_j sont des fonctions réelles pour les valeurs réelles assez petites des x_j, nulles pour $x_1 = \ldots = x_n = 0$ et holomorphes dans un voisinage de $(0, \ldots, 0)$;

/2/ La matrice formée des coefficients du premier degré dans les developpements des f_j peut se transformer à une matrice diagonale dont les éléments diagonaux sont $\lambda_1, \ldots, \lambda_n$;

/3/ Quant aux valeurs propres $\lambda_1, \ldots, \lambda_n$, nous supposons que l'on ait

$$\lambda_1 = i, \quad \lambda_2 = -i,$$
$$R\lambda_j < 0 \quad \text{pour} \quad j = 3, \ldots, m$$
$$R\lambda_j > 0 \quad \text{pour} \quad j = m+1, \ldots, n$$

et que l'on n'ait

$$\lambda_j = \kappa_1 \lambda_1 + \cdots \kappa_n \lambda_n \quad (\kappa_1 + \cdots + \kappa_n > 1)$$

que pour

(I) $j = 1$, $\kappa_1 = \kappa_2 + 1$, $\kappa_3 = \ldots = \kappa_n = 0$,

(II) $j = 2$, $\kappa_2 = \kappa_1 + 1$, $\kappa_3 = \ldots = \kappa_n = 0$,

(III) $j > 2$, $\kappa_1 = \kappa_2 > 0$, et $\kappa_3 = \ldots = \kappa_n = 0$ sauf $\kappa_j = 1$.

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